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Geometric constant on Riemannian manifold evolves by geometric flow

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ABSTRACT. In this article, we consider $(M^n, g(t))$ an n-dimensional closed Riemannian manifold whose metric $g(t)$ evolves by the abstract geometric flow and the geometric constant λ_a^b as the lowest constant such that the equation

$$-\Delta u + au\log u + bSu = \lambda_a^b u$$

with $\int_M u^2 d\mu = 1$ has a positive solution, where a (> 0) and b are two real constants. Here we find the evolution formula for λ_a^b on $(M^n, g(t))$ evolving along the abstract geometric flow.

1. INTRODUCTION

The study of geometric flows plays an important role in Riemannian manifold. There are many important geometric flows to gain information of the manifolds. Here we consider geometric flow in general way. Let M^n be a closed Riemannian manifold of dimension n whose Riemannian metric $g(t)$ evolves by the abstract geometric flow

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}, \ t \in [0, T), \tag{1.1}$$

where $T(> 0)$ is the maximum time of existence and S_{ij} is a time dependent symmetric $(0, 2)$-tensor on $(M^n, g(t))$. Some examples of geometric flows are the Ricci flow when $S_{ij} = R_{ij}$, i.e., the Ricci curvature tensor, the extended Ricci flow when $S_{ij} = R_{ij} - \alpha\nabla_i\phi \otimes \nabla_j\phi$ (where α is a positive constant depending only on n and $\phi = \phi(t)$ is a smooth scalar function) etc. Let us denote $S = g^{ij}S_{ij}$, i.e., the trace of S_{ij} with respect to $g(t)$.

Recently, many authors studied the evolution formula of the eigenvalues of different operators along many geometric flows such as the Ricci flow, extended Ricci flow, Ricci-Bourguignon flow, mean curvature flow etc., see [4, 5, 8, 12], after the work of Perelman in [17], where he showed that the lowest eigenvalue of $-\Delta + R/4$ (R is scalar curvature) is monotone nondecreasing along the Ricci flow. Recently in [3], Abolarinwa et al. studied the evolution, monotonicity and differentiability of the first eigenvalue of the p-Laplacian on $(M^n, g(t))$, whose metric $g(t)$ evolves by the generalized abstract geometric flow. It is mentioned that several estimates have been studied for nonlinear partial differential equations in [1, 2]. In [7], Daneshvar et al. constructed various monotonicity formulas for the lowest constant $\lambda_a^b(g)$ under the Ricci-Bourguignon flow such that the equation

$$-\Delta u + au\log u + bRu = \lambda_a^b(g)u \ \text{ with } \int_M u^2 d\mu = 1$$

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has a positive solution, where a and b are two real constants. In [11], Huang et al. studied the first variation formula for the lowest $\lambda_a^b(g)$ such that the following nonlinear equation

$$-\Delta u + au\log u + bRu = \lambda_a^b(g)u \text{ with } \int_M u^2 d\mu = 1$$

has a positive solution along the Ricci flow and the normalized Ricci flow. Recently, monotonicity of a geometric constant along the extended Ricci flow has been studied in [18]. Motivated by the aforementioned works, in this article we consider λ_a^b to be the lowest constant such that the following nonlinear equation

$$-\Delta u + au\log u + bSu = \lambda_a^b u \tag{1.2}$$

with the normalization condition

$$\int_M u^2 d\mu = 1, \tag{1.3}$$

where a (> 0) and b are two real constants, has a positive solution. Here we have obtained the evolution formula for the lowest constant $\lambda_a^b(g)$ such that equation (1.2) with the normalization condition (1.3) has a positive solution, when the Riemannian manifold $(M^n, g(t))$ evolves along the abstract geometric flow (1.1). Also, we generalize some results in [11]. We consider the following functional

$$\mathcal{F}_d^c(g,\phi) = \int_M (|\nabla\phi|^2 + cS - d(\phi+1))e^{-\phi}d\mu, \tag{1.4}$$

where c and d are two real constants, under the abstract geometric flow coupled to a nonlinear backward type heat equation

$$\begin{cases} \frac{\partial}{\partial t}g_{ij} = -2S_{ij}, \\ \frac{\partial}{\partial t}\phi = -\Delta\phi + |\nabla\phi|^2 + a\phi - S. \end{cases} \tag{1.5}$$

Finally we have obtained the first variation formula of the functional (1.4) under the system (1.5) and we proved that this functional is nondecreasing for $d = \frac{nac}{8}$ along the flow.

2. Preliminaries

Let us consider a local coordinate system $\{x^i\}$ in a neighborhood of every point $x \in M^n$, where M^n is an n-dimensional closed Riemannian manifold with Riemannian metric g_{ij}. We denote $g^{ij} = (g_{ij})^{-1}$, the inverse metric matrix and ∇ as the covariant derivative. Note that for a smooth function u

$$\nabla_i u = u_i = \frac{\partial u}{\partial x^i} \textit{ and } |\nabla u|^2 = g^{ij}\nabla_i u\nabla_j u = \nabla^i u\nabla_i u = u^i u_i.$$

The Riemannian volume measure $d\mu$ on M^n is given by $d\mu = \sqrt{|g_{ij}|}dx^i$. The divergence of a $(0,2)$-tensor $\mathcal{S}$ is defined by $\mathsf{div}(\mathcal{S})_k = g^{ij}\nabla_i S_{jk}$. We denote time derivative by $\frac{\partial}{\partial t}u = u_t$. In the following Lemma some geometric quantities are given when the Riemannian manifold $(M^n, g(t))$ evolves along the abstract geometric flow (1.1).

Lemma 2.1. [3] *Suppose $g(t)$ is a solution of the abstract geometric flow (1.1). Then the following equations hold:*

$$(i)\ \frac{\partial}{\partial t}g^{ij} = 2g^{ik}g^{jl}S_{kl}, \tag{2.6}$$

$$(ii)\ \frac{\partial}{\partial t}|\nabla u|^2 = 2S^{ij}u_i u_j + 2\langle\nabla u_t, \nabla u\rangle, \tag{2.7}$$

$$(iii)\ \frac{\partial}{\partial t}d\mu = -Sd\mu. \tag{2.8}$$

Along the abstract geometric flow (1.1), S evolves by

$$\frac{\partial}{\partial t}S = \frac{\partial}{\partial t}(g^{ij}S_{ij}) = 2|S_{ij}|^2 + T,$$

where we take $T_{ij} = \frac{\partial S_{ij}}{\partial t}$ and $T = g^{ij}\frac{\partial S_{ij}}{\partial t}$. Similar to an error term appeared in [15, Lemma 1.6], here we have used an error term for any time dependent vector X by

$$\mathcal{D}(X) = (R_{ij} - S_{ij})(X, X) + \langle \nabla S - 2\,\mathrm{div}\,\mathcal{S}, X\rangle + \frac{1}{2}(T - \Delta S). \tag{2.9}$$

Examples of Geometric flows:

A. **Hamilton's Ricci flow:** [10] For Hamilton's Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}, \tag{2.10}$$

we have $S_{ij} = R_{ij}$, the Ricci tensor and $S = R$, the scalar curvature on M. The evolution equation for R is given by $\frac{\partial R}{\partial t} = 2|Ric|^2 + \Delta R$, which shows that $T = \Delta R$. Now by using $\nabla R = 2\mathrm{div}(Ric)$ we get

$$\mathcal{D}(X) = 0,\ T - \Delta S = 0. \tag{2.11}$$

B. **Extended Ricci flow:**[14] Extended Ricci flow is given by the following equations

$$\begin{cases} \frac{\partial}{\partial t}g = -2Ric + 2\alpha\nabla\phi\otimes\nabla\phi, \\ \frac{\partial}{\partial t}\phi = \Delta\phi, \end{cases} \tag{2.12}$$

where α is a positive constant depending only on n, $\phi = \phi(t)$ is a smooth scalar function defined on M. Thus in our notations we have $\mathcal{S} = Ric - \alpha d\phi\otimes d\phi$ and $S = R - \alpha|\nabla\phi|^2$, which gives

$$\nabla S - 2\mathrm{div}(\mathcal{S}) = 2\alpha\Delta\phi\nabla\phi, \tag{2.13}$$

$$T = \Delta S + 2\alpha(\Delta\phi)^2. \tag{2.14}$$

Thus the error term is

$$\mathcal{D}(X) = \alpha(\langle\nabla\phi, X\rangle + \Delta\phi)^2. \tag{2.15}$$

C. **Ricci-harmonic map flow :** [16] Let (M^n, g) and (N^m, h) be two closed Riemannian manifolds of dimension n and m respectively and $\psi : M \to N$ a family of 1-parameter smooth maps. Then the couple $(g(t), \psi(t))$ is said to be a solution of the Ricci-harmonic map flow if it satisfies the following system of nonlinear parabolic equations

$$\begin{cases} \frac{\partial}{\partial t}g = -2Ric + 2\alpha(t)\nabla\psi\otimes\nabla\psi, \\ \frac{\partial}{\partial t}\psi = \tau_g\psi, \end{cases} \tag{2.16}$$

where τ_g denotes the tension field of the map ψ with respect to the Riemannian metric $g(t)$ and $\alpha(t) > 0$ is a time dependent constant. Here we have $S_{ij} = R_{ij} - \alpha\partial_i\psi\otimes\partial_j\psi$ and $S = R - \alpha|\nabla\psi|^2$ and

$$\frac{\partial}{\partial t}S = \Delta S + 2|S_{ij}|^2 + 2\alpha|\tau_g\psi|^2 - 2\alpha|\nabla\psi|^2. \tag{2.17}$$

Then after some computations, we get

$$\mathcal{D}(X) = \alpha|\tau_g\psi + \nabla_X\psi|^2 - \frac{1}{2}\alpha^{'}|\nabla\psi|^2. \tag{2.18}$$

D. **Lorentzian mean curvature flow:** [13] Let M^n be a family of space-like hypersurfaces of a Lorentzian manifold L^{n+1}. Let $F(t, x)$ be the position function of M^n in L^{n+1}. If at $F(t,x)$ the outer normal vector and mean curvature denoted by $\nu(t,x)$ and $H(t,x)$ respectively, then the Lorentzian mean curvature flow is given by the following equation

$$\frac{\partial}{\partial t}F(t,x) = H(t,x)\nu(t,x).$$

Then the induced metric $g(t)$ evolves by $\frac{\partial}{\partial t}g_{ij} = 2Hh_{ij}$, where h_{ij} are the components of the second fundamental form H on M^n. Let $\widetilde{Ric}$ and $\widetilde{Riem}$ be the Ricci and Riemannian curvature tensor of L^{n+1} respectively. Then we have

$$T - \Delta S = 2H^2|h_{ij}|^2 + 2|\nabla H|^2 + 2H^2\widetilde{Ric}(\nu,\nu),$$

$$\mathcal{D}(X) = |\nabla H + h(X,\cdot)|^2 + \widetilde{Ric}(H\nu + X, H\nu + X) + \widetilde{Riem}(X,\nu,\nu,X).$$

E. **Yamabe flow:** [6, 19] Yamabe flow is given by the following evolution equation

$$\frac{\partial}{\partial t}g_{ij} = -Rg_{ij},$$

where R is the scalar curvature. By [6], we have $\frac{\partial}{\partial t}R = (n-1)\Delta R + R^2$. By comparing with the flow (1.1) we have $S_{ij} = \frac{1}{2}Rg_{ij}$. Thus we have $\frac{\partial}{\partial t}S = \frac{n(n-1)}{2}\Delta R + \frac{n}{2}R^2$. Hence we can conclude

$$T - \Delta S = \frac{n(n-2)}{2}\Delta R \text{ and } \langle \nabla S - 2\mathrm{div}(\mathcal{S}), X\rangle = \frac{n-2}{2}\langle \nabla R, X\rangle.$$

3. Main Results

The lowest constant $\lambda_a^b(g)$, such that the equation (1.2) with (1.3) has a positive solution is defined by

$$\lambda_a^b(g) = \inf\{\mathcal{G}_a^b(g,u) : \int_M u^2 d\mu = 1,\ u > 0,\ u \in C^\infty(M)\},$$

where

$$\mathcal{G}_a^b(g,u) = \int_M (|\nabla u|^2 + au^2\log u + bSu^2)d\mu.$$

Now we show the existence of $\lambda_a^b(g)$ for any closed Riemannian manifold M^n. For this we only need to prove that the set

$$\{\mathcal{G}_a^b(g,u) : \int_M u^2 d\mu = 1,\ u > 0,\ u \in C^\infty(M)\}$$

is bounded below. Applying the trick used in [7], it is easy to see that the set $\{\int_M au^2 \log u d\mu$: $\int_M u^2 d\mu = 1,\ u > 0,\ u \in C^\infty(M)\}$ is bounded below for $a > 0$. Also from ([9], pg. 13, eq. (7.35)) we can say that the set $\{\int_M(|\nabla u|^2 + bSu^2)d\mu : \int_M u^2 d\mu = 1,\ u > 0,\ u \in C^\infty(M)\}$ takes its infimum. Thus we have that the set $\{\mathcal{G}_a^b(g,u) : \int_M u^2 d\mu = 1,\ u > 0,\ u \in C^\infty(M)\}$ is bounded below and consequently $\lambda_a^b(g)$ exists.
Using the method of [7, Lemma 2.1] and by taking the conditions $(1+\epsilon)^{-1}g_1 \leq g_2 \leq (1+\epsilon)g_1$ and $S(g_1) - \epsilon \leq S(g_2) \leq S(g_1) + \epsilon$, it can be shown that the geometric constant λ_a^b is continuous function with respect to the C^2-topology.

Till now it is not known that the lowest geometric constant λ_a^b such that the equation (1.2) with the normalization condition (1.3) has a positive solution, and corresponding function u are differentiable along t or not. To overcome this problem of differentiability we proceed further as [9]. According as [9, Theorem 7.2], for any $t_0 \in [0,T)$, there exists a smooth function $w(t) > 0$ satisfying $\int_M w^2(t)d\mu = 1$ and $w(t_0) = u(t_0)$. Let

$$\psi(t) = \int_M [-w(t)\Delta w(t) + aw^2(t)\log w(t) + bSw^2(t)]d\mu.$$

Then $\psi(t)$ is a smooth function by definition. And at a time t_0, we can conclude that $\lambda_a^b(t_0) = \psi(t_0)$.

Theorem 3.1. *We assume that the geometric flow (1.1) has a solution on the interval* $[0, T)$*, where* $T(>0)$ *is the maximum time of existence. Let* $g(t)$*,* $t \in [0, T)$*, be a solution of (1.1) on a closed Riemannian manifold* M^n *and* $\lambda_a^b(g)$ *be the lowest geometric constant such that the equation (1.2) with the normalization condition (1.3) has a positive solution. Suppose that* $u(t_0)$ *is the corresponding solution to* $\lambda_a^b(t_0)$*. Then we have*

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \frac{1}{2}\int_M |S_{ij} + \nabla_i\nabla_j f + \frac{a}{2}g_{ij}|^2 e^{-f}d\mu + (b - \frac{1}{4})\int_M (T - \Delta S)e^{-f}d\mu + (2b - \frac{1}{2})\int_M |S_{ij}|^2 e^{-f}d\mu + \int_M \frac{1}{2}\mathcal{D}(-\nabla f)e^{-f}d\mu - \frac{na^2}{8}, \tag{3.19}$$

where $f = -2\log w$.

Proof. By definition we have

$$\psi(t) = \int_M (|\nabla w|^2 + aw^2\log w + bSw^2)d\mu. \tag{3.20}$$

Using Lemma 2.1, we have

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \int_M (2S^{ij}w_iw_j + 2\langle\nabla w_t, \nabla w\rangle + 2aww_t\log w + aww_t + 2bSww_t + b\frac{\partial S}{\partial t}w^2)d\mu - \int_M (|\nabla w|^2 + aw^2\log w + bSw^2)Sd\mu. \tag{3.21}$$

Using integration by parts

$$\int_M \langle\nabla w_t, \nabla w\rangle d\mu = -\int_M w_t\Delta w d\mu. \tag{3.22}$$

Also, we have

$$-\int_M |\nabla w|^2 S d\mu = -\frac{1}{2}\int_M S\Delta w^2 d\mu + \int_M Sw\Delta w d\mu. \tag{3.23}$$

Applying (3.22) and (3.23) in (3.21), we get

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \int_M (2S^{ij}w_iw_j + aww_t + b\frac{\partial S}{\partial t}w^2 - \frac{1}{2}S\Delta w^2)d\mu + \psi(t_0)\int_M (2ww_t - Sw^2)d\mu. \tag{3.24}$$

From the normalization condition $\int_M w^2 d\mu = 1$, we conclude that

$$2\int_M ww_t d\mu = \int_M Sw^2 d\mu. \tag{3.25}$$

Using (3.25) in (3.24) we get

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \int_M (2S^{ij}w_iw_j + 2b|S_{ij}|^2w^2 + \frac{a}{2}Sw^2 + b(T - \Delta S)w^2 + (b - \frac{1}{2})S\Delta w^2)d\mu.$$

Again we have by definition $\psi(t_0)w(t_0) = (-\Delta w + aw\log w + bSw)(t_0)$. Thus

$$(bS\Delta w^2)(t_0) = (\psi\Delta w^2 + 2(\Delta w)^2 + 2\frac{\Delta w|\nabla w|^2}{w} - a\log w\,\Delta w^2)(t_0). \tag{3.26}$$

Thus from (3.26), using integration by parts, we get

$$\int_M bS\Delta w^2 d\mu = -2\int_M \langle \nabla w, \nabla(\Delta w)\rangle d\mu - 2\int_M \left\langle \nabla w, \nabla\left(\frac{|\nabla w|^2}{w}\right)\right\rangle d\mu - \int_M a\log w\, \Delta w^2 d\mu. \tag{3.27}$$

From Bochner's formula we have $-2\langle \nabla w, \nabla \Delta w\rangle = 2|w_{ij}|^2 + 2R^{ij}w_i w_j - \Delta(|\nabla w|^2)$. Using $\nabla\nabla \log w = \frac{\nabla\nabla w}{w} - \nabla \log w \otimes \nabla \log w$, we get

$$\left\langle \nabla w, \nabla\left(\frac{|\nabla w|^2}{w}\right)\right\rangle = |\nabla\nabla w|^2 - w^2|\nabla\nabla \log w|^2. \tag{3.28}$$

Finding the value of the first integral of RHS of (3.27) using Bochner's formula and putting (3.28) in (3.27) we get

$$\int_M bS\Delta w^2 d\mu = 2\int_M w^2|\nabla\nabla \log w|^2 d\mu + 2\int_M R^{ij}w_i w_j d\mu - \int_M aw^2\Delta \log w\, d\mu. \tag{3.29}$$

Using the integration by parts

$$\int_M S^{ij}w_i w_j d\mu = -\int_M w\langle \mathrm{div}(\mathcal{S}), \nabla w\rangle d\mu - \int_M w^2\langle \mathcal{S}, \nabla\nabla \log w\rangle d\mu - \int_M S^{ij}w_i w_j d\mu. \tag{3.30}$$

Finally using (3.29) and (3.30), we obtain

$$\begin{aligned}\frac{d}{dt}\psi(t)|_{t=t_0} &= \int_M \Big\{ -2w\langle \mathrm{div}(\mathcal{S}), \nabla w\rangle - 2w^2\langle \mathcal{S}, \nabla\nabla \log w\rangle - 2S^{ij}w_i w_j + 2b|S_{ij}|^2 w^2 + \frac{a}{2}Sw^2 \\ &\quad + b(T-\Delta S)w^2 + 2w^2|\nabla\nabla \log w|^2 + 2R^{ij}w_i w_j - aw^2\Delta \log w - \frac{1}{2}S\Delta w^2\Big\} d\mu \\ &= \int_M \frac{1}{2}\Big\{|S_{ij}|^2 + 4|\nabla\nabla \log w|^2 + \frac{na^2}{4} - 4\langle \mathcal{S}, \nabla\nabla \log w\rangle - 2a\Delta \log w + aS\Big\} w^2 d\mu \\ &\quad + (2b - \frac{1}{2})\int_M |S_{ij}|^2 w^2 d\mu - \frac{na^2}{8} + (b - \frac{1}{4})\int_M (T-\Delta S)w^2 d\mu \\ &\quad + \int_M \frac{1}{2}\Big\{4Ric(\nabla \log w, \nabla \log w) - 4\mathcal{S}(\nabla \log w, \nabla \log w) - 4\langle \mathrm{div}(\mathcal{S}), \nabla \log w\rangle \\ &\qquad + 2\langle \nabla S, \nabla \log w\rangle + \frac{1}{2}(T-\Delta S)\Big\} w^2 d\mu.\end{aligned}$$

Now by putting $f = -2\log w$, in the above equality we obtain the desired result. □

Remark 3.1. Taking $b = \frac{1}{4}$ in (3.19) we have the following

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \frac{1}{2}\int_M |S_{ij} + \nabla_i\nabla_j f + \frac{a}{2}g_{ij}|^2 e^{-f} d\mu + \int_M \frac{1}{2}\mathcal{D}(-\nabla f)e^{-f} d\mu - \frac{na^2}{8}. \tag{3.31}$$

Remark 3.2. When we take the geometric flow as the Hamilton's Ricci flow, from (2.11) we have $\mathcal{D}(X) = 0$ and $T - \Delta S = 0$. Thus from (3.19) we get

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \frac{1}{2}\int_M |R_{ij} - 2\nabla_i\nabla_j \log w + \frac{a}{2}g_{ij}|^2 w^2 d\mu + (2b - \frac{1}{2})\int_M |R_{ij}|^2 w^2 d\mu - \frac{na^2}{8}.$$

If we take the transformation $w^2 = e^{-f}$, then the above equation reduces to

$$\frac{d}{dt}\psi(t)|_{t=t_0} = \frac{1}{2}\int_M |R_{ij} + f_{ij} + \frac{a}{2}g_{ij}|^2 e^{-f} d\mu + (2b - \frac{1}{2})\int_M |R_{ij}|^2 e^{-f} d\mu - \frac{na^2}{8},$$

which is exactly the Theorem 1.1 in [11].

Theorem 3.2. *We assume that the geometric flow (1.1) has a solution on the interval* $[0, T)$, *where* $T(>0)$ *is the maximum time of existence. Let* $g(t)$, $t \in [0, T)$, *be a solution of (1.1) on a closed Riemannian manifold* M^n *and* $\lambda_a^b(g)$ *be the lowest geometric constant such that the equation (1.2) with the normalization condition (1.3) has a positive solution. If* $\mathcal{D} \geq 0,\ b \geq \frac{1}{4}$ *and* $T - \Delta S \geq 0$ *along the flow (1.1) then* $\frac{d}{dt}\left(\lambda_a^b(t) + \frac{na^2}{8}t\right) \geq 0$, *and therefore the quantity* $\lambda_a^b(t) + \frac{na^2}{8}t$ *is nondecreasing along the flow (1.1).*

Proof. From Theorem 3.1, we have by using the assumptions $\mathcal{D} \geq 0,\ b \geq \frac{1}{4}$ and $T - \Delta S \geq 0$,

$$\frac{d}{dt}\left(\psi(t) + \frac{na^2}{8}t\right)|_{t=t_0} \geq 0.$$

By definition it is clear that $\psi(t)$ is a smooth function in t. Therefore in any small interval $(t_0 - \epsilon, t_0 + \epsilon)$ for sufficiently $\epsilon > 0$,

$$\frac{d}{dt}\left(\psi(t) + \frac{na^2}{8}t\right) \geq 0.$$

Thus

$$\psi(t_0) + \frac{na^2}{8}t_0 \geq \psi(t_1) + \frac{na^2}{8}t_1,$$

for $t_1 < t_0$ and $t_1 \in (t_0 - \epsilon, t_0 + \epsilon)$. Again by definition $\psi(t_0) = \lambda_a^b(t_0)$ and $\psi(t_1) \geq \lambda_a^b(t_1)$. Since $t_0 \in [0, T)$ is arbitrary, so we can conclude that the quantity $\lambda_a^b + \frac{na^2}{8}t$ is nondecreasing along the flow. □

Definition 3.1. A solution $(M^n, g(t))$ of the flow (1.1) is called a breather whenever there are a positive constant c, a diffeomorphism $\eta : M^n \to M^n$, and times $t_1 < t_2$ such that $g(t_2) = c\eta^* g(t_1)$. The breather is called shrinking, steady, or expanding if $c < 1$, $c = 1$ or $c > 1$, respectively.

Theorem 3.3. *We assume that the geometric flow (1.1) has a solution on the interval* $[0, T)$, *where* $T(>0)$ *is the maximum time of existence. Let* $(M^n, g(t))$ *be a steady breather to the flow (1.1) and* $\lambda_a^b(t)$ *be the lowest geometric constant such that the equation (1.2) with the normalization condition (1.3) has a positive solution. Suppose that* $\mathcal{D} \geq 0$ *and* $T - \Delta S \geq 0$ *along the flow (1.1). Then* $\frac{d}{dt}\left(\lambda_a^b(t) + \frac{na^2}{8}t\right) \geq 0$.

Proof. From the Corollary 4.3 of [9], we have $T - \Delta S = 0$ and $\mathcal{S} = 0$. Hence the inequality $\frac{d}{dt}\left(\lambda_a^b(t) + \frac{na^2}{8}t\right) \geq 0$ holds. □

Theorem 3.4. *We assume that the geometric flow (1.1) has a solution on the interval* $[0, T)$, *where* $T(>0)$ *is the maximum time of existence. Let* $(M^n, g(t))$, $t \in [0, T)$ *be a closed Riemannian manifold evolves along the abstract geometric flow (1.1). Then the functional* $\mathcal{F}_d^c(g, \phi)$, *defined in (1.4) under the system (1.5), satisfies the following equation*

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_d^c(g, \phi) &= \int_M 2\mathcal{D}(-\nabla\phi)e^{-\phi}d\mu + (c-1)\int_M (T - \Delta S)e^{-\phi}d\mu + a\int_M |\nabla\phi|^2 e^{-\phi}d\mu \\ &\quad -d\int_M (-a\phi^2 - |\nabla\phi|^2 - S)e^{-\phi}d\mu + 2\int_M |S_{ij} + \phi_{ij} - \frac{a}{4}\phi g_{ij}|^2 e^{-\phi}d\mu \\ &\quad +(2c-2)\int_M |S_{ij} - \frac{a}{4}\phi g_{ij}|^2 e^{-\phi}d\mu - \frac{na^2c}{8}\int_M \phi^2 e^{-\phi}d\mu. \qquad (3.32)\end{aligned}$$

Proof. Using the system (1.5) we have

$$\frac{d}{dt}\int_M (\phi+1)e^{-\phi}d\mu = \int_M -\phi\Delta e^{-\phi}d\mu + \int_M (-a\phi^2 - S)e^{-\phi}d\mu = \int_M (-a\phi^2 - \Delta\phi - S)e^{-\phi}d\mu, \tag{3.33}$$

$$\frac{d}{dt}\int_M Se^{-\phi}d\mu = \int_M (2|S_{ij}|^2 + T - a\phi S)e^{-\phi}d\mu + \int_M S(-\Delta e^{-\phi})d\mu = \int_M (2|S_{ij}|^2 + T - \Delta S - a\phi S)e^{-\phi}d\mu, \tag{3.34}$$

and

$$\frac{d}{dt}\int_M |\nabla\phi|^2 e^{-\phi}d\mu = \int_M [2S^{ij}\phi_i\phi_j - 2\phi^i(\Delta\phi)_i + 4\phi^{ij}\phi_i\phi_j + 2a|\nabla\phi|^2 - 2\langle\nabla\phi, \nabla S\rangle]e^{-\phi}d\mu - \int_M |\nabla\phi|^2(\Delta e^{-\phi})d\mu - \int_M a\phi|\nabla\phi|^2 e^{-\phi}d\mu.$$

Now using $\int_M |\nabla\phi|^2(\Delta e^{-\phi})d\mu = 2\int_M \phi^{ij}\phi_i\phi_j e^{-\phi}d\mu$, we have

$$\frac{d}{dt}\int_M |\nabla\phi|^2 e^{-\phi}d\mu = \int_M [2S^{ij}\phi_i\phi_j - 2\phi^i(\Delta\phi)_i + 2\phi^{ij}\phi_i\phi_j + 2a|\nabla\phi|^2 - 2\langle\nabla\phi, \nabla S\rangle]e^{-\phi}d\mu - \int_M a\phi|\nabla\phi|^2 e^{-\phi}d\mu.$$

From Weitzenböck formula we have

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |u_{ij}|^2 + u^i(\Delta_\phi u)_i + (R^{ij} + \phi^{ij})u_i u_j, \quad \forall u, \tag{3.35}$$

where $\Delta_\phi u = \Delta u - \langle\nabla\phi, \nabla u\rangle$. Integrating the above formula we get

$$\int_M (-\phi^i(\Delta\phi)_i + \phi^{ij}\phi_i\phi_j)e^{-\phi}d\mu = \int_M (|\phi_{ij}|^2 + R^{ij}\phi_i\phi_j)e^{-\phi}d\mu. \tag{3.36}$$

So,

$$\frac{d}{dt}\int_M |\nabla\phi|^2 e^{-\phi}d\mu = \int_M [2S^{ij}\phi_i\phi_j + 2|\phi_{ij}|^2 + 2R^{ij}\phi_i\phi_j + 2a|\nabla\phi|^2 - 2\langle\nabla\phi, \nabla S\rangle]e^{-\phi}d\mu - \int_M a\phi|\nabla\phi|^2 e^{-\phi}d\mu. \tag{3.37}$$

Now differentiating (1.4) with respect to t and using (3.33), (3.34) and (3.37), we get

$$\frac{d}{dt}\mathcal{F}_d^c(g,\phi) = \int_M [2S^{ij}\phi_i\phi_j + 2|\phi_{ij}|^2 + 2R^{ij}\phi_i\phi_j + 2a|\nabla\phi|^2 - 2\langle\nabla\phi, \nabla S\rangle]e^{-\phi}d\mu - \int_M a\phi|\nabla\phi|^2 e^{-\phi}d\mu + c\int_M (2|S_{ij}|^2 + T - \Delta S - a\phi S)e^{-\phi}d\mu - d\int_M (-a\phi^2 - \Delta\phi - S)e^{-\phi}d\mu. \tag{3.38}$$

Using the integration by parts

$$\int_M S^{ij}\phi_i\phi_j e^{-\phi}d\mu = \int_M \langle \mathsf{div}(\mathcal{S}), \nabla\phi\rangle e^{-\phi}d\mu + \int_M S^{ij}\phi_{ij}e^{-\phi}d\mu \tag{3.39}$$

and using the above equation in (3.38) we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^c_d(g,\phi) &= \int_M [2(Ric-\mathcal{S})(\nabla\phi,\nabla\phi) - 2\langle\nabla S - 2\mathrm{div}(\mathcal{S}),\nabla\phi\rangle + (T-\Delta S)]e^{-\phi}d\mu \\ &+(c-1)\int_M (T-\Delta S)e^{-\phi}d\mu - d\int_M(-a\phi^2-\Delta\phi - S)e^{-\phi}d\mu \\ &+2\int_M [|S_{ij}|^2 + |\phi_{ij}|^2 + 2S^{ij}\phi_{ij} - \frac{a}{2}\phi S - \frac{a}{2}\phi\Delta\phi + \frac{na^2}{16}\phi^2]e^{-\phi}d\mu \\ &-a\int_M \phi|\nabla\phi|^2 e^{-\phi}d\mu + 2a\int_M |\nabla\phi|^2 e^{-\phi}d\mu + a\int_M \phi\Delta\phi e^{-\phi}d\mu \\ &+(2c-2)\int_M [|S_{ij}|^2 - \frac{a}{2}\phi S + \frac{na^2}{16}\phi^2]e^{-\phi}d\mu - \frac{na^2c}{8}\int_M \phi^2 e^{-\phi}d\mu.\end{aligned}$$

Thus the proof is complete. □

Remark 3.3. The above Theorem 3.4 generalizes the Theorem 1.4 of [11].

If in formula (3.32) we consider $d = \frac{nac}{8}$, then we get the following result:

Corollary 3.1. *We assume that the geometric flow (1.1) has a solution on the interval* $[0, T)$, *where* $T(>0)$ *is the maximum time of existence. Let* $(M^n, g(t))$ *be a closed Riemannian manifold evolves along the flow (1.1). If* $\mathcal{D} \geq 0,\ S \geq 0$, *and* $T - \Delta S \geq 0$ *along the flow (1.1) then, for* $c \geq 1$, *we have* $\frac{d}{dt}\mathcal{F}^c_{\frac{nac}{8}}(g,\phi) \geq 0$ *along the flow (1.1).*

4. Conclusion

The study of entropy formulas, eigenvalues and their monotonicities in the regime of geometric flows have become more interesting topic since the invention of the Ricci flow [10] and the work of Perelman [17]. Their geometric and topological implications are numerous. A striking instance is the noncolapsing theorem and the removal of the obstructions in the way of Hamilton Ricci flow for proving Poincaré conjecture completely by Perelman. Here in this paper, we consider a geometric constant such that nonlinear equation (1.2), with the normalization condition (1.3), has a positive solution. We derive variation formula for this geometric constant along abstract geometric flow (1.1) and consequently obtain monotonic quantities involving the geometric constant. We also obtain variational formula of the functional defined in (1.4) under the system (1.5). Our results generalizes some results of [11]. As a further study, one can replace the usual Laplace operator in (1.2) by weighted Laplace operator and study the variational formula of the geometric constant on a metric measure space along abstract geometric flow.

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