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Unpredictable solutions of compartmental quasilinear differential equations

MARAT AKHMET[1], MADINA TLEUBERGENOVA[2] and AKYLBEK ZHAMANSHIN[1]

ABSTRACT. In the present research, quasilinear differential equations with compartmental periodic unpredictable coefficients and perturbations are investigated. The compartmental periodic unpredictable functions present a new type of recurrency. The problem of existence and uniqueness of unpredictable solutions is studied for the systems. Numerical simulations which illustrate chaotic and periodic behavior in outputs of the dynamics are provided.

1. INTRODUCTION

Many scientists have devoted their researches to the study of solutions, which are recurrent in their behavior. The most studied are periodic [9, 12] and almost periodic solutions [10, 13, 14]. The Poisson stable motions, which were introduced by H. Poincaré [15], represent most difficult for analysis type of recurrence [16, 17]. In our papers, *the method of included intervals* has been introduced and developed to approve Poisson stability in differential equations and neural networks [1, 2].

In paper [3], to make the recurrence a chaotic ingredient, the Poisson stability has been extended to the unpredictability. Thus, the Poincaré chaos was determined, and one can say now that *the unpredictability implies chaos* [3, 4]. The concept of unpredictability has been applied for various problems of differential equations [5] and neural networks [6]. It is a powerful instrument for chaos *indication.* In papers [7, 8], a special method to *extend* chaos presence has been developed.

Unlike paper [5], where quasilinear differential equations with constant coefficients and unpredictable perturbations were investigated, in the present paper, we consider differential equations with time-varying compartmental coefficients and compartmental perturbations. The newly introduced functions combine the properties of periodicity and unpredictability. To prove that the functions are unpredictable, a special condition of synchrony for characteristics, the kappa property, is applied, which has not been considered in literature at all.

The rest of the paper is organized as follows. In Section 2, the main definitions of the unpredictable and compartmental periodic unpredictable functions (simply compartmental functions, in what follows) are presented and preliminary results concerning the properties of such functions are provided. The main result of the study is given in Section 3. Under certain conditions it is rigorously proven that an unpredictable solution, which is asymptotically stable, takes place in the quasilinear differential equations with compartmental perturbations. In Section 4, examples supporting the theoretical results are given. A compartmental periodic unpredictable function and unpredictable solutions

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Corresponding author: Marat Akhmet; marat@metu.edu.tr

of differential equations are demonstrated. Moreover, in Section 4, a quantitative characteristic, the degree of periodicity, that strongly affects periodical and the unpredictable appearance the dynamics of the solution is defined.

2. Preliminaries

Let us start with the main definitions of the research. Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the set of real and natural numbers, respectively, and the Euclidean norm will be used.

Definition 2.1. [3] A uniformly continuous and bounded function $f : \mathbb{R} \to \mathbb{R}^n$ is unpredictable if there exist positive numbers ϵ_0, δ and sequences t_k, s_k both of which diverge to infinity such that $\|f(t+t_k) - f(t)\| \to 0$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|f(t+t_k) - f(t)\| > \epsilon_0$ for each $t \in [s_k - \delta, s_k + \delta]$ and $k \in \mathbb{N}$.

The sequence $t_k, k = 1, 2, \ldots,$ is said to be the *Poisson* or *convergence sequence* of the function $f(t)$. We call the uniform convergence on compact subsets of $\mathbb{R}$, *the convergence property*, and the existence of the sequence s_k and positive numbers ϵ_0, δ is called *the separation property*.

Definition 2.2. [5] A continuous and bounded function $f(t, x) : \mathbb{R} \times D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is an open and bounded set, is unpredictable in t uniformly with respect to $x \in D$, if it is uniformly continuous in t and there exist positive numbers ϵ_0, δ and sequences t_k, s_k both of which diverge to infinity such that $\sup_D \|f(t+t_k, x) - f(t, x)\| \to 0$ as $k \to \infty$ uniformly on bounded intervals of t and $x \in D$, and $\|f(t+t_k, x) - f(t, x)\| > \epsilon_0$ for $t \in [s_k - \delta, s_k + \delta]$, $x \in D$ and $k \in \mathbb{N}$.

Definition 2.3. A function $f(t) : \mathbb{R} \to \mathbb{R}^n$ is said to be *compartmental periodic unpredictable (CPU)* function, if $f(t) = G(t, t)$, where $G(u, s)$ is a continuous bounded function, periodic in u uniformly with respect to s, and unpredictable in s uniformly with respect to u.

The next definition is a particular case of the Definition 2.3.

Definition 2.4. A function $f(t) = \phi(t) + \psi(t)$ is said to be the *modulo periodic unpredictable (MPU)* function, if $\phi(t)$ is an $\omega-$periodic continuous function and $\psi(t)$ is an unpredictable function.

Definition 2.5. A function $f(t, x) : \mathbb{R} \times D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is an open and bounded set, is said to be *compartmental periodic unpredictable in* t uniformly for x function, if $f(t, x) = G(t, t, x)$, where $G(u, s, x)$ is a continuous bounded function, periodic in u uniformly with respect to arguments s, x, and unpredictable in s uniformly with respect to u and x.

Consider an unpredictable function $\psi(t)$ with convergence sequence t_k. For fixed $\omega > 0$ there exist a subsequence t_{k_l} and a number τ_ω such that $t_{k_l} \to \tau_\omega (mod\ \omega)$ as $l \to \infty$. In what follows, we shall call the number τ_ω as the *Poisson shift* for the function $\psi(t)$ with respect to the ω. The set of Poisson shifts $\mathcal{T}_\omega$ is not empty, in general case, it can consist of several or even an infinite number of elements. The number $\kappa_\omega = inf\ \mathcal{T}_\omega$, $0 \le \kappa_\omega < \omega$, is said to be *Poisson number for the function* $\psi(t)$ *with respect to the number* ω. We say that the sequence t_k satisfies *kappa property with respect to the number* ω if $\kappa_\omega = 0$.

Remark 2.1. A compartmental function is not necessary unpredictable. It is unpredictable in the sense of the Definition 2.1 under a special condition, for example, if its convergence sequence satisfies the kappa property.

Lemma 2.1. *Assume that $G(u, v, x) : \mathbb{R} \times \mathbb{R} \times D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is an open and bounded set, is a continuous $\omega-$periodic in u uniformly with respect to v and x function. Then the function $g(t, x) = G(t, t, x)$, is unpredictable in t, if the following conditions are valid:*

(i) for each $\epsilon > 0$ there exists a positive number η such that $\|G(t + s, t, x) - G(t, t, x)\| < \epsilon$ if $|s| < \eta, t \in \mathbb{R}, x \in D$;

there exist sequences t_k, s_k both of which diverges to infinity as $k \to \infty$, and positive numbers ϵ_0, δ such that

(ii) the sequence t_k satisfies the kappa property with respect to the period ω;

(iii) $\sup_{I\times D} \|G(t, t + t_k, x) - G(t, t, x)\| \to 0$ on each bounded interval $I \subset R$;

(iv) $\inf_{[s_k-\delta,s_k+\delta]\times D} \|G(t, t + t_k, x) - G(t, t, x)\| > \epsilon_0$, $k \in \mathbb{N}$.

Proof. Let us fix a positive number ϵ, and a bounded interval $I \in \mathbb{R}$. Since the sequence t_k satisfies the kappa property, one can write, without loss of generality, that $t_k \to 0(mod\,\omega)$ as $k \to \infty$. Therefore, by conditions (i) and (iii), the following inequalities are valid

$$\sup_{\mathbb{R}\times D} \|G(t + t_k, t, x) - G(t, t, x)\| < \frac{\epsilon}{2} \tag{2.1}$$

and

$$\sup_{I\times D} \|G(t, t + t_k, x) - G(t, t, x)\| < \frac{\epsilon}{2}, \tag{2.2}$$

for sufficiently large k.

Using inequalities (2.1) and (2.2), we get that

$$\begin{aligned}
&\|g(t + t_k, x) - g(t, x)\| = \|G(t + t_k, t + t_k, x) - G(t, t, x)\| \le \\
&\|G(t + t_k, t + t_k, x) - G(t, t + t_k, x)\| + \|G(t, t + t_k, x) - G(t, t, x)\| < \\
&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

for all $t \in I$, $x \in D$. That is, $g(t + t_k, x)$ converges to $g(t, x)$ on each arbitrary bounded time interval uniformly for $x \in D$. Moreover, conditions (i) and (ii) imply that $\sup_{\mathbb{R}\times D} \|G(t+ t_k, t, x)-G(t, t, x)\| < \frac{\epsilon_0}{2}$ for sufficiently large k. Applying assumption (iv), one can obtain that

$$\begin{aligned}
&\|g(t + t_k, x) - g(t, x)\| = \|G(t + t_k, t + t_k, x) - G(t, t, x)\| \ge \\
&\|G(t + t_k, t + t_k, x) - G(t + t_k, t, x)\| - \|G(t + t_k, t, x) - G(t, t, x)\| > \\
&\epsilon_0 - \frac{\epsilon_0}{2} = \frac{\epsilon_0}{2},
\end{aligned}$$

for all $t \in [s_k - \delta, s_k + \delta]$, $x \in D$, $k \in \mathbb{N}$. The lemma is proved. □

Along with Lemma 2.1, in what follows, the next assertion, which can be proved in the similar way, will be needed.

Lemma 2.2. *A compartmental periodic unpredictable in t function $g(t, x)$ is unpredictable in t provided that its convergence sequence t_k admits the kappa property.*

3. Main results

3.1. **A case of compartmental periodic unpredictable coefficients and perturbations.** The main object of the present section is the system of quasilinear differential equations,

$$x'(t) = (A(t) + B(t))x + g(t, x), \tag{3.3}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^n$, n is a fixed natural number; $A(t)$ and $B(t)$ are $n-$dimensional square matrices; $g : \mathbb{R} \times D \to \mathbb{R}^n$, $D = \{x \in \mathbb{R}^n, \|x\| < H\}$, where H is a fixed positive number.

We assume that the following conditions are satisfied.

(C1) $A(t)$ is an $\omega-$periodic matrix for a fixed positive ω;
(C2) $B(t)$ is an unpredictable matrix, and $g(t, x)$ is compartmental function;
(C3) the entries of the matrix $B(t)$ and function $g(t, x)$ are with common convergence sequence t_k;
(C4) the convergence sequence t_k satisfies the kappa property with respect to the period ω.

Let us consider the homogeneous system, associated with (3.3),

$$x'(t) = A(t)x(t). \tag{3.4}$$

Let $X(t)$, $t \in \mathbb{R}$, is the fundamental matrix of the system (3.4) such that $X(0) = I$, and I is the $n \times n$ identical matrix. Moreover, $X(t, s)$ is transition matrix of the system (3.4), which equal to $X(t)X^{-1}(s)$, and $X(t + \omega, s + \omega) = X(t, s)$ for all $t, s \in \mathbb{R}$.

Assume that the following assumption is valid.

(C5) All multipliers of the system (3.4) in modulus are less than one.

It follows from the last condition that there exist positive numbers K and α such that

$$\|X(t, s)\| \le Ke^{-\alpha(t-s)}, \tag{3.5}$$

for $t \ge s$ [11].

The next lemma is necessary for further reasoning.

Lemma 3.3. [1] *If the inequality (3.5) is satisfied, then the following estimation is correct*

$$\|X(t + \tau, s + \tau) - X(t, s)\| \le \max_{t\in\mathbb{R}} \|A(t + \tau) - A(t)\| \frac{2K^2}{\alpha^2 e} e^{-\frac{\alpha}{2}(t-s)}, \tag{3.6}$$

for $t \ge s$ and arbitrary real number τ.

The additional conditions on system (3.3) are required:

(C6) there exists a positive constant L such that $\|g(t, x_1) - g(t, x_2)\| \le L \|x_1 - x_2\|$ for all $t \in \mathbb{R}, x_1, x_2 \in D$;
(C7) $K(b + \frac{m_g}{H}) < \alpha$;
(C8) $K(b + L) < \alpha$,

where $b = \sup_{t\in\mathbb{R}} \|B(t)\|$ and $m_g = \sup_{\mathbb{R}\times D} \|g(t, x)\|$.

According to [11], a bounded on the real axis function $y(t)$ is a solution of (3.3), if and only if it satisfies the equation

$$y(t) = \int_{-\infty}^{t} X(t, s)[B(s)y(s) + g(s, y(s))]ds, \ t \in \mathbb{R}. \tag{3.7}$$

Theorem 3.1. *If conditions (C1)-(C8) are valid, then the system (3.3) possesses a unique asymptotically stable unpredictable solution.*

Proof. Let t_k is the convergence sequence of the function $g(t, x)$ in the system (3.3). We denote by $\mathcal{B}$ the set of functions $\psi(t) = \mathbb{R} \to \mathbb{R}^n$, satisfy convergence property with common sequence t_k, and $\|\psi\|_1 < H$.

Let us show that the $\mathcal{B}$ is a complete space. Consider a Cauchy sequence $\theta_m(t)$ in $\mathcal{B}$, which converges to a limit function $\theta(t)$ on $\mathbb{R}$. We have that

$$\begin{aligned} &\|\theta(t + t_k) - \theta(t)\| < \|\theta(t + t_k) - \theta_m(t + t_k)\| + \|\theta_m(t + t_k) - \theta_m(t)\| + \\ &\|\theta_m(t) - \theta(t)\|. \end{aligned} \tag{3.8}$$

for a fixed closed and bounded interval $I \subset \mathbb{R}$. Now, one can take sufficiently large m and k such that each term on the right hand-side of (3.8) is smaller than $\frac{\epsilon}{3}$ for a fixed positive ϵ and $t \in I$. That is, the sequence $\theta(t+t_k)$ uniformly converges to $\theta(t)$ on I. Likewise, one can check that the limit function is uniformly continuous [11]. The completeness of $\mathcal{B}$ is shown.

Define the operator Π on $\mathcal{B}$ such that

$$\Pi\nu(t) = \int_{-\infty}^{t} X(t,s)[B(s)\psi(s) + g(s,\psi(s))]ds, \ t \in \mathbb{R}. \tag{3.9}$$

Fix a function $\psi(t)$ that belongs to $\mathcal{B}$. We have that

$$\|\Pi\psi(t)\| \le \int_{-\infty}^{t} \|X(t,s)\|(\|B(s)\|\|\psi(s)\| + \|g(s,\psi(s))\|)ds \le \frac{K(bH+m_g)}{\alpha}$$

for all $t \in \mathbb{R}$. Therefore, by the condition (C7) it is true that $\|\Pi\psi\|_1 < H$.

Now, applying the method of included intervals [5], we will show that $\Pi\psi(t+t_k) \to \Pi\psi(t)$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{R}$. Let us fix a positive number ϵ and an interval $[a,b]$, $-\infty < a < b < \infty$. There exist numbers c, ξ such that $c < a$ and $\xi > 0$, which satisfy the following inequalities:

$$\frac{4K^2\xi}{\alpha^3 e}(bH + m_g) < \frac{\epsilon}{3}, \tag{3.10}$$

$$\frac{2K}{\alpha}(2bH + m_g)e^{-\alpha(a-c)} < \frac{\epsilon}{3}, \tag{3.11}$$

and

$$\frac{K(H+b+L)\xi}{\alpha}[1 - e^{-\alpha(b-c)}] < \frac{\epsilon}{3}. \tag{3.12}$$

By using the condition (C4), for periodic matrix $A(t)$ one can attain that $\|A(t+t_k) - A(t)\| < \xi$, $t \in \mathbb{R}$. Moreover, since $\psi(t)$ belongs to the set $\mathcal{B}$, the matrix $B(t)$ is unpredictable and $g(t,x)$ is unpredictable in t then for sufficiently large k we have that $\|\psi(t+t_k) - \psi(t)\| < \xi$, $\|B(t+t_k) - B(t)\| < \xi$, and $\|g(t+t_k,\psi(t)) - g(t,\psi(t))\| < \xi$ for $t \in [c,b]$. Applying Lemma

(3.3), we get that

$$\|\Pi\psi(t+t_k)-\Pi\psi(t)\| =$$
$$\|\int_{-\infty}^{t} X(t+t_k,s+t_k)\Big(B(s+t_k)\psi(s+t_k)+g(s+t_k,\psi(s+t_k))\Big)ds -$$
$$\int_{-\infty}^{t} X(t,s)\Big(B(s)\psi(s)+g(s,\psi(s))\Big)ds\| \leq$$
$$\int_{-\infty}^{t} \|X(t+t_k,s+t_k)-X(t,s)\|\Big(\|B(s+t_k)\|\|\psi(s+t_k)\|+\|g(s+t_k,\psi(s+t_k))\|\Big)ds +$$
$$\int_{-\infty}^{t} \|X(t,s)\|\Big(\|B(s+t_k)-B(s)\|\|\psi(s+t_k)\|+\|B(s)\|\|\psi(s+t_k)-\psi(s)\|+$$
$$\|g(s+t_k,\psi(s+t_k))-g(s,\psi(s))\|\Big)ds \leq$$
$$\int_{-\infty}^{t} \|X(t+t_k,s+t_k)-X(t,s)\|\Big(\|B(s+t_k)\|\|\psi(s+t_k)\|+\|g(s+t_k,\psi(s+t_k))\|\Big)ds +$$
$$\int_{-\infty}^{c} \|X(t,s)\|\Big(\|B(s+t_k)-B(s)\|\|\psi(s+t_k)\|+\|B(s)\|\|\psi(s+t_k)-\psi(s)\|+$$
$$\|g(s+t_k,\psi(s+t_k))-g(s,\psi(s))\|\Big)ds \leq$$
$$\int_{c}^{t} \|X(t,s)\|\Big(\|B(s+t_k)-B(s)\|\|\psi(s+t_k)\|+\|B(s)\|\|\psi(s+t_k)-\psi(s)\|+$$
$$\|g(s+t_k,\psi(s+t_k))-g(s,\psi(s))\|\Big)ds \leq \int_{-\infty}^{t} \frac{2K^2\xi}{\alpha^2 e} e^{-\frac{\alpha}{2}(t-s)}(bH+m_g)ds +$$
$$\int_{-\infty}^{c} Ke^{-\alpha(t-s)}(2bH+2bH+2m_g)ds + \int_{c}^{t} Ke^{-\alpha(t-s)}(\xi H+b\xi+L\xi)ds \leq$$
$$\frac{4K\xi}{\alpha^3 e}(bH+m_g)+\frac{2K}{\alpha}(2bH+m_g)e^{-\alpha(a-c)}+\frac{K(H+b+L)\xi}{\alpha}[1-e^{-\alpha(b-c)}],$$

for all $t\in[a,b]$. From inequalities (3.10) to (3.12) it follows that $\|\Pi\psi(t+t_k)-\Pi\psi(t)\|<\epsilon$ for $t\in[a,b]$. Therefore, $\Pi\psi(t+t_k)$ uniformly converges to $\Pi\psi(t)$ on bounded interval of $\mathbb{R}$.

It is easy to verify that $\Pi\psi(t)$ is a uniformly continuous function, since its derivative is a uniformly bounded function on the real axis. Summarizing the above discussion, the set $\mathcal{B}$ is invariant for the operator Π.

We proceed to show that the operator $\Pi:\mathcal{B}\to\mathcal{B}$ is contractive. Let $u(t)$ and $v(t)$ be members of $\mathcal{B}$. Then, we obtain that

$$\|\Pi u(t)-\Pi v(t)\| \leq \int_{-\infty}^{t} \|X(t,s)\|\Big(\|B(s)\|\|u(s)-v(s)\|+g(s,u(s))-g(s,v(s))\|\Big)ds \leq$$
$$\int_{-\infty}^{t} Ke^{-\alpha(t-s)}(b+L)\|u(s)-v(s)\|ds \leq \frac{K(b+L)}{\alpha}\|u(t)-v(t)\|_1,$$

for all $t\in\mathbb{R}$. Therefore, the inequality $\|\Pi u-\Pi v\|_1 \leq \dfrac{K(b+L)}{\alpha}\|u-v\|_1$ holds, and according to the condition $(C8)$ the operator $\Pi:\mathcal{B}\to\mathcal{B}$ is contractive.

By the contraction mapping theorem there exists the unique fixed point, $z(t)\in\mathcal{B}$, of the operator Π, which is the unique bounded solution of the system (3.3).

Next, we will show that the solution $z(t)$ is unpredictable. According condition (C2) and Lemma 2.2, the function $g(t,z(t))$ is unpredictable in t, and there exists positive numbers δ and ϵ_0 such that $\|g(t+t_k,z(t))-g(t,z(t))\|>\epsilon_0$ for $t\in[s_k-\delta,s_k+\delta]$. Denote $K_0=\inf\{\|X(t,s):t>s\}\|$. One can find numbers $l\in\mathbb{N}$ and $\delta_1>0$ which satisfy the following inequalities

$$\delta_1<\delta, \tag{3.13}$$

$$\|X(t+t_k,s+t_k)-X(t,s)\|\le\frac{\epsilon_0}{l}, t\in\mathbb{R}, |s|<\delta_1, \tag{3.14}$$

$$\delta_1(K_0-\frac{bH+m_g}{l})>\frac{1}{2l}(3+\frac{K(H+b+L)}{2\alpha}), \tag{3.15}$$

$$\|z(t+s)-z(t)\|\le\frac{\epsilon_0}{4l}, t\in\mathbb{R}, |s|<\delta_1, \tag{3.16}$$

$$\|B(t+s)-B(t)\|\le\frac{\epsilon_0}{4l}, t\in\mathbb{R}, |s|<\delta_1, \tag{3.17}$$

Assume that the numbers l,δ_1 and $k\in\mathbb{N}$ are fixed. Denote by Δ the value of $\|z(s_k+t_k)-z(s_k)\|$, and consider two alternative cases: (1) $\Delta\ge\frac{\epsilon_0}{l}$, and (2) $\Delta<\frac{\epsilon_0}{l}$.

(1) If $\Delta\ge\frac{\epsilon_0}{l}$, it is easily find, from (3.16), that

$$\|z(t+t_k)-z(t)\|\ge\|z(s_k+t_k)-z(s_k)\|-\|z(s_k)-z(t)\|-$$
$$\|z(t+t_k)-z(s_k+t_k)\|>\frac{\epsilon_0}{l}-\frac{\epsilon_0}{4l}-\frac{\epsilon_0}{4l}=\frac{\epsilon_0}{2l},$$

for $t\in[s_k-\delta_1,s_k+\delta_1]$.

(2) Applying the relation

$$z(t+t_k)-z(t)=z(s_k+t_k)-z(s_k)+$$
$$\int_{s_k}^t X(t+t_k,s+t_k)(B(s+t_k)z(s+t_k)+g(s+t_k,z(s+t_k)))ds-$$
$$\int_{s_k}^t X(t,s)(B(s)z(s)+g(s,z(s)))ds=z(s_k+t_k)-z(s_k)+$$
$$\int_{s_k}^t X(t+t_k,s+t_k)[(B(s+t_k)-B(s))z(s+t_k))+B(s)(z(s+t_k)-z(s))]ds+$$
$$\int_{s_k}^t [X(t+t_k,s+t_k)-X(t,s)]B(s)z(s)ds+$$
$$\int_{s_k}^t X(t+t_k,s+t_k)[g(s+t_k,z(s+t_k))-g(s,z(s+t_k))]ds+$$
$$\int_{s_k}^t X(t+t_k,s+t_k)[g(s,z(s+t_k))-g(s,z(s))]ds+$$
$$\int_{s_k}^t [X(t+t_k,s+t_k)-X(t,s)]g(s,z(s))ds,$$

and inequalities (3.13)-(3.17), one can obtain that

$$
\begin{aligned}
&\|z(t+t_k)-z(t)\| \geq \int_{s_k}^{t} \|X(t+t_k,s+t_k)\|\|g(s+t_k,z(s+t_k))-g(s,z(s+t_k))\|ds - \\
&\int_{s_k}^{t} \|X(t+t_k,s+t_k)\|[\|B(s+t_k)-B(s)\|\|z(s+t_k)\|+\|B(s)\|\|z(s+t_k)-z(s)\|]ds - \\
&\int_{s_k}^{t} \|X(t+t_k,s+t_k)-X(t,s)\|\|B(s)\|z(s)\|ds - \|z(s_k+t_k)-z(s_k)\| - \\
&\int_{s_k}^{t} \|X(t+t_k,s+t_k)\|\|g(s,z(s+t_k))-g(s,z(s))\|ds - \\
&\int_{s_k}^{t} \|X(t+t_k,s+t_k)-X(t,s)\|\|g(s,z(s))\|ds \geq \int_{s_k}^{t} K_0\epsilon_0 ds - \\
&\int_{s_k}^{t} Ke^{-\alpha(t-s)}(H+b)\frac{\epsilon_0}{4l}ds - \int_{s_k}^{t}\frac{\epsilon_0}{l}bHds - \frac{\epsilon_0}{l} - \int_{s_k}^{t} Ke^{-\alpha(t-s)}L\frac{\epsilon_0}{4l}ds - \int_{s_k}^{t}\frac{\epsilon_0}{l}m_g ds \geq \\
&\delta_1 K_0\epsilon_0 - \frac{K(H+b+L)\epsilon_0}{4\alpha l} - \delta_1\frac{\epsilon_0}{l}(bH+m_g) - \frac{\epsilon_0}{l} > \frac{\epsilon_0}{2l},
\end{aligned}
$$

for each $t \in [s_k, s_k+\delta_1]$. Thus, the solution $z(t)$ is unpredictable.

Finally, we will study the asymptotic stability of the solution $z(t)$ of the system (3.3). It is true that

$$z(t) = X(t,t_0)z(t_0) + \int_{t_0}^{t} X(t,s)(B(s)z(s)+g(s,z(s)))ds,$$

for $t \geq t_0$.

Let $x(t)$ be another solution of system (3.3). One can write

$$x(t) = X(t,t_0)x(t_0) + \int_{t_0}^{t} X(t,s)(B(s)x(s)+g(s,x(s)))ds.$$

Making use of the relation

$$
\begin{aligned}
&z(t)-x(t) = X(t,t_0)(z(t_0)-x(t_0)) + \\
&\int_{t_0}^{t} X(t,s)\Big(B(s)(z(s)-x(s))+g(s,z(s))-g(s,x(s))\Big)ds,
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
&\|z(t)-x(t)\| \leq \|X(t,t_0)\|\|z(t_0)-x(t_0)\| + \\
&\int_{t_0}^{t} \|X(t,s)\|\Big(\|B(s)\|\|z(s)-x(s)\| + \|g(s,z(s))-g(s,x(s)\|ds \leq \\
&Ke^{-\alpha(t-t_0)}\|z(t_0)-x(t_0)\| + \int_{t_0}^{t} K(b+L)e^{-\alpha(t-s)}\|z(s)-x(s)\|ds.
\end{aligned}
$$

Now, applying Gronwall-Bellman Lemma, one can attain that

$$\|z(t)-x(t)\| \leq Ke^{-(\alpha-K(b+L))(t-t_0)}\|z(t_0)-x(t_0)\|, t \geq t_0. \tag{3.18}$$

The last inequality and condition (C8) confirm that the unpredictable solution $z(t)$ is asymptotically stable. The theorem is proved. □

3.2. **A case of compartmental periodic unpredictable perturbations and periodic coefficients.** Let us consider the following quasilinear system,

$$x'(t) = A(t)x + g(t,x), \tag{3.19}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^n$, n is a fixed natural number; the $A(t)$ is a continuous square matrix; $g : \mathbb{R} \times D \to \mathbb{R}^n$, $D = \{x \in \mathbb{R}^n, \|x\| < H\}$, where H is a fixed positive number.

Assume that the conditions (C1), (C5) and (C6) are satisfied, and the following assumptions are valid:

- **(C9)** the function $g(t,x)$ is compartmental in t;
- **(C10)** the convergence sequence of the function $g(t,x)$ satisfies the kappa property with respect to ω;
- **(C11)** $KL < \alpha$;
- **(C12)** $\dfrac{Km_g}{H} < \alpha$.

The next theorem is a corollary of the Theorem 3.1.

Theorem 3.2. *If conditions (C1),(C5),(C6), (C9)-(C12) are hold, then system (3.19) admits a unique asymptotically stable unpredictable solution.*

4. Numerical examples

Let $\psi_i, i \in \mathbb{Z}$, be a solution of the logistic map

$$\lambda_{i+1} = \mu\lambda_i(1 - \lambda_i), \tag{4.20}$$

where $i \in \mathbb{Z}$, and $\mu \in [3 + (2/3)^{1/2}, 4]$ is a fixed parameter. The section $[0,1]$ is invariant with respect to (4.20) for the considered values of μ.

Now, consider the following integral function

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds, \tag{4.21}$$

where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$. It is worth noting that $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t\in\mathbb{R}} |\Theta(t)| \leq \frac{1}{3}$. In [3], it was proved that the function $\Theta(t)$ is unpredictable.

In what follows, the piecewise constant function, $\Omega(t)$, will be defined for $t \in [hi, h(i+1))$, where $i \in \mathbb{Z}$, and h is a positive real number. The number h is said to be *the length of step* of the functions $\Omega(t)$ and $\Theta(t)$. The ratio of the period and the length of step, $\nabla = \omega/h$, we call *the degree of periodicity*. The dependence of the dynamics of the compartmental functions on the degree of periodicity is shown below.

Example 4.1. Before considering samples of unpredictable systems, let us show how the degree of periodicity can affect shape of graphs trajectory of unpredictable functions.

Consider the following modulo periodic unpredictable function,

$$f(t) = a\sin(\frac{2\pi t}{\omega}) + b\Theta(t), \tag{4.22}$$

where a and b are real coefficients, $\Theta(t)$ is the unpredictable function with step h, and the first component is an $\omega-$periodic function. In Figure 1 the graph of function $f(t)$ with degree of periodicity $\nabla = 0.25 < 1$ is shown. It is seen that periodicity appears locally on intervals of 10 units length approximately.

Figure 2 depicts the graph of unpredictable function $f(t)$ with $\nabla = 1$. In this case, the unpredictability dominates in the dynamics, and the periodicity is not seen.

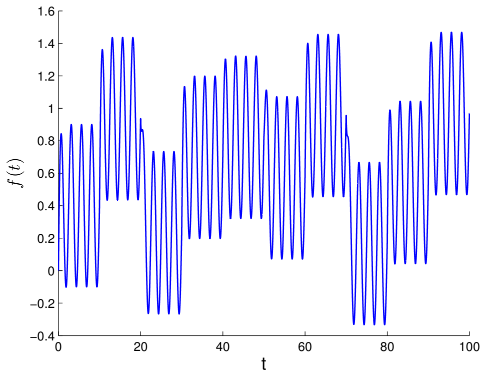

FIGURE 1. The graph of function $f(t)$ with $a = 0.5$, $b = 3$, $\omega = 2.5$, and degree of periodicity $\nabla = 0.25$.

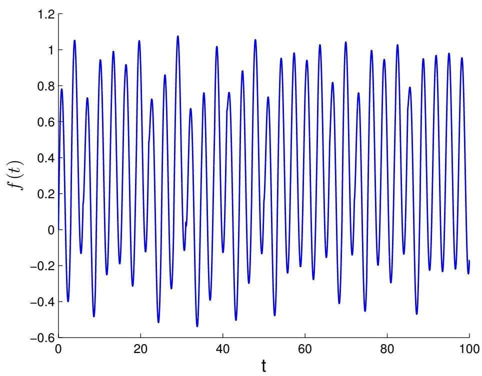

FIGURE 2. The graph of function $f(t)$ with $a = 0.6$, $b = 1.5$, $\omega = \pi$, and degree of periodicity $\nabla = 1$.

Figure 3 demonstrates the trajectory of the function $f(t)$ with degree of periodicity $\nabla = 60 > 1$. We obtain the function that admits a periodic shape which is enveloped by

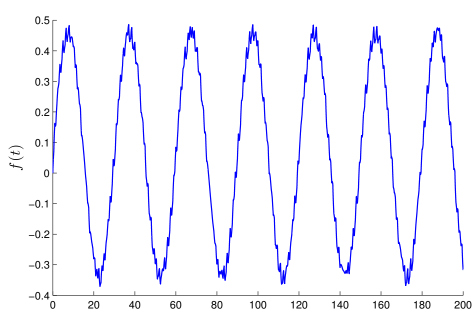

FIGURE 3. The graph of function $f(t)$ with $a = 0.4$, $b = 0.3$, $\omega = 30$, and degree of periodicity $\nabla = 60$.

unpredictability.

Analyzing simulations with different degrees of periodicity, we can say that there is a loss of global periodicity and irregularity dominates if ∇ is less than or equal to one. If ∇ is larger than one, we have the opposite effect, when periodicity in the trajectory is observed and envelopes the irregularity.

Example 4.2. Now, let us consider the following unpredictable system,

$$\begin{aligned} x_1' &= (-3+\cos(t)+0.3\Theta(t))x_1 - 0.4\sin(2t) + 0.25arctg(x_2) + 1.2\Theta(t) \\ x_2' &= (-2+\sin(2t)-0.2\Theta(t))x_2 + 0.2\cos(2t) + 0.3arctg(x_1) + 0.9\Theta(t), \end{aligned} \tag{4.23}$$

where $\Theta(t)$ is the unpredictable function with the step of length $h = 8\pi$, described above in (4.21),

$$A(t) = \begin{pmatrix} -3+\cos(t) & 0 \\ 0 & -2+\sin(2t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0.3\Theta(t) & 0 \\ 0 & -0.2\Theta(t) \end{pmatrix},$$

$$g(t,x) = \begin{pmatrix} -0.4\sin(2t) + 0.25arctg(x_2) + 1.2\Theta(t) \\ 0.2\cos(2t) + 0.3arctg(x_1) + 0.9\Theta(t) \end{pmatrix}.$$

The common period of the matrix and the components of $g(t,x)$ is equal to 2π. The degree of periodicity, ∇, is equal to 0.25. All conditions of the Theorem 3.2 are satisfied with $L = 0.3$, $K = 1$, $\alpha = 4\pi$, $b = 0.12$, $m_g = 1.54$ and $H = 0.5$. The coordinates of the solution $x(t)$, which asymptotically converges to the unpredictable solution are shown in Figure 4.

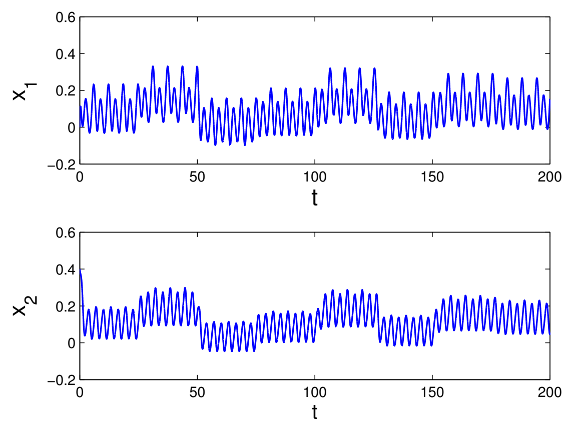

FIGURE 4. The coordinates of the solution $x(t)$ of system (4.23), which asymptotically converge to the coordinates of the unpredictable solution. The degree of periodicity is equal to $\nabla = 0.25$.

Figure 5 demonstrates the time series of the coordinates $x_1(t)$, $x_2(t)$ of the solution $x(t)$ of the system (4.23) with initial values $x_1(0) = 0$, $x_2(0) = 0.4$, when the degree of periodicity $\nabla = 1$.

Now, let us obtain simulation results for the following system,

$$\begin{aligned} x_1' &= (-3+\cos(t)+0.3\Theta(t))x_1 - 0.4\sin(0.2t) + 0.25arctg(x_2) + 1.2\Theta(t), \\ x_2' &= (-2+\sin(2t)-0.2\Theta(t))x_2 + 0.2\cos(0.2t) + 0.3arctg(x_1) + 0.9\Theta(t). \end{aligned} \tag{4.24}$$

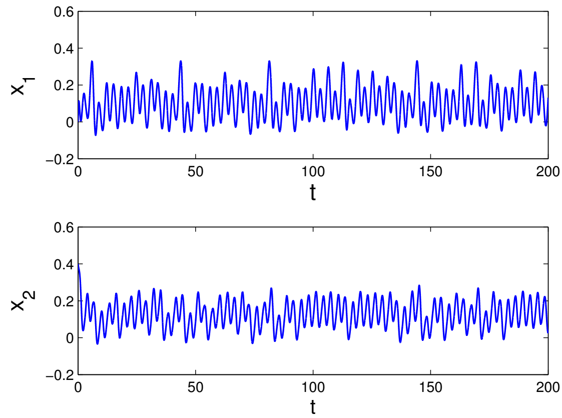

FIGURE 5. The time series of the coordinates $x_1(t), x_2(t)$ of the solution of system (4.23) with $\nabla = 1$, which asymptotically approach the coordinates of the unpredictable solution.

All coefficients are the same as in system (4.23), with the only difference that the common period of the coefficients and perturbations is equal to 10π. The unpredictable function $\Theta(t)$ with the step of length $h = 0.1\pi$. Thereby, the degree of periodicity, ∇, is equal to 100. In this case, the solution admits periodic shape, which envelopes the unpredictability, it is seen in Figure 6.

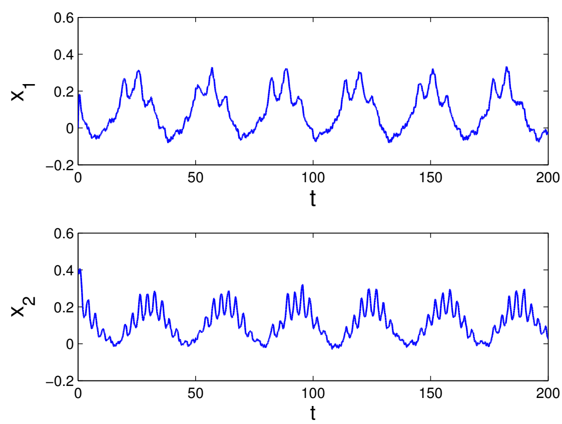

FIGURE 6. The coordinates $x_1(t)$, $x_2(t)$ of the solution of system (4.24) with initial values $x_1(0) = 0$, $x_2(0) = 0.4$. The degree of periodicity is equal to $\nabla = 100$.

Observation of the Figures 4, 5 and 6 helps to make the conclusion that the degree of periodicity is useful to decide how large the presence of periodicity and/or irregularity in the chaotic processes. That is, if the degree is less than one then irregularity dominates and envelopes a periodicity, while for the degree larger than one periodicity is vividly seen and it envelopes an irregularity. One can suppose that the interaction of periodicity and irregularity can provide rich industrial opportunities as well as can be extended to interactions of quasiperiodicity and almost periodicity with irregularity in further investigations.

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[1]DEPARTMENT OF MATHEMATICS
MIDDLE EAST TECHNICAL UNIVERSITY
06800, ANKARA, TURKEY
Email address: marat@metu.edu.tr
Email address: akylbek78@mail.ru

[2]DEPARTMENT OF MATHEMATICS
KTOBE REGIONAL STATE UNIVERSITY
030000, AKTOBE, KAZAKHSTAN
Email address: madina_1970@mail.ru