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A strongly convergent simultaneous cutter method for finding the minimal norm solution to common fixed point problem

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ABSTRACT. In this paper, we propose a strongly convergent simultaneous cutter method for finding the minimal norm solution over the intersection of fixed point sets of cutters. The proposed method is the combination of the simultaneous cutter method and the strongly variance of Krasnosel'skiĭ-Mann method. We show a strong convergence result of the sequence generated by the proposed method to the unique minimal norm solution. We finally present the numerical experiments on the minimal norm solution over a finite number of linear feasibility problem.

1. INTRODUCTION

Let $\mathcal{H}$ a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $T_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, \ldots, m$, be cutters with $\bigcap_{i=1}^{m} \operatorname{Fix} T_i \neq \emptyset$, where $\operatorname{Fix} T_i := \{x \in \mathcal{H} : T_i x = x\}$ denotes the set of fixed points of T_i. In this work, we focus on the finding of the minimal norm solution to the common fixed points of cutters of the following form:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & x \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i, \end{array} \tag{1.1}$$

This considered problem (1.1) is basically seen as a norm minimizing problem over the set of common fixed points of cutters. Note that the minimal norm solution over the solution sets of nonlinear problems has been studied in many aspects, for instance, in finding a minimal norm solution of convex optimization problems [7, 24], and in finding a minimal norm solution (in a general setting of variational inequalities) over the set of common fixed points of cutters [21, 22, 23]. It is worth noting that some practical applications, for instance, the classification problems via the support vector machine learning [21, Section 4], linear inverse problems [24, Section 5] and Markowitz portfolio optimization problem [7, Section 5.1], can be written as the problem (1.1) by transforming their constrained sets in the corresponding minimization problems to the (common) fixed point sets.

Focusing on the common fixed point problem (in short, CFP) linked to the constrained set of the considered problem (1.1), the powerful iterative methods for solving the CFP is known as the so-called simultaneous cutter method (in short, SCM), see [11, Sections 4.4, 4.8 and 4.9]. The formal form of SCM is given by the recurrence:

$$x_{k+1} := x_k + \lambda_k \left(\sum_{i=1}^{m} \omega_i(x_k) T_i(x_k) - x_k \right), \tag{1.2}$$

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where $x_1 \in \mathcal{H}$ is arbitrarily chosen, $(\lambda_k)_{k\geq 1} \subset [0, 2]$ is a sequence of relaxation parameters and $w : \mathcal{H} \to \Delta_m$ is a weight function of the form $w(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_m(x))$ for all $x \in \mathcal{H}$. The weak convergence result of the sequence generated by (1.2) to a solution of the CFP was given in [11, Section 5.8.2] and [12, Section 9.5]. A bit history: The most classical simultaneous type method in the setting of finite dimensional space $\mathbb{R}^n$ is due to the simultaneous projection method, which was introduced by Cimmino [14] for solving systems of linear equations by setting $\lambda_k = 2$ for all $k \geq 1$. Some particular situations of (1.2) were considered by many authors. For instance, Auslender [2] considered SCM for finding the common point in the intersection of nonempty closed and convex sets, where $\lambda_k = 1$ for all $k \geq 1$. De Pierro and Iusem [17] studied SCM for solving systems of linear inequalities, where $\lambda_k = \lambda$ for all $k \geq 1$ with a fixed parameter $\lambda \in (0, 2)$. Iusem and De Pierro [18] also investigated an extrapolated variance scheme of SCM, where $\lambda_k = 1$ for all $k \geq 1$. Combettes [15] proved the weak convergence result of SCM to the common point in the intersection of nonempty closed and convex sets, where $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for all $k \geq 1$ for some $\varepsilon \in (0, 1)$. See [11, Sections 5.4 and 5.8] for more convergence results and some literature reviews.

Even if the weak convergence results of simultaneous type methods have been studied by many authors, it should be noted that there is a constructive counterexample showing that the sequence generated by SCM may not converge strongly in general, see Bauschke, Matouskova and Reich [6] for more details. Moreover, the weak convergence results of SCM appear to be inadequate when dealing with applications that involve infinite-dimensional functional spaces. To achieve strong convergence results, it is usually necessary to impose more restrictive conditions, such as bounded regular properties of operators (see [13, 4, 16, 19, 3]). Furthermore, since the intersection of fixed point sets is a closed and convex set, it may be a singleton set; otherwise it must contain infinitely many points. In this situation, it is natural to find a common fixed point which is better than any other common fixed points. A typical strategy is to consider the minimal norm solution of the CFP as in the considered problem (1.1).

On the other hand, for a certain nonexpansive operator $T : \mathcal{H} \to \mathcal{H}$ with $\operatorname{Fix} T \neq \emptyset$, the celebrated Krasnosel'skiĭ-Mann method [10, Theorem 2.2] for finding a point in $\operatorname{Fix} T$ has the following form:

$$x_{k+1} := x_k + \lambda_k \left(T(x_k) - x_k \right),$$

where $x_1 \in \mathcal{H}$ is arbitrarily chosen, $(\lambda_k)_{k\geq 1} \subset (0, 1)$ is a real sequence. It is well known that the sequence generated by Krasnosel'skiĭ-Mann method converges weakly to a point in $\operatorname{Fix} T$. In order to deal with strong convergence result of Krasnosel'skiĭ-Mann type method, Boţ, Csetnek and Meier [8] proposed a modified Krasnosel'skiĭ-Mann method [8, Scheme (2)] of the following form:

$$x_{k+1} := \delta_k x_k + \lambda_k \left(T(\delta_k x_k) - \delta_k x_k \right), \tag{1.3}$$

where $x_1 \in \mathcal{H}$ is arbitrary and $(\lambda_k)_{k\geq 1}, (\delta_k)_{k\geq 1} \subset (0, 1]$ are sequences of real numbers which are suitably chosen. They proved that the generated sequence converges strongly to a point $x^* \in \operatorname{Fix} T$. It is worth noting that such a point x^* has a special feature in the sense that it captures the minimal norm value compared to other fixed points of T. The modified Krasnosel'skiĭ-Mann method (1.3) has been studied and generalized extensively in some aspects, see for example [1, 9, 26]. Recently, many researchers have proposed iterative methods to solve fixed point problems; see, e.g., [27, 28, 29] and the references therein.

The main contribution of this work is to propose an iterative method for finding the minimal norm solution to the common fixed point set as in the problem (1.1). The based ideas of the constructed method are the simultaneous cutter method (1.2) and the modified Krasnosel'skiĭ-Mann method (1.3). Under some appropriate conditions on the control sequences, we establish strong convergence of the proposed algorithm to the considered problem (1.1). To demonstrate the performance of the proposed method, we present some numerical experiments on the minimal norm solution to the linear feasibility problem.

The remaining of this work is organized as follows. In Section 2, we recall and collect some useful definitions and properties required in the work. In Section 3, we present an iterative method and subsequently prove its convergence results. In Section 4, we provide some important particular situations of the problem (1.1). After that, we examine the performance of the proposed method by numerical experiments in Section 5. Finally, we close this work by some concluding remarks in Section 6.

2. Preliminaries

In this section, we recall some elements of cutters along with their properties and some convergence tools. More comprehensive details can be found in, for instance, [5, 11, 25].

For a sequence $(x_k)_{k\geq 1}$, the strong convergence and the weak convergence to some element $x \in \mathcal{H}$ are written by the expressions $x_k \to x$ and $x_k \rightharpoonup x$, respectively. We denote by I the identity operator on $\mathcal{H}$. .

For an operator $T : \mathcal{H} \to \mathcal{H}$, $\operatorname{Fix} T := \{x \in \mathcal{H} : Tx = x\}$ denotes the set of fixed points of T. We recall further that an operator $T : \mathcal{H} \to \mathcal{H}$ with $\operatorname{Fix} T \neq \emptyset$ is said to be a *cutter* if,

$$\langle x - Tx, z - Tx\rangle \leq 0, \text{ for all } x \in \mathcal{H}, z \in \operatorname{Fix} T.$$

We collect some important properties of a cutter in the following proposition.

Proposition 2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a cutter with $\operatorname{Fix} T \neq \emptyset$. Then*

(i) $\operatorname{Fix} T$ *is closed and convex,*
(ii) *T is quasi-nonexpansive, i.e., $\|Tx - z\| \leq \|x - z\|$ for all $x \in \mathcal{H}$ and $z \in \operatorname{Fix} T$.*
(iii) *It holds that $\langle Tx - x, z - x\rangle \geq \|Tx - x\|^2$ for all $x \in \mathcal{H}$ and $z \in \operatorname{Fix} T$.*

The following definition is also a key tool for obtaining the convergence result.

Definition 2.1. An operator $S : \mathcal{H} \to \mathcal{H}$ is said to be *demi-closed* at 0 if for any sequence $(x_k)_{k\geq 1}$ in $\mathcal{H}$ and $x \in \mathcal{H}$ such that $x_k \rightharpoonup x$ and $Sx_k \to 0$, we have $Sx = 0$.

For dealing with the finite family of nonlinear operators, we denote the standard simplex by the set

$$\Delta_m := \left\{ v \in \mathbb{R}^m : v_i \geq 0, i = 1, 2, \ldots, m, \text{and} \sum_{i=1}^{m} v_i = 1 \right\}.$$

Next, we recall an important definition of a weight function $w : \mathcal{H} \to \Delta_m$ which is defined by $w(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_m(x))$ for each $x \in \mathcal{H}$.

Definition 2.2. For a finite family of nonlinear operators $T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, \ldots, m$, and a constant $\rho > 0$, we call the dynamic weight function $w : \mathcal{H} \to \Delta_m$ is *ρ-regular* with respect to $\{T_i\}_{i=1}^m$ if, for any $x \in \mathcal{H}$ there exists $j \in \{1, 2, \ldots, m\}$ in which

$$\omega_j(x)\|T_j x - x\|^2 \geq \rho \max_{1\leq i\leq m} \|T_i x - x\|^2.$$

Next, we recall some important examples of regular weight functions with respect to $\{T_i\}_{i=1}^m$:

(i) The positive constant weights

$$\omega_i(x) := \omega_i,$$

where $\omega_i > 0$ for all $i = 1, 2, \dots, m$, and $\sum_{i=1}^{m} \omega_i = 1$.

(ii) The weight functions $w : \mathcal{H} \to \Delta_m$, which is defined by

$$\omega_i(x) := \begin{cases} \dfrac{\|T_i x - x\|}{\sum_{i=1}^{m} \|T_i x - x\|}, & \text{for } x \notin \bigcap_{i=1}^{m} \operatorname{Fix} T_i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

For more examples of regular weight functions, the readers can consult [11, Example 5.8.3].

Next, we recall a useful fact used for proving the main result. The proof can be found in [30, Lemma 2.5].

Proposition 2.2. *Let $(a_k)_{k\geq 1}$ be a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \beta_k)a_k + \beta_k \xi_k,$$

where $(\beta_k)_{k\geq 1}$ and $(\xi_k)_{k\geq 1}$ are sequences satisfying the conditions:

(i) $(\beta_k)_{k\geq 1} \subset [0, 1]$ *and* $\sum_{k=1}^{\infty} \beta_k = \infty$;

(ii) $(\xi_k)_{k\geq 1} \subset \mathbb{R}$ *and* $\limsup_{k\to\infty} \xi_k \leq 0$.

Then $\lim_{k\to\infty} a_k = 0$.

We close this section by the important fact in proving the convergence of the generated sequence, where its proof can be found in [20, Lemma 3.1].

Proposition 2.3. *Let $(a_k)_{k\geq 1}$ be a sequence of nonnegative real numbers such that there exists a subsequence $(a_{k_j})_{j\geq 1}$ of $(a_k)_{k\geq 1}$ with $a_{k_j} < a_{k_j+1}$ for all $j \in \mathbb{N}$. If, for all $k \geq k_0$, we define*

$$\nu(k) = \max\left\{\bar{k} \in \mathbb{N} : k_0 \leq \bar{k} \leq k, a_{\bar{k}} < a_{\bar{k}+1}\right\},$$

then the sequence $(\nu(k))_{k\geq k_0}$ is nondecreasing, $\lim_{k\to\infty} \nu(k) = \infty$, $a_{\nu(k)} \leq a_{\nu(k)+1}$ and $a_k \leq a_{\nu(k)+1}$ for every $k \geq k_0$.

3. Algorithm and its convergence

In this section, we will state the proposed iterative method for solving the problem (1.1) and subsequently discuss some important convergence properties of the proposed algorithm.

Proposition 3.4. *The existence and uniqueness of the optimal solution to the problem (1.1) is guaranteed.*

Proof. Since all of the operators T_i, $i = 1, 2, \dots, m$, is cutter, we note from Proposition 2.1 that the intersection $\bigcap_{i=1}^{m} \operatorname{Fix} T_i$ is closed and convex. Hence, by applying [11, Theorem 1.3.1], the strictly convexity of the objective function $\frac{1}{2}\|\cdot\|^2$ and the closedness and the convexity of $\bigcap_{i=1}^{m} \operatorname{Fix} T_i$, we can conclude that the problem (1.1) has the unique optimal solution. □

Next, we present an iterative method for solving the problem (1.1) as the following algorithm.

Algorithm 1 A strongly convergent simultaneous cutter method

Initialization: Given two real sequences $(\lambda_k)_{k\geq 1} \subseteq (0, +\infty)$ and $(\delta_k)_{k\geq 1} \subseteq [0, 1)$. Given a dynamic weight function $w : \mathcal{H} \to \Delta_m$ in which $w(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_m(x))$ for all $x \in \mathcal{H}$. Choose an arbitrary initial point $x_1 \in \mathcal{H}$.
Iterative Steps: For an iterate $x_k \in \mathcal{H}$, define $x_{k+1} \in \mathcal{H}$ as

$$x_{k+1} := \delta_k x_k + \lambda_k \left(\sum_{i=1}^{m} \omega_i(\delta_k x_k) T_i(\delta_k x_k) - \delta_k x_k \right).$$

Update $k := k + 1$.

To prove the convergence of Algorithm 1, we assume the following assumption throughout this work.

Assumption 3.1. *The sequences $(\lambda_k)_{k\geq 1}$ and $(\delta_k)_{k\geq 1}$ satisfy the following conditions:*

(i) $(\lambda_k)_{k\geq 1} \subseteq (\varepsilon, 2 - \varepsilon)$ *for some* $\varepsilon \in (0, 1)$;

(ii) $\lim\limits_{k\to+\infty} \delta_k = 1$ *and* $\sum_{k\geq 1}(1 - \delta_k) = +\infty$.

Remark 3.1. Some remarks relating to Algorithm 1 and Assumption 3.1 are in order:

(i) In the case of $m = 1$, Algorithm 1 is reduced to the modified Krasnosel'skiĭ-Mann iterative method studied in [8, Scheme (2)]. It is worth noting that if the sequence $(\lambda_k)_{k\geq 1}$ considered in this work can be extended to $(0, 2)$, which is wider than the interval $(0, 1)$ which is considered in [8, Theorem 3].

(ii) We note that the sequence $(\delta_k)_{k\geq 1}$ considered in this work need to be less than 1 due to the proving lines, whereas the one considered in [8] can include 1. Even if the length of the sequence $(\delta_k)_{k\geq 1}$ is short, in this work, we remove the assumption $\sum_{k\geq 2} |\delta_k - \delta_{k-1}| < +\infty$ which was supposed in [8].

(iii) An example of the sequence $(\delta_k)_{k\geq 1}$ satisfying Assumption 3.1 is $\delta_k = 1 - \frac{a}{k+1}$ for all $k \geq 1$, where $a \in (0, 1]$.

We prove the boundedness of a sequence $(x_k)_{k\geq 1}$ generated by Algorithm 1 as the following lemma.

Lemma 3.1. *Let $(x_k)_{k\geq 1}$ be a sequence generated by Algorithm 1. Then, $(x_k)_{k\geq 1}$ is a bounded sequence.*

Proof. Let $x \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$ be given. We firstly note that from the definition of x_{k+1}, the property of the dynamic weight, the convexity of $\|\cdot\|$ and Proposition 2.1 (ii) that for every $k \geq 1$

$$\begin{aligned}
\|x_{k+1} - x\| &= \left\| \delta_k x_k + \lambda_k \left(\sum_{i=1}^{m} \omega_i(\delta_k x_k) T_i(\delta_k x_k) - \delta_k x_k \right) - x \right\| \\
&\leq (1 - \lambda_k) \|\delta_k x_k - x\| + \lambda_k \left\| \sum_{i=1}^{m} \omega_i(\delta_k x_k) \left(T_i(\delta_k x_k) - x \right) \right\| \\
&\leq (1 - \lambda_k) \|\delta_k x_k - x\| + \lambda_k \sum_{i=1}^{m} \omega_i(\delta_k x_k) \|T_i(\delta_k x_k) - x\|
\end{aligned}$$

$$\begin{aligned} &\leq (1-\lambda_k)\,\|\delta_k x_k - x\| + \lambda_k \sum_{i=1}^{m} \omega_i(\delta_k x_k)\,\|\delta_k x_k - x\| \\ &\leq \|\delta_k x_k - x\| \leq \delta_k\|x_k - x\| + (1-\delta_k)\|x\|. \end{aligned}$$

To deduce the boundedness of the sequence $(x_k)_{k\geq 1}$, let us notice that for every $k \geq 1$

$$\begin{aligned} \|x_{k+1} - x\| &\leq \delta_k\|x_k - x\| + (1-\delta_k)\|x\| \\ &\leq \max\{\|x_k - x\|, \|x\|\} \leq \cdots \leq \max\{\|x_0 - x\|, \|x\|\}. \end{aligned}$$

By applying the induction argument, we obtain that the sequence $(x_k)_{k\geq 1}$ is bounded as required. □

Now, we are in a position to address the strong convergence of the sequence $(x_k)_{k\geq 1}$ generated by Algorithm 1 as the following theorem.

Theorem 3.2. *Let $(x_k)_{k\geq 1}$ be a sequence generated by Algorithm 1. Assume that Assumption 3.1 holds, the dynamic weight function $w : \mathcal{H} \to \Delta_m$ is ρ-regular with respect to $\{T_i\}_{i=1}^m$ and the operator $T_i - I$ is demi-colsed at 0, for all $i = 1, \ldots, m$. Then, the sequence $(x_k)_{k\geq 1}$ converges strongly to x^*, the unique solution of the problem (1.1).*

Proof. Let x^* be the unique solution of the problem (1.1). By the definition of x_{k+1} together with the convexity of the function $\|\cdot\|^2$ and Proposition 2.1 (iii), we note that for every $k \geq 1$

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \left\|\delta_k x_k + \lambda_k \left(\sum_{i=1}^{m} \omega_i(\delta_k x_k) T_i(\delta_k x_k) - \delta_k x_k\right) - x^*\right\|^2 \\ &= \|\delta_k x_k - x^*\|^2 + \lambda_k^2 \left\|\sum_{i=1}^{m} \omega_i(\delta_k x_k)(T_i(\delta_k x_k) - \delta_k x_k)\right\|^2 \\ &\quad -2\lambda_k \sum_{i=1}^{m} \omega_i(\delta_k x_k)\langle x^* - \delta_k x_k, T_i(\delta_k x_k) - \delta_k x_k\rangle \\ &\leq \|\delta_k x_k - x^*\|^2 + \lambda_k^2 \sum_{i=1}^{m} \omega_i(\delta_k x_k)\|T_i(\delta_k x_k) - \delta_k x_k\|^2 \\ &\quad -2\lambda_k \sum_{i=1}^{m} \omega_i(\delta_k x_k)\|T_i(\delta_k x_k) - \delta_k x_k\|^2 \\ &= \|\delta_k x_k - x^*\|^2 - \lambda_k(2-\lambda_k) \sum_{i=1}^{m} \omega_i(\delta_k x_k)\|T_i(\delta_k x_k) - \delta_k x_k\|^2. \end{aligned} \tag{3.5}$$

We will consider the first term in the right-hand side of (3.5) as follows. For every $k \geq 1$, we note that

$$\begin{aligned} \|\delta_k x_k - x^*\|^2 &= \|\delta_k(x_k - x^*) + (\delta_k - 1)x^*\|^2 \\ &= \delta_k^2\|x_k - x^*\|^2 + 2\delta_k(1-\delta_k)\langle -x^*, x_k - x^*\rangle + (1-\delta_k)^2\|x^*\|^2 \\ &\leq \delta_k\|x_k - x^*\|^2 + (1-\delta_k)(2\delta_k\langle -x^*, x_k - x^*\rangle + (1-\delta_k)\|x^*\|^2). \end{aligned}$$

Invoking this obtained relation in the inequality (3.5), we obtain that for every $k \geq 1$

$$\|x_{k+1} - x^*\|^2 \leq \delta_k\|x_k - x^*\|^2 + (1-\delta_k)(2\delta_k\langle -x^*, x_k - x^*\rangle + (1-\delta_k)\|x^*\|^2), \tag{3.6}$$

and

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + (1-\delta_k)(2\delta_k\langle -x^*, x_k - x^*\rangle + (1-\delta_k)\|x^*\|^2)$$

$$-\lambda_k(2-\lambda_k)\sum_{i=1}^{m}\omega_i(\delta_k x_k)\|T_i(\delta_k x_k)-\delta_k x_k\|^2. \tag{3.7}$$

For simplicity, we denote

$$a_k := \|x_k - x^*\|^2 \text{ and } \xi_k := 2\delta_k\langle -x^*, x_k - x^*\rangle + (1-\delta_k)\|x^*\|^2, \qquad \text{for all } k \geq 1.$$

By using the boundedness of $(x_k)_{k\geq 1}$ obtained in Lemma 3.1 and the assumption that $\lim_{k\to+\infty}\delta_k = 1$, we have that $(\xi_k)_{k\geq 1}$ is a bounded sequence, subsequently,

$$\lim_{k\to+\infty}(1-\delta_k)\xi_k = 0. \tag{3.8}$$

To obtain the strong convergence of $(x_k)_{k\geq 1}$ to the unique solution x^*, we will divide the proof into two cases related to the behavior of the sequence $(a_k)_{k\geq 1}$.

Case 1. Suppose that there exists $k_0 \in \mathbb{N}$ such that $(a_k)_{k\geq k_0}$ is nonincreasing. In this case, we immediately note that $\lim_{k\to+\infty} a_k$ exists.

Since the sequence $(x_k)_{k\geq 1}$ is bounded by Lemma 3.1, there are a subsequence $(x_{k_j})_{j\geq 1}$ of $(x_k)_{k\geq 1}$ and a point $z \in \mathcal{H}$ such that $x_{k_j} \rightharpoonup z$ and

$$\limsup_{k\to+\infty}\langle -x^*, x_k - x^*\rangle = \lim_{j\to+\infty}\langle -x^*, x_{k_j} - x^*\rangle = \lim_{j\to+\infty}\langle -x^*, \delta_{k_j}x_{k_j} - x^*\rangle. \tag{3.9}$$

On the other hand, the relation (3.7) yields that

$$\begin{aligned} 0 &\leq \limsup_{j\to+\infty}\lambda_{k_j}(2-\lambda_{k_j})\sum_{i=1}^{m}\omega_i(\delta_{k_j}x_{k_j})\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2 \\ &\leq \limsup_{j\to+\infty}\left(a_{k_j} - a_{k_j+1} + (1-\delta_{k_j})\xi_{k_j}\right) \\ &= \lim_{j\to+\infty}a_{k_j} - \lim_{j\to+\infty}a_{k_j+1} + \lim_{j\to+\infty}(1-\delta_{k_j})\xi_{k_j} = 0, \end{aligned}$$

which implies that

$$\lim_{j\to+\infty}\lambda_{k_j}(2-\lambda_{k_j})\sum_{i=1}^{m}\omega_i(\delta_{k_j}x_{k_j})\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2 = 0.$$

Since $\lambda_k \in (\varepsilon, 2-\varepsilon)$ for an arbitrary constant $\varepsilon \in (0,1)$, we have $\varepsilon^2 < \lambda_{k_j}(2-\lambda_{k_j})$. Consequently, the above relation leads to

$$\lim_{j\to+\infty}\sum_{i=1}^{m}\omega_i(\delta_{k_j}x_{k_j})\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2 = 0. \tag{3.10}$$

The ρ-regularity of the dynamic weight function w implies that

$$\sum_{i=1}^{m}\omega_i(\delta_{k_j}x_{k_j})\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2 \geq \rho\max_{1\leq i\leq m}\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2.$$

By using this inequality and the relation (3.10), we have

$$0 = \lim_{j\to+\infty}\sum_{i=1}^{m}\omega_i(\delta_{k_j}x_{k_j})\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2 \geq \rho\lim_{j\to+\infty}\max_{1\leq i\leq m}\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\|^2 \geq 0,$$

which leads to

$$\lim_{j\to+\infty}\|T_i(\delta_{k_j}x_{k_j})-\delta_{k_j}x_{k_j}\| = 0, \qquad \text{for all } i = 1,2,\ldots,m. \tag{3.11}$$

We note that $\delta_{k_j}x_{k_j} \rightharpoonup z$, together with the relation (3.11) and the assumption that each $T_i - I$ is demi-closed at 0 yield that $z \in \operatorname{Fix} T_i$, for all $i = 1,2,\ldots,m$, and hence $z \in \bigcap_{i=1}^{m}\operatorname{Fix} T_i$.

Furthermore, we obtain from the relation (3.9) that

$$\limsup_{k\to+\infty}\langle -x^*, x_k - x^*\rangle = \lim_{j\to+\infty}\langle -x^*, \delta_{k_j}x_{k_j} - x^*\rangle = \langle -x^*, z - x^*\rangle \le 0. \tag{3.12}$$

Now, by the boundedness of the sequences $(x_k)_{k\geq 1}$ and the inequality (3.12), we have

$$\limsup_{k\to+\infty}\xi_k = \limsup_{k\to+\infty}\left(2\delta_k\langle -x^*, x_k - x^*\rangle + (1-\delta_k)\|x^*\|^2\right) \le 0. \tag{3.13}$$

Notice that the relation (3.6) can be written the form of

$$a_{k+1} \le (1-(1-\delta_k))a_k + (1-\delta_k)\xi_k.$$

By applying this relation together with (3.13), Assumption 3.1 (ii) and Proposition 2.2, we obtain that

$$\lim_{k\to+\infty}\|x_k - x^*\| = 0.$$

Case 2. Suppose that there exists a subsequence $(a_{k_r})_{r\geq 1}$ of $(a_k)_{k\geq 1}$ such that $a_{k_r} < a_{k_r+1}$ for all $r\in\mathbb{N}$.

By considering the sequence $(\nu(k))_{k\geq k_0}$ defined in Proposition 2.3, we have

$$a_{\nu(k)} \le a_{\nu(k)+1} \tag{3.14}$$

and

$$a_k \le a_{\nu(k)+1} \tag{3.15}$$

for all $k\geq k_0$. Now, let $(x_{\nu(k_j)})_{j\geq 1}$ be a subsequence of $(x_{\nu(k)})_{k\geq 1}$ and $z\in\mathcal{H}$ such that $x_{\nu(k_j)} \rightharpoonup z$. It follows that $\delta_{\nu(k_j)}x_{\nu(k_j)} \rightharpoonup z$ and

$$\begin{aligned}\limsup_{k\to+\infty}\langle -x^*, x_{\nu(k)} - x^*\rangle &= \lim_{k\to\infty}\langle -x^*, x_{\nu(k_j)} - x^*\rangle \\ &= \lim_{j\to+\infty}\langle -x^*, \delta_{\nu(k_j)}x_{\nu(k_j)} - x^*\rangle.\end{aligned}$$

By using the inequalities (3.7) and (3.14), we obtain that, for all $k_j\geq k_0$,

$$\begin{aligned}0 &\le a_{\nu(k_j)+1} - a_{\nu(k_j)} \\ &\le -\lambda_{\nu(k_j)}(2-\lambda_{\nu(k_j)})\sum_{i=1}^m \omega_i(\delta_{\nu(k_j)}x_{\nu(k_j)})\|T_i(\delta_{\nu(k_j)}x_{\nu(k_j)}) - \delta_{\nu(k_j)}x_{\nu(k_j)}\|^2 + \xi_{\nu(k_j)},\end{aligned}$$

and hence

$$\lambda_{\nu(k_j)}(2-\lambda_{\nu(k_j)})\sum_{i=1}^m \omega_i(\delta_{\nu(k_j)}x_{\nu(k_j)})\|T_i(\delta_{\nu(k_j)}x_{\nu(k_j)}) - \delta_{\nu(k_j)}x_{\nu(k_j)}\|^2 \le \xi_{\nu(k_j)}.$$

By using the relation (3.8), we obtain that

$$\lim_{j\to+\infty}\sum_{i=1}^m \omega_i(\delta_{\nu(k_j)}x_{\nu(k_j)})\|T_i(\delta_{\nu(k_j)}x_{\nu(k_j)}) - \delta_{\nu(k_j)}x_{\nu(k_j)}\|^2 = 0.$$

By carrying on the similar context to those used in **Case 1**, we have

$$\lim_{j\to+\infty}\|T_i(\delta_{\nu(k_j)}x_{\nu(k_j)}) - \delta_{\nu(k_j)}x_{\nu(k_j)}\| = 0, \quad \text{for all } i = 1,2,\ldots,m.$$

This implies that $z\in\bigcap_{i=1}^m \operatorname{Fix} T_i$. Moreover, we also have that

$$\limsup_{k\to+\infty}\xi_{\nu(k)} = \limsup_{k\to+\infty}\left(2\delta_{\nu(k)}\langle -x^*, x_{\nu(k)} - x^*\rangle + (1-\delta_{\nu(k)})\|x^*\|^2\right) \le 0. \tag{3.16}$$

In the view of the inequality (3.6), we have

$$0 \le a_{\nu(k)+1} \le \delta_{\nu(k)}a_{\nu(k)} + (1-\delta_{\nu(k)})\xi_{\nu(k)}, \tag{3.17}$$

and hence

$$0 \leq a_{\nu(k)+1} - a_{\nu(k)} \leq (1 - \delta_{\nu(k)}) \left(\xi_{\nu(k)} - a_{\nu(k)}\right).$$

According to $(1 - \delta_{\nu(k)}) > 0$, we have

$$0 \leq a_{\nu(k)} \leq \xi_{\nu(k)},$$

and by (3.17), we also have

$$0 \leq a_{\nu(k)+1} \leq \xi_{\nu(k)}.$$

In virtue of (3.16), we obtain that

$$\lim_{k \to +\infty} a_{\nu(k)+1} = 0,$$

and hence, by combining this together with inequality (3.15), we have

$$0 \leq \limsup_{k \to +\infty} a_k \leq \limsup_{k \to +\infty} a_{\nu(k)+1} = 0.$$

Therefore, we conclude that $\lim_{k \to +\infty} \|x_k - x^*\| = 0.$

□

4. Minimal norm solution to convex feasibility problems

In this section, we discuss the minimal norm solution to convex feasibility problems which includes solving methods, convergence results, and the convergence behavior of the proposed method.

Firstly, let us consider the problem of the form:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & x \in \bigcap_{i=1}^{m} C_i, \end{array} \tag{4.18}$$

where the subset C_i is given in the form $C_i = \{x \in \mathcal{H} : f_i(x) \leq 0\}$, the sublevel set of continuous convex functions $f_i : \mathcal{H} \to \mathbb{R}, i = 1, 2, ..., m$, and we assume that the intersection $\bigcap_{i=1}^{m} C_i$ is a nonempty set. Now, we recall the subgradient projection relative to f_i as the operator $P_{f_i} : \mathcal{H} \to \mathcal{H}$ defined by

$$P_{f_i}(x) := \begin{cases} x - \dfrac{\max\{f_i(x), 0\}}{\|s_{f_i}(x)\|^2} s_{f_i}(x) & \text{if } s_{f_i}(x) \neq 0, \\ x & \text{if } s_{f_i}(x) = 0, \end{cases}$$

where $s_{f_i}(x) \in \partial f_i(x) := \{u \in \mathcal{H} : \langle u, y - x\rangle \leq f_i(y) - f_i(x), \forall y \in \mathcal{H}\}$ is a subgradient of the function f_i at x see [11, Definition 4.2.4]. Note that the nonemptiness of $\partial f_i(x)$ is quaranteed by the continuity and convexity of f_i, see [11, Theorem 1.1.56]. It is worth noting that the subgradient projection P_{f_i} is a cutter with $\operatorname{Fix} P_{f_i} = C_i$ for all $i = 1, 2, ..., m$, see [11, Lemma 4.2.5 and Corollary 4.2.6]. Thus, the problem (4.18) is a particular situation of the problem (1.1).

The iterative method for solving the problem (4.18) can be stated as the following algorithm.

Algorithm 2 A strongly convergent simultaneous subgradient projection method

Initialization: Given two real sequences $(\lambda_k)_{k\geq 1} \subseteq (0,+\infty)$ and $(\delta_k)_{k\geq 1} \subseteq [0,1)$. Given a dynamic weight function $w : \mathcal{H} \to \Delta_m$ in which $w(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_m(x))$ for all $x \in \mathcal{H}$. Choose an arbitrary initial point $x_1 \in \mathcal{H}$.
Iterative Steps: For an iterate $x_k \in \mathcal{H}$, define $x_{k+1} \in \mathcal{H}$ as

$$x_{k+1} := \delta_k x_k + \lambda_k \left(\sum_{i=1}^{m} \omega_i(\delta_k x_k) P_{f_i}(\delta_k x_k) - \delta_k x_k \right).$$

Update $k := k+1$.

The following corollary is directly obtained from Theorem 3.2.

Corollary 4.1. *Let $(x_k)_{k\geq 1}$ be a sequence generated by Algorithm 2. Assume that Assumption 3.1 holds, the dynamic weight function $w : \mathcal{H} \to \Delta_m$ is ρ-regular with respect to $\{P_{f_i}\}_{i=1}^m$ and the functions $f_i, i = 1, 2, ..., m$, are Lipschitz continuous relative to every bounded subset of $\mathcal{H}$. Then, the sequence $(x_k)_{k\geq 1}$ converges strongly to x^*, the unique solution of the problem* (4.18).

Proof. Accoding to [11, Theorem 4.2.7], we note that the assumption that each function f_i is Lipschitz continuous relative to every bouned subset of $\mathcal{H}$ implies that $P_{f_i} - I$ is demi-closed at 0. Hence, the assumptions of Theorem 3.2 are satisfied, and we conclude that the sequence $(x_k)_{k\geq 1}$ converges strongly to the unique solution of the problem (4.18). □

Remark 4.2. Note that the Lipschitz continuity of each f_i holds true when $\mathcal{H}$ is a finite-dimensional space, see [5, Proposition 16.20].

5. Numerical experiments

In this section, we provide a numerical illustration for solving the minimal norm solution to the linear feasibility problem which is written in the form of a finite number of half-space constraints. Let $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ be given for all $i = 1, 2, ..., m$. We consider the following the minimal norm solution:

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\|x\|^2 \\ &\text{subject to} \quad \langle a_i, x\rangle \leq b_i, i = 1, 2, ..., m. \end{aligned} \tag{5.19}$$

Certainly, this minimum norm solution (5.19) can be expressed in the form of the problem (1.1) as:

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\|x\|^2 \\ &\text{subject to} \quad x \in \bigcap_{i=1}^{m} \mathrm{Fix}\,(P_{C_i}), \end{aligned}$$

where the constrained sets $C_i := \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq b_i\}, i = 1, 2, ..., m$, are half spaces and P_{C_i} is a metric projection on C_i , which is defined by

$$P_{C_i}(x) = \begin{cases} x - \dfrac{\langle a_i, x\rangle - b_i}{\|a_i\|^2} a_i & \text{if } \langle a_i, x\rangle > b_i, \\ x & \text{if } \langle a_i, x\rangle \leq b_i, \end{cases}$$

for all $i = 1, 2, ..., m$. Note that the metric projection P_{C_i} is a cutter with $\mathrm{Fix}\, P_{C_i} = C_i$, and the operator $P_{C_i} - I$ is demi-closed at 0, for all $i = 1, 2, ..., m$.

We generate each component of $a_i \in \mathbb{R}^n$ by uniformly distributed random generating between $(0,1)$, and we generate each component $b_i \in \mathbb{R}^m$ by normally distributed randomly chosen in $(-1,1)$ for all $i = 1,2,...,m$. Moreover, the initial point x_1 is a vector whose all coordinates are 1.

All experiments were conducted by using MATLAB 9.19 (R2022b). All computations were performed on a MacBook Pro 14-inch 2021 with an Apple M1 Pro processor and 16 GB memory. Moreover, all computational running times are given in seconds. In this numerical experiment, we terminate the algorithm based on the stopping criterion that the computational running time exceeds 5000 seconds or

$$\frac{\|x_k - x_{k-1}\|}{\|x_k\| + 1} \leq t,$$

where t is an error tolerance.

We start the numerical investigation in solving problem (5.19) with different sizes of m and n. We present the numerical comparison of Algorithm 1 to the modified Krasnosel'skiĭ-Mann method (1.3). To perform Algorithm 1, we set the operator $T_i = P_{C_i}$ for all $i = 1,2,...,m$, and put the dynamic weight function defined by (2.4). For the modified Krasnosel'skiĭ-Mann method (1.3), we set the operator T in [8, The equation (2)] to be $T := \frac{1}{m}\sum_{i=1}^{m} P_{C_i}$, which is a nonexpansive operator with $\operatorname{Fix} T = \bigcap\limits_{i=1}^{m} C_i$ and [8, Theorem 3] can also be applied when solving problem (5.19). We choose the possibly best parameters of these two methods as:

- Algorithm 1: $\delta_k = 1 - \frac{1.6}{k+1}$ and $\lambda_k = 1.8$.
- The modified Krasnosel'skiĭ-Mann method (1.3): $\delta_k = 1 - \frac{1.0}{k+1}$ and $\lambda_k = 1.9$.

The involved parameters' combinations are given in Appendices 1 and 2.

In Table 1, we compare Algorithm 1 and the modified Krasnosel'skiĭ-Mann method (1.3) for solving problem (5.19) with different sizes of m and n. We set the error tolerance $t = 10^{-6}$. We conducted 10 independent tests for each parameter combination (m,n). The table presents the average number of iterations and average computational running time for each parameter set. The results showed that Algorithm 1 consistently outperformed the modified Krasnosel'skiĭ-Mann method (1.3) for all dimensions. In every case, the average number of iterations and computational time for Algorithm 1 were 4-24 times lower than those of the modified Krasnosel'skiĭ-Mann method (1.3). The best performance of Algorithm 1 compared to the modified Krasnosel'skiĭ-Mann method (1.3) was observed for the case $(m,n) = (500, 50000)$. In the case where $(m,n) = (1000, 100000)$, we observe that Algorithm 1 had a computational running time of only 515.36 seconds, while the modified Krasnosel'skiĭ-Mann method (1.3) required more than 5000 seconds. This significant difference in computational running time further emphasizes the superior performance of Algorithm 1 compared to the modified Krasnosel'skiĭ-Mann method (1.3).

TABLE 1. Comparisons between Algorithm 1 and the modified Krasnosel'skiĭ-Mann method (1.3) for different sizes of m and n.

m	n	Algorithm 1		Method (1.3)	
		#(iters)	Time	#(iters)	Time
50	100	476	0.26	2561	1.09
	200	420	0.20	2487	1.14
	500	368	0.24	2314	1.48
	1000	344	0.29	2485	2.12
	5000	308	1.18	3621	13.05
100	200	457	0.46	3328	2.94
	400	416	0.46	3457	3.79
	1000	362	0.58	2743	4.37
	2000	311	0.83	2871	7.62
	10000	297	3.82	3983	49.42
200	400	434	0.96	4614	9.65
	800	379	1.10	3592	10.28
	2000	306	1.60	3378	17.61
	4000	301	3.08	3327	33.57
	20000	282	15.72	4643	246.04
500	1000	402	3.29	6270	50.64
	2000	344	4.86	5042	71.69
	5000	287	10.68	4151	150.77
	10000	260	16.24	4170	253.45
	50000	275	126.18	6397	2887.57
1000	2000	347	9.98	6840	193.14
	4000	353	19.17	5523	295.04
	10000	275	35.34	4651	571.91
	20000	248	72.33	4856	1352.70
	100000	261	515.36	$-$	> 5000.00

For each optimal tolerance t, we notice that the case $(m, n) = (500, 1000)$ required computational running time less than the case $(m, n) = (1000, 2000)$ approximately at least 3 times. In a similar fashion, the case $(m, n) = (1000, 2000)$ reached the tolerance faster than the case $(m, n) = (2000, 4000)$ at least 2 times. Notice that the case $(m, n) = (500, 1000)$ required computational running time less than the case $(m, n) = (1000, 2000)$ approximately at least 3 times.

Next, we investigate the behavior of Algorithm 1 for different sizes of (m, n) with various error tolerances t. We put the size (m, n) as $(500, 1000)$, $(1000, 2000)$, and $(2000, 4000)$. The results are given in Figure 1.

6. CONCLUSION

We proposed in this paper the so-called a strongly convergent simutaneous cutter method for finding the minimal norm solution to the common fixed point problem. We proved that the proposed method converged strongly to the solution of the considered

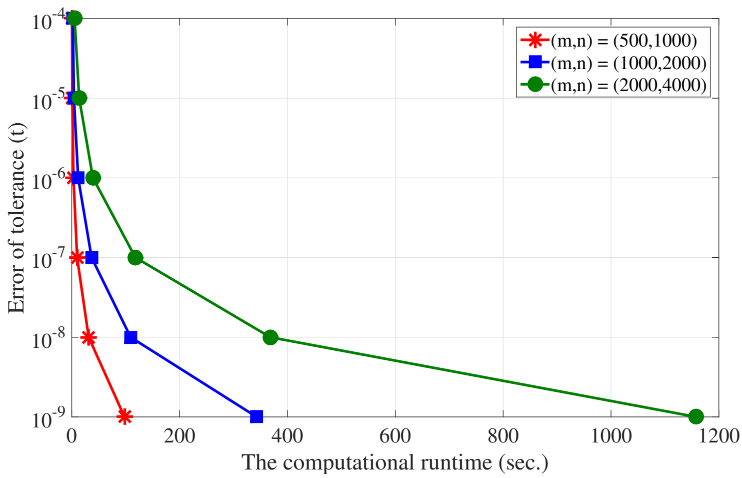

FIGURE 1. Behaviors of Algorithm 1 for three different sizes of (m, n) and various error tolerances t.

problem under some suitable conditions on control sequence $(\delta_k)_{k\geq 1}$ and the sequence of relaxation parameters $(\lambda_k)_{k\geq 1}$. We showed in the numerical examples that the proposed method with a suitably selected weight function achieved a superiority than the existing method for both number of the iterations and the computational running times.

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APPENDIX 1 PARAMETER COMBINATIONS OF ALGORITHM 1

We start by investigating several parameter combinations, chosen as in Algorithm 1 by setting $(m, n) = (500, 1000)$ and utilizing the stopping criterion that either the number of iterations exceeds 10000 or $\frac{\|x_k - x_{k-1}\|}{\|x_k\|+1} \leq 10^{-5}$. We present the average number of iterations and the average computational running time for different choices of parameters λ_k and δ_k in Table 2.

TABLE 2. The average number of iterations and average computational running time of Algorithm 1 for several choices of parameters $\lambda_k \in (1, 2)$ and $\delta_k = 1 - \frac{\delta}{k+1}$ where $\delta \in (0, 1)$.

δ	0.1	0.5	1.0	1.5	1.6	1.7	1.8	1.9
$\lambda_k = 0.1$	2704	1192	569	629	643	657	673	688
	(22.99)	(10.36)	(5.32)	(5.37)	(5.47)	(5.56)	(5.62)	(5.82)
$\lambda_k = 0.2$	2703	1173	417	429	440	453	464	476
	(22.17)	(9.85)	(3.62)	(3.65)	(3.68)	(3.81)	(3.85)	(4.05)
$\lambda_k = 0.3$	2703	1167	357	340	348	358	367	376
	(22.11)	(9.69)	(3.08)	(2.87)	(2.94)	(3.10)	(3.03)	(3.10)
$\lambda_k = 0.4$	2703	1164	325	288	294	301	308	317
	(22.06)	(9.58)	(2.81)	(92.43)	(2.44)	(2.51)	(2.53)	(2.63)
$\lambda_k = 0.5$	2703	1162	307	251	257	263	269	275
	(22.03)	(9.51)	(2.77)	(2.11)	(2.17)	(2.16)	(2.25)	(2.29)
$\lambda_k = 0.6$	2703	1161	296	225	229	235	240	246
	(22.04)	(9.50)	(2.50)	(1.90)	(1.94)	(1.92)	(1.98)	(2.06)
$\lambda_k = 0.7$	2703	1160	289	204	208	212	218	222
	(21.90)	(9.46)	(2.42)	(1.75)	(1.77)	(1.74)	(1.79)	(1.86)
$\lambda_k = 0.8$	2703	1159	283	188	191	194	199	204
	(22.24)	(9.47)	(2.37)	(1.60)	(1.61)	(1.60)	(1.64)	(1.70)
$\lambda_k = 0.9$	2704	1159	280	175	177	180	184	189
	(21.91)	(9.48)	(2.37)	(1.49)	(1.50)	(1.49)	(1.52)	(1.56)
$\lambda_k = 1.0$	2704	1158	277	165	166	169	172	176
	(21.92)	(9.47)	(2.31)	(1.39)	(1.40)	(1.39)	(1.42)	(1.47)
$\lambda_k = 1.1$	2273	1158	275	156	157	160	163	166
	(18.66)	(9.41)	(2.28)	(1.31)	(1.32)	(1.32)	(1.33)	(1.37)
$\lambda_k = 1.2$	>10000	1158	273	149	150	152	155	158
	-	(9.43)	(2.26)	(1.25)	(1.26)	(1.25)	(1.26)	(1.31)
$\lambda_k = 1.3$	>10000	1158	271	143	144	146	148	152
	-	(9.37)	(2.25)	(1.20)	(1.20)	(1.19)	(1.21)	(1.24)
$\lambda_k = 1.4$	>10000	638	193	137	138	140	143	146
	-	(5.16)	(1.59)	(1.14)	(1.16)	(1.14)	(1.17)	(1.20)
$\lambda_k = 1.5$	>10000	737	205	134	135	137	139	142
	-	(5.99)	(1.68)	(1.13)	(1.14)	(1.12)	(1.13)	(1.16)
$\lambda_k = 1.6$	>10000	1543	217	131	132	134	137	139
	-	(14.69)	(1.77)	(1.09)	(1.15)	(1.10)	(1.12)	(1.13)
$\lambda_k = 1.7$	>10000	735	209	129	130	132	134	137
	-	(6.00)	(1.74)	(1.07)	(1.08)	(1.09)	(1.10)	(1.12)
$\lambda_k = 1.8$	>10000	3523	230	130	**128**	130	133	135
	-	(37.04)	(1.88)	(1.09)	**(1.07)**	(1.07)	(1.08)	(1.11)
$\lambda_k = 1.9$	>10000	4531	1221	131	129	129	132	135
	-	(47.11)	(12.36)	(1.12)	(1.07)	(1.05)	(1.07)	(1.11)

From Table 2, we observe that the combination of $\delta_k = 1 - \frac{1.6}{k+1}$ with the relaxation parameter $\lambda_k = 1.8$ led to the fewest number of iterations (128 iterations) and the shortest computational running time (1.07 seconds).

Appendix 2 Parameter combinations of the modified Krasnosel′skiĭ-Mann method (1.3)

In this section, we present some parameter combinations of the modified Krasnosel′skiĭ-Mann method (1.3). All experimental settings are the same as mentioned above.

Table 3. The average number of iterations and average computational running time of the modified Krasnosel′skiĭ-Mann method (1.3) for several choices of parameters $\lambda_k \in (1,2)$ and $\delta_k = 1 - \frac{\delta}{k+1}$ where $\delta \in (0,1)$

δ	0.1	0.5	1.0	1.5	1.6	1.7	1.8	1.9
$\lambda_k = 0.1$	3009	2019	1950	1951	1949	1956	1965	1970
	(26.31)	(18.06)	(17.87)	(17.97)	(17.75)	(18.02)	(18.13)	(18.57)
$\lambda_k = 0.2$	2870	1951	1858	1885	1903	1908	1911	1927
	(24.29)	(16.65)	(16.24)	(16.49)	(17.11)	(17.22)	(16.78)	(17.12)
$\lambda_k = 0.3$	2824	1853	1805	1866	1875	1879	1874	1871
	(23.48)	(15.59)	(15.60)	(16.24)	(16.25)	(16.68)	(16.10)	(16.49)
$\lambda_k = 0.4$	2797	1796	1753	1820	1833	1837	1842	1857
	(23.00)	(15.09)	(15.15)	(15.32)	(15.53)	(15.38)	(15.76)	(15.88)
$\lambda_k = 0.5$	2778	1742	1698	1806	1819	1827	1835	1832
	(22.99)	(14.55)	(14.62)	(15.48)	(15.48)	(15.49)	(15.65)	(15.91)
$\lambda_k = 0.6$	2764	1699	1682	1765	1781	1794	1811	1823
	(22.70)	(14.11)	(14.19)	(14.82)	(14.99)	(14.86)	(15.35)	(15.90)
$\lambda_k = 0.7$	2753	1663	1685	1731	1750	1761	1776	1786
	(22.50)	(14.24)	(14.11)	(14.45)	(14.73)	(14.83)	(14.94)	(15.06)
$\lambda_k = 0.8$	2745	1638	1669	1716	1726	1735	1749	1763
	(22.39)	(13.51)	(14.06)	(14.46)	(14.68)	(14.34)	(15.03)	(14.85)
$\lambda_k = 0.9$	2738	1617	1643	1714	1713	1723	1731	1739
	(22.41)	(13.54)	(13.78)	(14.34)	(14.35)	(14.23)	(14.79)	(14.59)
$\lambda_k = 1.0$	2734	1596	1618	1713	1718	1721	1721	1728
	(22.19)	(13.12)	(13.56)	(14.31)	(14.36)	(14.10)	(14.40)	(14.43)
$\lambda_k = 1.1$	2731	1579	1598	1707	1711	1721	1723	1724
	(22.22)	(12.99)	(13.50)	(14.18)	(14.21)	(14.17)	(14.33)	(14.68)
$\lambda_k = 1.2$	2728	1562	1580	1684	1702	1714	1719	1728
	(22.11)	(12.82)	(13.20)	(14.18)	(14.16)	(14.32)	(14.37)	(14.62)
$\lambda_k = 1.3$	2727	1544	1562	1661	1681	1698	1711	1717
	(22.19)	(12.82)	(13.03)	(13.76)	(13.93)	(13.86)	(14.49)	(14.22)
$\lambda_k = 1.4$	4464	1577	1542	1640	1658	1676	1693	1708
	(36.00)	(12.74)	(12.72)	(13.90)	(13.82)	(13.62)	(14.13)	(14.19)
$\lambda_k = 1.5$	6373	1969	1522	1620	1638	1654	1673	1688
	(51.82)	(15.86)	(12.93)	(13.37)	(13.81)	(13.47)	(14.03)	(14.01)
$\lambda_k = 1.6$	7321	2618	**1507**	1604	1619	1635	1651	1669
	(58.70)	(21.20)	**(12.25)**	(13.54)	(13.34)	(13.33)	(13.71)	(13.82)
$\lambda_k = 1.7$	7878	3696	1491	1588	1603	1617	1633	1648
	(63.70)	(29.78)	(12.17)	(13.07)	(13.59)	(13.26)	(13.52)	(13.62)
$\lambda_k = 1.8$	8243	5316	1475	1570	1587	1601	1615	1630
	(66.90)	(42.92)	(12.26)	(12.88)	(13.13)	(13.33)	(13.34)	(13.53)
$\lambda_k = 1.9$	8501	7021	1460	1555	1570	1587	1599	1613
	(68.69)	(57.09)	(11.95)	(12.57)	(12.74)	(12.89)	(13.26)	(13.58)

Table 3 displays the average number of iterations and average computational running time for various choices of parameters λ_k and δ_k. The combination of $\delta_k = 1 - \frac{1.0}{k+1}$ and $\lambda_k = 1.6$ achieved the best number of iterations and the shortest computational running time of 12.25 seconds.

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