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The crossing numbers of join products of seven graphs of order six with paths and cycles

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ABSTRACT. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. The main aim of this paper is to give the crossing numbers of the join products of seven graphs on six vertices with paths and cycles on n vertices. The proofs are done with the help of several well-known auxiliary statements, the idea of which is extended by a suitable classification of subgraphs that do not cross the edges of the examined graphs. Finally, for m at least three and $n = 5$, we also establish the validity of a conjecture introduced by Staš and Valiska concerning the crossings numbers of the join products of the wheels on $m + 1$ vertices with the paths on n vertices.

1. INTRODUCTION

The problem of reducing the number of crossings on edges of graphs is interesting in many areas. One of the most popular areas is the implementation of the VLSI layout, which has revolutionized circuit design and had a strong impact on parallel computing. Crossing numbers were also studied to improve the readability of hierarchical structures and automated graphs. The visualized graph should be easy to read and understand. For the sake of clarity of the graphical drawings, the reduction of crossings is likely the most important. Therefore, the investigation on the crossing number of simple graphs is a classical, but very difficult problem. Garey and Johnson [7] proved that determining $\mathrm{cr}(G)$ is an NP-complete problem. Nevertheless, many researchers are trying to solve this problem. Note that the exact values of the crossing numbers are known for some families of graphs, see Clancy *et al.* [4].

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane (for the definition of a *drawing* see Klešč [11]). A drawing with a minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G by [11], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

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Some parts of proofs will be based on Kleitman's result [10] on the crossing numbers for some complete bipartite graphs $K_{m,n}$. He showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6. \tag{1.1}$$

The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. The crossings numbers of the join products of the paths and the cycles with all graphs of order at most four have been well-known for a long time by Klešč [12, 13], and Klešč and Schrötter [16], and therefore it is understandable that our immediate goal is to establish the exact values for the crossing numbers of $G + P_n$ and $G + C_n$ also for all graphs G of order five and six. Of course, the crossing numbers of $G + P_n$ and $G + C_n$ are already known for a lot of graphs G of order five and six [2, 5, 6, 11, 14, 17, 19, 21, 22, 23, 24, 26]. In all these cases, the graph G is connected and contains usually at least one cycle. Note that the crossing numbers of the join product $G + P_n$ and $G + C_n$ are known only for some disconnected graphs G on five or six vertices [3, 18, 25].

For this purpose, we present a new technique regarding the use of knowledge from the subgraphs whose values of crossing numbers are already known. Due to several possible isomorphisms, the results on the smaller graphs are important to confirm the validity of many conjectures, e.g., Corollary 5.11 in which the crossings numbers of the join products of the wheels W_m on $m + 1$ vertices with the paths P_n are established for m at least three and $n = 5$.

In this paper, we will use definitions and notation of the crossing numbers of graphs presented by Klešč [12]. We will also use special designation of seven graphs of order six that are represented by lower indexes in the order originally designated by Clancy *et al.* [4]. Their planar drawings are shown in Fig. 1, 2, and 9.

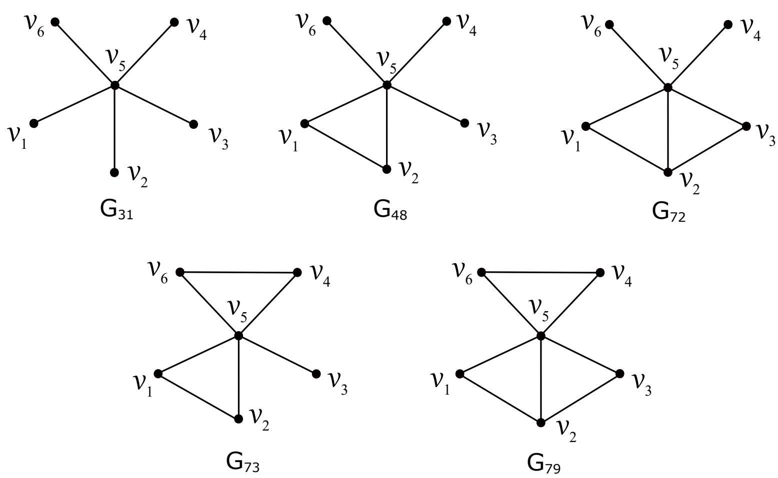


FIGURE 1. Planar drawings of five graphs G_{31}, G_{48}, G_{72}, G_{73}, and G_{79}.

Let G_{80} be the connected graph consisting of the complete bipartite graph $K_{1,5}$ and three edges which form the path P_4 on four leaves of $K_{1,5}$. The crossing number of $G_{80} + D_n$ is determined in Corollary 3.1 as some consequence of the result $\mathrm{cr}(K_{1,5,n})$

by Mei and Huang [20] if we add the four mentioned edges without additional crossings on them in some optimal drawing of $K_{1,5,n}$. The main aim of the paper is to establish the crossing numbers of $G_{80} + P_n$ and $G_{80} + C_n$ presented in Theorems 3.2 and 4.4, respectively. The paper concludes by giving the crossing numbers of the join products of one other graph G_{104} with D_n, P_n, and C_n in Corollaries 5.8, 5.9, and 5.10, respectively, where the graph G_{104} is obtained from G_{80} by adding new edge joining one vertex of order two with the leaf in G_{80}. In certain parts of the presented proofs, it is also possible to simplify the procedure with the help of software COGA generating all cyclic permutations of six elements. Its description can be found in Berežný and Buša [1]. In the proofs of the paper, we will often use the term "region" also in nonplanar subdrawings. In this case, crossings are considered to be vertices of the "map".

2. Cyclic Permutations and Possible Drawings of G_{80}

In the rest of the paper, let $V(G_{80}) = \{v_1, v_2, \dots, v_6\}$, and let v_5 and v_6 be the vertex notation of the dominating vertex and the leaf of G_{80} in all considered good subdrawings of the graph G_{80}, respectively. We consider the join product of the graph G_{80} with the discrete graph D_n, which yields that the graph $G_{80} + D_n$ consists of just one copy of G_{80} and n vertices $t_1, t_2, \dots, t_n$. Here, each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of the graph G_{80}. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the fixed vertex t_i. This means that the graph $T^1 \cup \cdots \cup T^n$ is isomorphic to the complete bipartite graph $K_{6,n}$ and

$$G_{80} + D_n = G_{80} \cup K_{6,n} = G_{80} \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.2}$$

The graph $G_{80} + P_n$ contains $G_{80} + D_n$ as a subgraph, and therefore let P_n^* denote the path induced on n vertices of $G_{80} + P_n$ not belonging to the subgraph G_{80}. The path P_n^* consists of the vertices $t_1, t_2, \dots, t_n$ and of the edges $\{t_i, t_{i+1}\}$ for $i = 1, 2, \dots, n-1$, and thus

$$G_{80} + P_n = G_{80} \cup K_{6,n} \cup P_n^* = G_{80} \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*. \tag{2.3}$$

Similarly, the graph $G_{80} + C_n$ contains both $G_{80} + D_n$ and $G_{80} + P_n$ as subgraphs. Let C_n^* denote the subgraph of $G_{80} + C_n$ induced on the vertices $t_1, t_2, \dots, t_n$. Therefore,

$$G_{80} + C_n = G_{80} \cup K_{6,n} \cup C_n^* = G_{80} \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup C_n^*. \tag{2.4}$$

Let D be a good drawing of the graph $G_{80} + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave t_i, as defined by Hernández-Vélez *et al.* [8] or Woodall [27]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5, and t_iv_6. Recall that a rotation is a cyclic permutation; that is, (123456), (234561), (345612), (456123), (561234), and (612345) denote the same rotation. We separate all subgraphs T^i, $i = 1, 2, \dots, n$, of the graph $G_{80} + D_n$ into four mutually-disjoint families of subgraphs depending on how many times the considered T^i crosses the edges of G_{80} in D. Let $R_D = \{T^i : \mathrm{cr}_D(G_{80}, T^i) = 0\}$, $S_D = \{T^i : \mathrm{cr}_D(G_{80}, T^i) = 1\}$, and $T_D = \{T^i : \mathrm{cr}_D(G_{80}, T^i) = 2\}$. Every other subgraph T^i crosses the edges of G_{80}

at least three times in D. For $T^i \in R_D \cup S_D \cup T_D$, let F^i denote the subgraph $G_{80} \cup T^i$, $i \in \{1, 2, \ldots, n\}$, of $G_{80} + D_n$ and let $D(F^i)$ be its good subdrawing induced by D. Clearly, this idea of dividing all subgraphs T^i into four mentioned families of subgraphs will be also retained in all drawings of the graphs $G_{80} + P_n$ and $G_{80} + C_n$.

Lemma 2.1. *In an effort to obtain a drawing D of the join product of the graph G_{80} with paths or cycles with the smallest numbers of crossings, the edges of G_{80} do not cross each other. Moreover, the planar subdrawing of G_{80} induced by D is isomorphic to one of the five drawings depicted in Fig. 2.*

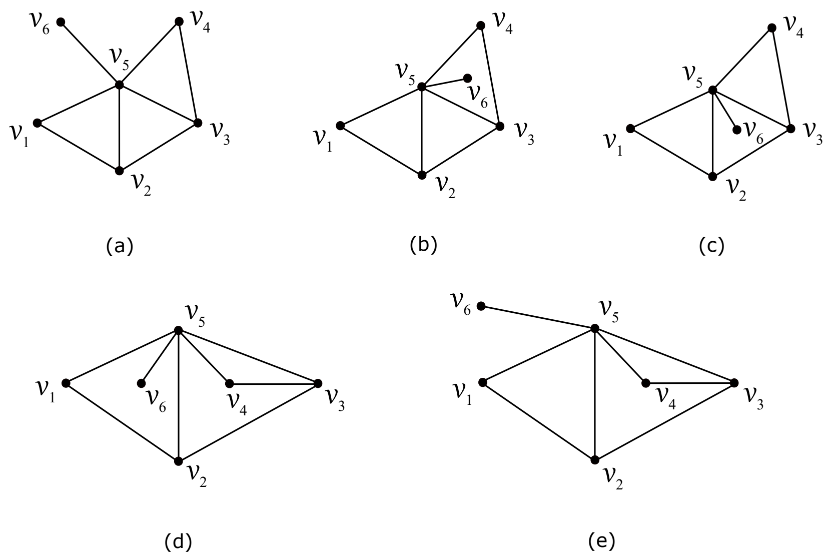


FIGURE 2. Five possible non isomorphic planar drawings of the graph G_{80}.

According to the relevant Lemmas 3.2 and 4.5, let us discuss all possible non isomorphic planar good drawings of G_{80}. The graph G_{80} consists of three edge disjoint subgraphs, namely $K_{1,1,2}$, P_3, and P_2. There is only one possibility of planar good subdrawing of $K_{1,1,2}$ (denote this subdrawing by $K^*_{1,1,2}$). In the next, we have two possibilities to add two new edges with common inner vertex of P_3 in $K^*_{1,1,2}$. If we consider a good subdrawing in which P_3 is placed in a quadrangular region of $K^*_{1,1,2}$, we have three possibilities for adding one leaf with corresponding edge of P_2 and there are three possible different drawings of G_{80} (Fig. 2(a), (b), and (c)). If we consider a good subdrawing in which P_3 is placed in a triangular region of $K^*_{1,1,2}$, we have four possibilities for adding P_2, but only two new non isomorphic drawings of G_{80} are obtained (Fig. 2(d) and (e)).

3. THE CROSSING NUMBER OF $G_{80} + P_n$

In the order originally designated by Clancy *et al.* [4], let G_{31} be the graph isomorphic to the complete bipartite graph $K_{1,5}$. The crossing numbers of the join products of $K_{1,5}$ with the discrete graphs D_n have been well known by Mei and Huang [20].

Theorem 3.1 ([20], Theorem 1). *If $n \geq 1$, then* $\mathrm{cr}(G_{31} + D_n) = \mathrm{cr}(K_{1,5,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$.

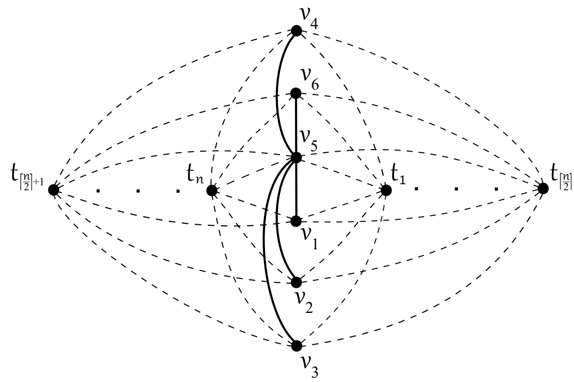


FIGURE 3. The good drawing of $G_{31} + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ crossings.

Due to Theorem 3.1, the good drawing of $G_{31} + D_n$ in Fig. 3 is optimal. We can add new edges v_1v_2, v_2v_3, v_4v_6, t_it_{i+1} for $i = 1, \dots, n-1$ into this drawing without additional crossings, and therefore, the drawings of the join products of G_{48}, G_{72}, G_{73}, G_{79} with D_n and G_{31}, G_{48}, G_{72}, G_{73}, G_{79} with P_n are obtained, respectively. Moreover, the same edge crossings can be also obtained by adding three edges v_1v_2, v_2v_3, and v_3v_4. So, the following results are obvious.

Corollary 3.1. *If $n \geq 1$, then $\mathrm{cr}(G_k + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ for any $k = 48, 72, 73, 79, 80$.*

Corollary 3.2. *If $n \geq 2$, then $\mathrm{cr}(G_k + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ for any $k = 31, 48, 72, 73, 79$.*

The assumption of a planar subdrawing of the subgraph G_{80} will be very strongly used in an effort to determine the crossing number of $G_{80} + P_n$ according to Lemma 3.2.

Lemma 3.2. *For $n \geq 2$, if D is any good drawing of the join product $G_{80}+P_n$ with $\mathrm{cr}_D(G_{80}) \geq 1$, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 1$ crossings in D.*

Proof. Let us consider any good drawing D of $G_{80} + P_n$ with $\mathrm{cr}_D(G_{80}) \geq 1$. In the rest of the proof, suppose that let v_1, v_4, and v_2, v_3 be the vertex notation of two vertices of degree 2 and two vertices of degree 3 in the considered good subdrawing of the graph G_{80}, respectively. Since no two edges incident with the same vertex cross, there is at least one crossing on the edge v_1v_2, v_2v_3, or v_3v_4 in the subdrawing of G_{80} induced by D. By removing three mentioned edges from the graph G_{80}, we obtain a subgraph isomorphic to the graph G_{31}. The exact value for the crossing number of $G_{31} + P_n$ is given by Corollary 3.2, which yields that there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 1$ crossings in D. □

As the same argument with the removing of the edges v_1v_2, v_2v_3, and v_3v_4 from the graph G_{80} can be also applied for all possible planar subdrawings of G_{80} in D, the proof of Corollary 3.3 can be omitted.

Corollary 3.3. *For $n \geq 2$, let D be any good drawing of the join product $G_{80} + P_n$ with $\mathrm{cr}_D(G_{80}) = 0$ and also with one vertex notation of G_{80} given in Fig. 2(a) – (e). If any of the edges v_1v_2, v_2v_3, or v_3v_4 is crossed in D, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 1$ crossings in the drawing D.*

Lemma 3.3. *For $n \geq 2$, let D be a good drawing of $G_{80} + P_n$ in which for some i, $i \in \{1, \dots, n\}$, and for all $j = 1, \dots, n$, $j \neq i$, $\mathrm{cr}_D(G_{80} \cup T^i, T^j) \geq 5$. If $\mathrm{cr}_D(G_{80} \cup T^i, T^j) > 5$ for p different subgraphs T^j, then D has at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + p + \mathrm{cr}_D(G_{80}, T^i)$ crossings.*

Proof. Assume, without loss of generality, that the edges of $F^n = G_{80} \cup T^n$ are crossed in D at least five times by the edges of every subgraph T^j, $j = 1, \dots, n-1$, and that p of the subgraphs T^j cross the edges of F^n more than five times. As $G_{80} + D_n = K_{6,n-1} \cup F^n$, we have

$$\mathrm{cr}_D(G_{80} + P_n) \geq \mathrm{cr}_D\,(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, F^n) + \mathrm{cr}_D(F^n) \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor +$$

$$+5(n-1) + p + \mathrm{cr}_D(G_{80}, T^n) \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n}{2}\right\rfloor + p + \mathrm{cr}_D(G_{80}, T^n).$$

□

Note that the last estimate used in the proof of Lemma 3.3 offers at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 1$ crossings for n even if $p = 0$ and $T^n \in R_D$.

Corollary 3.4. *For $n \geq 2$, let D be a good drawing of $G_{80} + P_n$ with $|T_D| = n$. If some subgraph T^i is crossed at least three times by any subgraph T^j, $j = 1, \dots, n$, $j \neq i$, then there are at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 2$ crossings in D.*

Lemma 3.4 ([27]). *For $m \geq 3$, let t_1 and t_2 be two different vertices of degree m in any good drawing D of the graph $K_{m,2}$. Let T^1 and T^2 be two considered subgraphs represented by their rot(t_1) and rot(t_2) of the length m, respectively. If the minimum number of interchanges of adjacent elements of rot(t_1) required to produce rot(t_2) is at most z, then $\mathrm{cr}_D(T^1, T^2) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor - z$.*

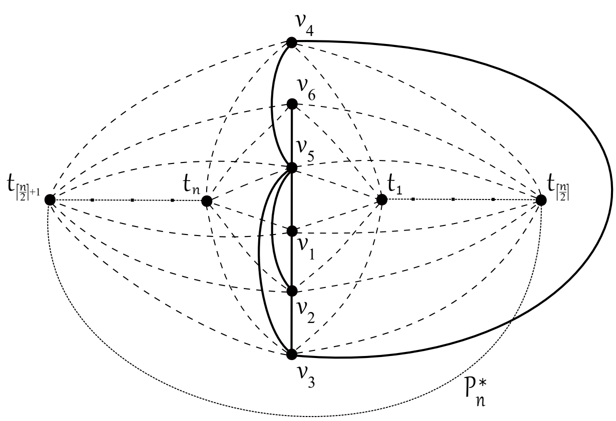


FIGURE 4. The good drawing of $G_{80} + P_n$ with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 1$ crossings.

Theorem 3.2. *If $n \geq 2$, then* $\mathrm{cr}(G_{80} + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 1$.

Proof. In Fig. 4, the edges of $K_{6,n}$ cross each other $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i, $i = 1, \dots, \lceil \frac{n}{2} \rceil$ on the right side does not cross the edges of the graph G_{80} and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \dots, n$ on the left side crosses the edges of G_{80} exactly four times. The path P_n^* crosses G_{80} once, and so $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 1$ crossings appear among the edges of the graph $G_{80} + P_n$ in this drawing. Thus, $\mathrm{cr}(G_{80} + P_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 1$. To prove the reverse inequality, let us suppose that for some $n \geq 2$, there is a drawing D such that

$$\mathrm{cr}_D(G_{80} + P_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4\left\lfloor \frac{n}{2} \right\rfloor. \tag{3.5}$$

Since the graph $G_{80} + D_n$ is a subgraph of $G_{80} + P_n$, the edges of $G_{80} + P_n$ are crossed exactly $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ times by Corollary 3.1. This enforces that no edge of the path P_n^* is crossed in D, and therefore, all vertices t_i of the path P_n^* have to be placed in the same region of the considered good subdrawing $D(G_{80})$. Lemmas 2.1 and 3.2 together with the assumption (3.5) offer only five possible planar subdrawings of the graph G_{80} presented in Fig. 2 that can be induced by D. Corollary 3.3 also implies no crossing on the edges v_1v_2, v_2v_3, and v_3v_4 in any such drawing D. Moreover, if $r = |R_D|$, $s = |S_D|$ and $t = |T_D|$, the assumption (3.5) together with the well-known fact $\mathrm{cr}_D(K_{6,n}) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ by (1.1) imply the relation $4\lfloor \frac{n}{2} \rfloor \geq \mathrm{cr}_D(G_{80}) + \mathrm{cr}_D(G_{80}, K_{6,n})$ with respect to the edge crossings of the subgraph G_{80} in D. More precisely

$$6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4\left\lfloor \frac{n}{2} \right\rfloor = \mathrm{cr}_D(G_{80}) + \mathrm{cr}_D(K_{6,n}) + \mathrm{cr}_D(P_n^*) + \mathrm{cr}_D(G_{80}, K_{6,n}) +$$

$$+ \mathrm{cr}_D(G_{80}, P_n^*) + \mathrm{cr}_D(K_{6,n}, P_n^*) \geq \mathrm{cr}_D(G_{80}) + 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \mathrm{cr}_D(G_{80}, K_{6,n})$$

i.e.,

$$4\left\lfloor \frac{n}{2} \right\rfloor \geq 0 + 0r + 1s + 2t + 3(n - r - s - t). \tag{3.6}$$

The obtained inequality (3.6) forces $3r + 2s + t \geq 2\lceil \frac{n}{2} \rceil$, that is, $r + s + t \geq 1$, and so there is at least one subgraph T^i whose edges cross the edges of G_{80} at most twice in D. However, if $r = s = 0$, then $t = n$. Now, we will show that a contradiction with the assumption (3.5) can be obtained in all subcases:

Case 1: $r \geq 1$. In this case, we can only suppose the planar subdrawing of the graph G_{80} given in Fig. 2(a). Since the set R_D is nonempty and no edge of the path P_n^* is crossed in the drawing D, all vertices t_i of P_n^* are placed in the region of subdrawing $D(G_{80})$ with all six vertices of G_{80} on its boundary.

Now, let us turn to list all possible rotations $\mathrm{rot}_D(t_i)$ that can appear in the drawing D if the edges of the graph G_{80} are not crossed by the edges of T^i. For $T^i \in R_D$, there is only one possible subdrawing of $F^i \setminus v_5$ represented by the subrotation (16432). This offers two ways of obtaining the subdrawing of F^i depending on which of the two regions of $D(F^i \setminus v_5)$ the edge t_iv_5 is placed in. For both these subdrawings of F^i, we can easily verify in six possible regions of $D(G_{80} \cup T^i)$ that $\mathrm{cr}_D(G_{80} \cup T^i, T^j) \geq 5$ is fulfilling for any T^j, $j \neq i$. If there is a subgraph T^j by which are crossed the edges of $G_{80} \cup T^i$ at least six times, then Lemma 3.3 confirms a contradiction with the assumption (3.5) in D. The same contradiction is also obtained for n even using the same fixation in the proof of Lemma 3.3. In the following, let us assume that $\mathrm{cr}_D(G_{80} \cup T^i, T^j) = 5$ holds for each T^j, $j \neq i$, and let n be odd. In the rest of the proof, for $T^i \in R_D$ with $\mathrm{rot}_D(t_i) = (165432)$ (based on their symmetry), there are seven possible different subgraphs $T^j \notin R_D$ with respect to the restriction $\mathrm{cr}_D(G_{80} \cup T^i, T^j) = 5$ only if the vertices t_j are placed in the quadrangular region of $D(G_{80} \cup T^i)$ with three vertices v_1, v_5, and v_6 of the graph G_{80} on its boundary, see Fig. 5.

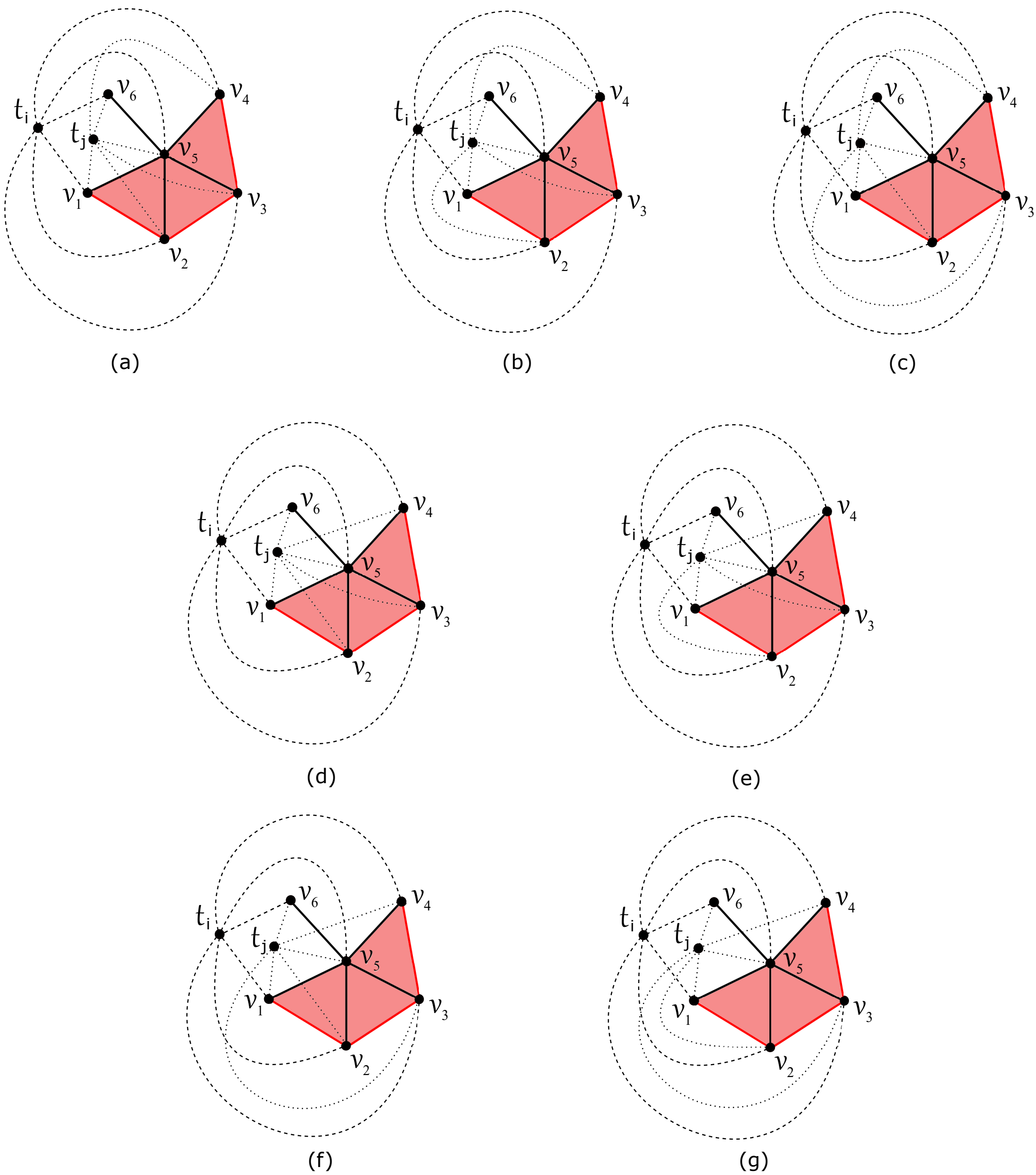


FIGURE 5. Seven possible subdrawings of $G_{80} \cup T^i \cup T^j$ for $T^j \notin R_D$ with $\mathrm{cr}_D(G_{80} \cup T^i, T^j) = 5$.

For all their presented subdrawings it is not difficult to show over considered regions of $D(G_{80} \cup T^i \cup T^j)$ that $\mathrm{cr}_D(G_{80} \cup T^i \cup T^j, T^k) \geq 8$ holds for any other $T^k \notin R_D$, $k \neq j$. Moreover, if there is some subgraph $T^l \in R_D$ with $\mathrm{rot}_D(t_l) = (156432)$ and $\mathrm{cr}_D(T^i, T^l) = 5$, then the edges of $G_{80} \cup T^l$ would be crossed at least six times by each possible considered subgraph T^j mentioned above (this verification can be performed using the properties of cyclic permutations with the help of Lemma 3.4). The obtained contradiction again by Lemma 3.3 forces $r = 1$, and so by fixing the subgraph $G_{80} \cup T^i \cup T^j$, we have

$$\mathrm{cr}_D(G_{80} + P_n) \geq 6\,\frac{n-3}{2}\,\frac{n-3}{2} + 8(n-2) + 5 \geq 6\,\frac{n-1}{2}\,\frac{n-1}{2} + 4\,\frac{n-1}{2} + 1.$$

This also confirms a contradiction with the assumption in D.

Case 2: $r = 0$ and $s \geq 1$. Clearly, the vertex t_j of any subgraph $T^j \in S_D$ must be placed in some region of subdrawing $D(G_{80})$ with at least five vertices of G_{80} on its boundary. Assume the planar subdrawing of the graph G_{80} given in Fig. 2(a). For a subgraph $T^j \in S_D$, there are six ways how to obtain the subdrawing of $F^j = G_{80} \cup T^j$ depending on which of three edges v_1v_5, v_4v_5, and v_5v_6 of the graph G_{80} is crossed by the edge t_jv_2, t_jv_3, and either t_jv_1 or t_jv_4 of T^j, respectively. For all these six possible subdrawings in Fig. 6, we can show that $\mathrm{cr}_D(G_{80} \cup T^j, T^k) \geq 5$ holds for any T^k, $k \neq j$, over all considered regions of $D(G_{80} \cup T^j)$. As $\mathrm{cr}_D(G_{80}, T^j) = 1$, Lemma 3.3 again contradicts the assumption (3.5).

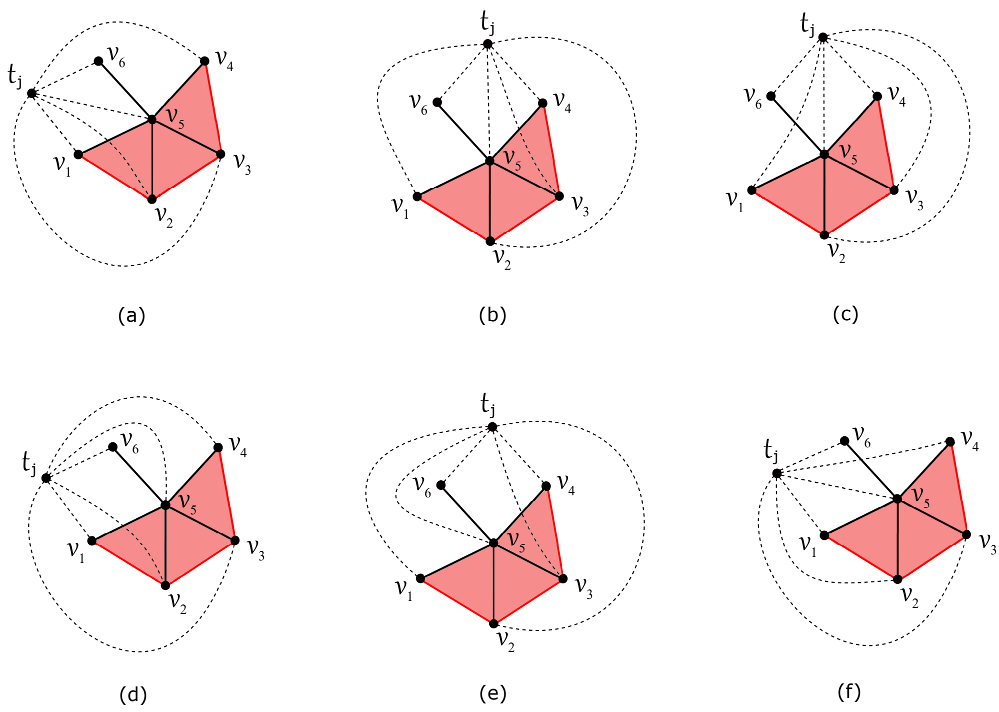


FIGURE 6. Six possible subdrawings of $F^j = G_{80} \cup T^j$ for $T^j \in S_D$.

Now, we consider the planar subdrawing of G_{80} in D given in Fig. 2(b). For a subgraph $T^j \in S_D$, only the edge v_4v_5 of G_{80} can be crossed by the the edge t_jv_6. We can easily verify in six considered regions of $D(G_{80} \cup T^j)$ that $\mathrm{cr}_D(G_{80} \cup T^j, T^k) \geq 5$ holds for any T^k, $k \neq j$, and so Lemma 3.3 also confirms a contradiction with the assumption of D. For both subdrawings of G_{80} in D given in Fig. 2(c) and (d), there is no possibility to obtain a subdrawing of $G_{80} \cup T^j$ for some $T^j \in S_D$. Finally, the subdrawing of the graph G_{80} given in Fig. 2(e) offers two ways of obtaining the subdrawing of $F^j = G_{80} \cup T^j$ with $T^j \in S_D$ depending on which of the two regions of $D(F^j \setminus v_5)$ the edge t_jv_5 is placed in. For both such possibilities of F^j, Lemma 3.3 again contradicts the assumption in D because the edges of F^j are crossed at least five times by any other subgraph T^k, $k \neq j$, based on the discussion of an inserting the vertex t_k with corresponding edges over six considered regions of $D(G_{80} \cup T^j)$.

Case 3: $r = 0$ and $s = 0$, that is, $t = n$ according to the inequality (3.6). Assume the planar subdrawing of the graph G_{80} given in Fig. 2(a), which yields that all vertices t_j of the path P_n^* are placed in the region of subdrawing $D(G_{80})$ with all six vertices of G_{80} on its boundary. By Berežný and Staš [3], it was proved that $\mathrm{cr}(C_5 \cup \{v\} + P_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor + 1$, and thus there are less than $3\lfloor\frac{n}{2}\rfloor$ crossings on the edge v_5v_6 in D. Since each subgraph T^k cannot cross only the edge v_5v_6 of G_{80}, at least one of the edges v_1v_5 and v_4v_5 must be crossed by some subgraph T^k. In the rest of the proof, based on their symmetry, let the edge v_4v_5 be crossed at most as many times as the edge v_1v_5 in D. There are six ways how to obtain the subdrawing of $F^k = G_{80} \cup T^k$ depending on which of three edges v_2v_5, v_4v_5, and v_5v_6 of the graph G_{80} is also crossed by some edge of T^k, see Fig. 7. As $\mathrm{cr}_D(T^k, T^l) \geq 3$ for any other subgraphs T^l with $l \neq k$, Corollary 3.4 also forces a contradiction with the assumption in D.

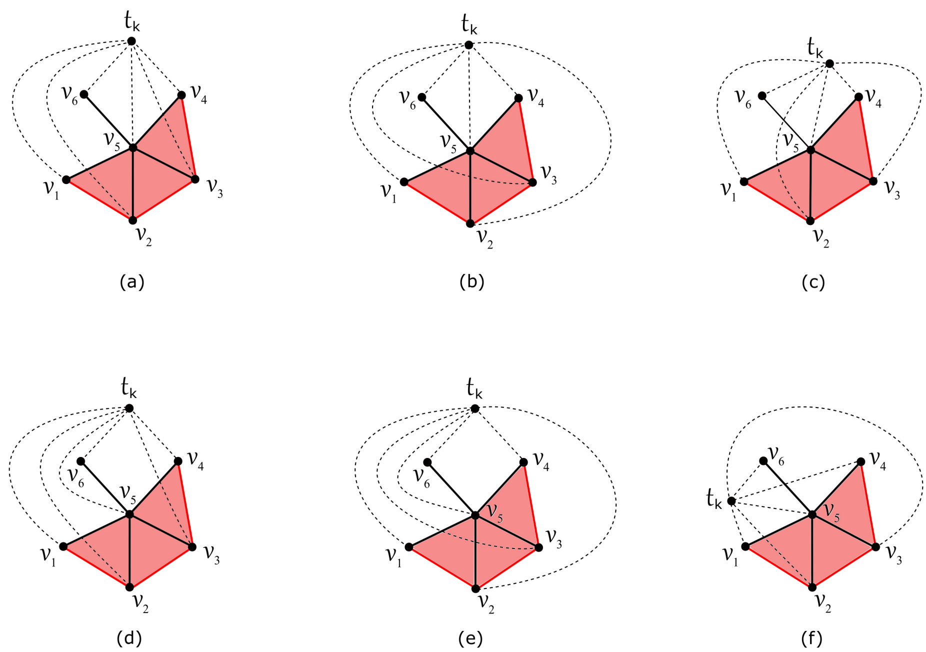


FIGURE 7. Six possible subdrawings of $F^k = G_{80} \cup T^k$ with one crossing on the edge v_1v_5 and exactly one crossing on one of the edges v_2v_5, v_4v_5, and v_5v_6.

Now, we consider the planar subdrawing of G_{80} in D given in Fig. 2(b). If all vertices of the path P_n^* are placed in the region of $D(G_{80})$ with four vertices v_3, v_4, v_5, and v_6 of G_{80} on its boundary, then the edges t_kv_1 and t_kv_2 cross the edges v_4v_5 and v_3v_5 of G_{80}, respectively. This enforces that there are only two ways of obtaining the subdrawing of $F^k = G_{80} \cup T^k$ with $T^k \in T_D$ depending on which of the two regions of $D(F^k \setminus v_5)$ the edge t_kv_5 is placed in. Using their rotations by Lemma 3.4, $\mathrm{cr}_D(T^k, T^l) \geq 5$ holds for all two different subgraphs T^k and T^l, which yields that Corollary 3.4 contradicts the assumption (3.5) of D. If all vertices t_k of P_n^* are placed in the region of $D(G_{80})$ with five vertices v_1, v_2, v_3, v_4, and v_5 of G_{80} on its boundary, then t_kv_6 crosses v_4v_5 and either t_kv_2 crosses v_1v_5 or t_kv_3 crosses v_4v_5. Again using their rotations, Corollary 3.4 together with Lemma 3.4 contradict the assumption (3.5) of D provided by there at least four crossings on edges of $D(T^k \cup T^l)$ for any two different subgraphs $T^k, T^l \in T_D$.

If we assume the planar subdrawing of G_{80} in D given in Fig. 2(c), then either all edges t_kv_6 cross just two of the edges v_1v_5, v_2v_5 and v_4v_5, v_3v_5 (if t_k is placed in the region of subdrawing $D(G_{80})$ with the five vertices v_1, v_2, v_3, v_4, and v_5 of G_{80} on its boundary) or there are only two ways of obtaining the subdrawing of $F^k = G_{80} \cup T^k$ (if t_k is placed in the region of subdrawing $D(G_{80})$ with the four vertices v_2, v_3, v_5, and v_6 of G_{80} on its boundary) depending on which of the two regions of $D(F^k \setminus v_5)$ the edge t_kv_5 is placed in if the edges t_kv_1 and t_kv_4 cross the edges v_2v_5 and v_3v_5 of G_{80}, respectively. For both such possibilities, Corollary 3.4 together with Lemma 3.4 again imply a contradiction with the assumption in D because there are at least five crossings on edges of $D(T^k \cup T^l)$ for any two different subgraphs $T^k, T^l \in T_D$. The same idea of discussions for three quadrangular regions in the subdrawing of the graph G_{80} given in Fig. 2(d) forces the same contradictions.

Finally, if we consider the subdrawing of G_{80} in D given in Fig. 2(e), then all vertices t_k of P_n^* must be placed in the region of $D(G_{80})$ with five vertices v_1, v_2, v_3, v_5, and v_6 of G_{80} on its boundary.

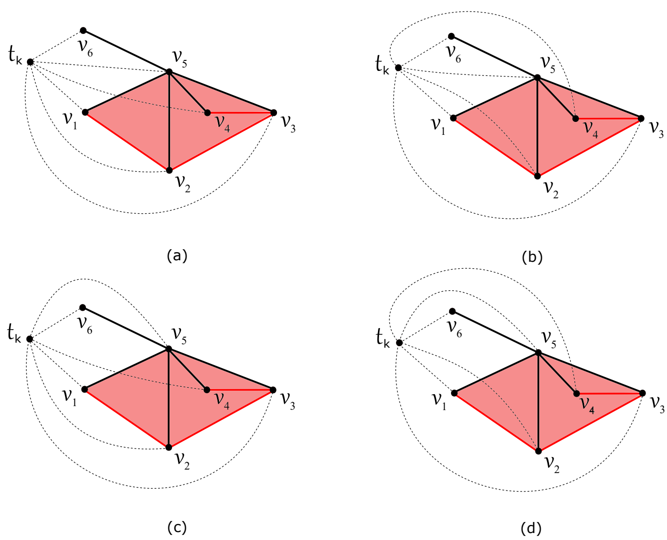


FIGURE 8. Four possible subdrawings of $F^k = G_{80} \cup T^k$ with one crossing on the edge v_1v_5 and one crossing on one of the edges v_2v_5 and v_3v_5.

For any $T^k \in T_D$, at least one edge of G_{80} is crossed by t_kv_4, which yields that no edge of G_{80} is crossed by t_kv_3. Each subgraph T^k cannot cross both edges v_3v_5 and v_5v_6 using at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{n}{2}\rfloor + 1$ crossings on the edges of $G_{80} \setminus \{v_2v_5, v_3v_5, v_5v_6\} + P_n$ isomorphic to $C_5 \cup \{v\} + P_n$. Thus, there is at least one subgraph T^k by which both edges either v_1v_5, v_2v_5 or v_1v_5, v_3v_5 are crossed. Since the edge t_kv_5 can be placed in two regions of $D(F^k \setminus v_5)$, we obtain four possible subdrawings of $F^k = G_{80} \cup T^k$ shown in Fig. 8. For all such possibilities of F^k, Corollary 3.4 again contradicts the assumption in D because the edges of T^k are crossed at least three times by any other subgraph T^l based on the discussion of an inserting the vertex t_l with corresponding edges over six considered regions of $D(G_{80} \cup T^k)$.

We have shown, in all cases, that there are at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 1$ crossings in each good drawing D of the graph $G_{80} + P_n$. The proof of Theorem 3.2 is done. □

4. The Crossing Number of $G_{80} + C_n$

Our aim in this section is to give the crossing number of the join product $G_{80} + C_n$ for n at least three. Let S_m denote the star on $m+1$ vertices. Using the results of Klešč et al. [15], the crossing numbers of the graphs $S_m + C_n$ for $m = 3, 4, 5$ and $n \geq 3$ were established. As the graph G_{31} is isomorphic to the star S_5, the crossing number of the join product $G_{31} + C_n$ is given in Theorem 4.3.

Theorem 4.3 ([15], Theorem 9). *If $n \geq 3$, then $\mathrm{cr}(G_{31} + C_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 3$.*

For any $k = 48, 72, 73, 79$, we can also obtain the good drawings of $G_k + C_n$ with exactly $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 3$ crossings by adding new edges v_1v_2, v_2v_3, v_4v_6, t_nt_1, t_it_{i+1} for $i = 1, \dots, n-1$ only with three additional crossings on the edge t_nt_1 in Fig. 3. So, the following results are obvious.

Corollary 4.5. *If $n \geq 3$, then $\mathrm{cr}(G_k + C_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 3$ for any $k = 48, 72, 73, 79$.*

The exact value for the crossing number of $G_{31} + C_n$ is given by $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 3$. Given the use of arguments similar to those in the proof of Lemma 3.2, the proofs of Lemma 4.5 and Corollary 4.6 can be omitted.

Lemma 4.5. *For $n \geq 3$, if D is any good drawing of the join product $G_{80}+C_n$ with $\mathrm{cr}_D(G_{80}) \geq 1$, then there are at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 4$ crossings in D.*

Corollary 4.6. *For $n \geq 3$, let D be any good drawing of the join product $G_{80} + C_n$ with $\mathrm{cr}_D(G_{80}) = 0$ and also with the vertex notation of G_{80} given in Fig. 2(a) – (e). If any of the edges v_1v_2, v_2v_3, or v_3v_4 is crossed in D, then there are at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 4$ crossings in the drawing D.*

Let $t_1, t_2, \dots, t_n, t_1$ be the vertex notation of the n-cycle C_n for $n \geq 3$. The join product $G_{80} + C_n$ consists of one copy of the graph G_{80}, one copy of the cycle C_n, and the edges joining each vertex of G_{80} with each vertex of C_n. Let C_n^* denote the cycle as a subgraph of $G_{80} + C_n$ induced on the vertices of C_n not belonging to the subgraph G_{80}. The subdrawing $D(C_n^*)$ induced by any good drawing D of $G_{80} + C_n$ represents some drawing of C_n. For the vertices $v_1, v_2, \dots, v_6$ of the graph G_{80}, let T^{v_i} denote the subgraph induced by n edges joining the vertex v_i with n vertices of C_n^*. The edges joining the vertices of G_{80} with the vertices of C_n^* form the complete bipartite graph $K_{6,n}$, and so

$$G_{80} + C_n = G_{80} \cup K_{6,n} \cup C_n^* = G_{80} \cup \left(\bigcup_{i=1}^{6} T^{v_i} \right) \cup C_n^*. \tag{4.7}$$

In the proof of the main theorem of this section, the following three statements related to some restricted subdrawings of the graphs $G + C_n$ will be helpful.

Lemma 4.6 ([12], Lemma 2.2)**.** *For $m \geq 2$ and $n \geq 3$, let D be a good drawing of $D_m + C_n$ in which no edge of C_n^* is crossed, and C_n^* does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \dots, m$, two different subgraphs T^{v_i} and T^{v_j} cross each other in D at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

Corollary 4.7 ([15], Corollary 4)**.** *For $m \geq 2$ and $n \geq 3$, let D be a good drawing of the join product $D_m + C_n$ in which the edges of C_n^* do not cross each other and C_n^* does not separate p vertices $v_1, v_2, \dots, v_p$, $2 \leq p \leq m$. Let $T^{v_1}, T^{v_2}, \dots, T^{v_q}$, $q < p$, be the subgraphs induced on the edges incident with the vertices $v_1, v_2, \dots, v_q$ that do not cross C_n^*. If k edges of some subgraph T^{v_j} induced on the edges incident with the vertex v_j, $j \in \{q+1, q+2, \dots, p\}$, cross the cycle C_n^*, then the subgraph T^{v_j} crosses each of the subgraphs $T^{v_1}, T^{v_2}, \dots, T^{v_q}$ at least $\lfloor \frac{n-k}{2} \rfloor \lfloor \frac{(n-k)-1}{2} \rfloor$ times in D.*

Lemma 4.7 ([15], Lemma 1)**.** *For $m \geq 1$, let G be a graph of order m. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of C_n^* do not cross each other.*

In the following, we are able to compute the exact values of crossing numbers of the join products of the graph G_{80} with both cycles C_3 and C_4 using the algorithm located on the website `http://crossings.uos.de/`. This algorithm can find the crossing numbers of small undirected graphs. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs, as described by Chimani and Wiedera [9]. Unfortunately, the capacity of this system is restricted.

Lemma 4.8. $\mathrm{cr}(G_{80} + C_3) = 14$ *and* $\mathrm{cr}(G_{80} + C_4) = 24$.

Theorem 4.4. *If $n \geq 3$, then* $\mathrm{cr}(G_{80} + C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 4$.

Proof. By Lemma 4.8, the result holds for $n = 3$ and $n = 4$. Into the drawing in Fig. 4, it is possible to add the edge $t_1 t_n$ which forms the cycle C_n^* on the vertices of P_n^* with just three another crossings, i.e., C_n^* is crossed by three edges $v_1 v_5$, $v_2 v_5$, and $v_3 v_5$ of the graph G_{80}. Thus, $\mathrm{cr}(G_{80} + C_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 4$, and let us suppose that there is a drawing D such that

$$\mathrm{cr}_D(G_{80} + C_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4\left\lfloor \frac{n}{2} \right\rfloor + 3 \quad \text{for some } n \geq 5. \tag{4.8}$$

By Corollary 3.1, at most three edges of the cycle C_n^* can be crossed in D, and we can also suppose that the edges of C_n^* do not cross each other using Lemma 4.7. The subdrawing of C_n^* induced by D divides the plane into two regions with at least three vertices of G_{80} in one of them, and so three possible cases may occur:

Case 1: There is at most one crossing on the edges of C_n^*. Since at least five vertices of G_{80} are placed in one region of $D(C_n^*)$, any two different considered subgraphs T^{v_i} and T^{v_j} cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times by Lemma 4.6, and therefore, there are at least $\binom{5}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor + 4$ crossings in D.

Case 2: There are exactly two crossings on the edges of C_n^*. If $\mathrm{cr}_D(G_{80}, C_n^*) = 2$, then only one edge of G_{80} can be crossed by the edges of C_n^* using Corollary 4.6. All vertices of G_{80} are placed in one region of $D(C_n^*)$, and so there are at least $\binom{6}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2$ crossings in D. But, for $n \geq 5$, $\binom{6}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2 \geq 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor+4$ which confirms a contradiction in D. If $\mathrm{cr}_D(T^{v_i}, C_n^*) = 2$ for some subgraph T^{v_i}, the same idea as in Case 1 contradicts again the assumption (4.8) of D. Now, let us turn to the possibility of an existence of two different subgraphs T^{v_i} and T^{v_j} with $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_j}, C_n^*) = 1$. This, by Corollary 4.7 for $p = 6$, $q = 4$, and $k = 1$, enforces at least

$$\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+1+1 \tag{4.9}$$

crossings in D. This also confirms a contradiction with the assumption (4.8) in D.

Now, assume $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ for only one $i \in \{1, \ldots, 6\}$ and $\mathrm{cr}_D(G_{80}, C_n^*) = 1$. All five vertices v_1, v_2, v_3, v_4, and v_5 of the graph G_{80} must be placed in one region of $D(C_n^*)$. For $i = 6$, we obtain at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2$ crossings in D. If $i \neq 6$, then there at least

$$\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+1+1+1 \tag{4.10}$$

crossings in D provided by $\mathrm{cr}_D(T^{v_i}, T^{v_6}) \geq 1$ and using Corollary 4.7 again for $p = 5$, $q = 4$, and $k = 1$. The number of crossings obtained in (4.10) can confirm a contradiction in D for all n at least 6. For $n = 5$, if there is at least one subgraph $T^j \notin R_D$, then we can add at least one additional crossing on edges of G_{80} in (4.10) with a contradiction in D. Finally, for $n = |R_D| = 5$, suppose only planar subdrawing of G_{80} in D given in Fig. 2(a). Any $T^j \in R_D$ can be represented by one from two possible rotations (156432) and (165432). Since $\mathrm{cr}_D(T^j, T^k) \geq 5$ holds for any subgraphs $T^j, T^k \in R_D$ with $\mathrm{rot}_D(t_j) \neq \mathrm{rot}_D(t_k)$ and $\mathrm{cr}_D(T^j, T^k) \geq 6$ with $\mathrm{rot}_D(t_j) = \mathrm{rot}_D(t_k)$, $j \neq k$, we obtain at least 54 crossings in D. This also contradicts the assumption of D.

Case 3: There are exactly three crossings on the edges of C_n^*. If $\mathrm{cr}_D(G_{80}, C_n^*) = 0$, then all vertices of the cycle C_n^* have to be placed in one region of $D(G_{80})$. Let D' be the subdrawing of $G_{80}+D_n$ induced by D without the edges of C_n^*. Clearly, the subdrawing D' is some optimal drawing of the graph $G_{80}+D_n$ with exactly $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor$ crossings and all vertices t_i are also placed in the same region of $D'(G_{80})$. But, the same idea of discussions as in the proof of Theorem 3.2 enforces at least $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor+1$ crossings in D'. In the subcase of $\mathrm{cr}_D(G_{80}, C_n^*) = 1$, the proof proceeds in a similar way provided that the edge v_5v_6 is crossed by C_n^*, and therefore, all vertices of C_n^* are only placed in the region of $D'(G_{80})$ with the vertex v_6 on its boundary. Finally, if $2 \leq \mathrm{cr}_D(G_{80}, C_n^*) \leq 3$, then either only one edge of G_{80} is crossed twice, or the edge v_5v_6 and the edge v_5v_i, $i \in \{1, \ldots, 4\}$, is crossed once and twice, respectively, by the edges of C_n^* using Corollary 4.6. Both subcases confirm a contradiction with the assumption in D using Lemma 4.6, because imply at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3$ crossings in D.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $G_{80} + C_n$ with fewer than $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 4\lfloor\frac{n}{2}\rfloor + 4$ crossings. This completes the proof of Theorem 4.4. □

5. Some Consequences of the Main Result

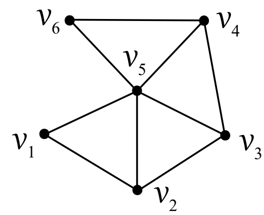


FIGURE 9. Planar drawing of one graph G_{104}.

In Fig. 9, let G_{104} be the connected graph of order six obtained from G_{80} by adding the edge v_4v_6 to the drawing in Fig. 2(a). Since we can add the mentioned edge v_4v_6 to the graph G_{80} without additional crossings in Fig. 4, the drawings of the graphs $G_{104}+P_n$ and $G_{104}+C_n$ with exactly $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor+1$ and $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor+4$ crossings can be obtained, respectively. Further, the graph G_{80} is some subgraph of G_{104}, and therefore, $\mathrm{cr}(G_{104}+P_n)\geq\mathrm{cr}(G_{80}+P_n)$ and $\mathrm{cr}(G_{104}+C_n)\geq\mathrm{cr}(G_{80}+C_n)$. Thus, the following results are obvious.

Corollary 5.8. *If* $n\geq 1$, *then* $\mathrm{cr}(G_{104}+D_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor$.

Corollary 5.9. *If* $n\geq 2$, *then* $\mathrm{cr}(G_{104}+P_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor+1$.

Corollary 5.10. *If* $n\geq 3$, *then* $\mathrm{cr}(G_{104}+C_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+4\lfloor\frac{n}{2}\rfloor+4$.

Finally, Staš and Valiska [26] conjectured that the crossing numbers of W_m+P_n are given by $(Z(m)-1)\lfloor\frac{n}{2}\rfloor+Z(m+1)Z(n)+n+1$, for all $m\geq 3$ and $n\geq 2$, where W_m denotes the wheel on $m+1$ vertices and the Zarankiewicz's number $Z(n)=\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ is defined for all positive integers n. Recently, this conjecture was proved for W_3+P_n, W_4+P_n, and W_5+P_n by Klešč and Schrötter [16], Staš and Valiska [26], and Berežný and Staš [2], respectively. On the other hand, the graphs W_m+P_2 and W_m+P_3 are isomorphic to the join product of the cycle C_m with the cycle C_3 and with the graph $K_4\setminus e$ obtained by removing one edge from K_4, respectively. The exact values for the crossing numbers of the graphs C_m+C_n and $K_4\setminus e+C_m$ are given by Klešč [12] and [13], respectively, and so the graphs W_m+P_2 and W_m+P_3 confirm the validity of this conjecture. This conjecture was also proved for W_m+P_4 by Staš [23] again due to some isomorphism. Since the graph W_m+P_5 is isomorphic to the graph $G_{104}+C_m$, we establish the validity of this conjecture also for the graph W_m+P_5.

Corollary 5.11. *If* $m\geq 3$, *then* $\mathrm{cr}(W_m+P_5)=6\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+4\lfloor\frac{m}{2}\rfloor+4$.

6. Conclusions

All values of crossing numbers of the join products for all seven considered graphs G_k on six vertices with the paths P_n and with the cycles C_n are collected in Table 1 (here, the Zarankiewicz's number $Z(m,n)=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ is defined for all positive integers m,n.) We suppose that similar forms of discussions can be used to estimate the unknown values of the crossing numbers of the remaining graphs on six vertices with a much larger number of edges in the join products with the paths, and also with the cycles.

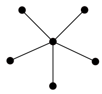
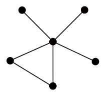
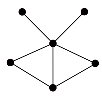
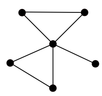
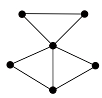
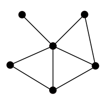
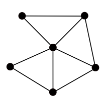
G_k	$\mathrm{cr}(G_k + P_n),\, n \geq 2$	$\mathrm{cr}(G_k + C_n),\, n \geq 3$
G_{31} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 3$
G_{48} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 3$
G_{72} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 3$
G_{73} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 3$
G_{79} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 3$
G_{80} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 1$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 4$
G_{104} 	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 1$	$Z(6,n) + 4\lfloor \frac{n}{2} \rfloor + 4$

TABLE 1. Summary of crossing numbers for $G_k + P_n$ and $G_k + C_n$.

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