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Asymptotic properties of even-order functional differential equations with deviating argument

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ABSTRACT. In this paper, new effective technique for investigation of the higher order differential equation

$$(E) \qquad y^{(n)}(t) = p(t)y(\tau(t)).$$

is established. We offer new criteria for excluding certain types of nonoscillatory solutions which improve all existing results from the literature. Examples are provided to illustrate the importance of the main results.

1. INTRODUCTION

In this article, we consider the linear differential equation with deviating argument of the form

$$(E) \qquad y^{(n)}(t) = p(t)y(\tau(t)),$$

where n is even and the following conditions hold

(H_1) $p(t) \in C([t_0, \infty))$, $p(t) > 0$,
(H_2) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a proper solution of Eq. (E) we mean a function $y : [T_y, \infty) \to R$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (E) does possess proper solutions.

The oscillatory nature of the solutions is understood in the usual way, that is, a proper solution is termed oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros.

In recent years, there has been increasing interest in studying oscillation of solutions to different classes of differential equations due to the fact that they have numerous applications in natural sciences and engineering, see, for instance, the papers [16] – [17] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. The problem of establishing oscillation criteria for differential equations with deviating arguments has been a very active research area over the past decades (see [1]–[14] and [19]–[15]) and several references and reviews of known results can be found in the monographs by Agarwal et al. [1], Došly and Řehák [5] and Ladde et al. [15].

By the well-known result of Kiguradze [7] (Lemma 1), one can easily classify the possible nonoscillatory solutions of (E). As a matter of fact, the set $\mathcal{N}$ of all nonoscillatory solutions of (E) has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n,$$

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where $y(t) \in \mathcal{N}_\ell$ means that there exists $t_0 \ge T_y$ such that

$$\begin{aligned} y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0,\infty) \ \text{ for } \ 0 \le i \le \ell, \\ (-1)^i y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0,\infty) \ \text{ for } \ \ell \le i \le n. \end{aligned} \tag{1.1}$$

Such a $y(t)$ is said to be a solution of degree ℓ.

Following the classical results of Kiguradze [7], we say that equation (E) enjoys property (B) if $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$. This definition formulates the fact that (E) with $\tau(t) \equiv t$ always possesses solutions of degrees 0 and n, that is $\mathcal{N}_0 \neq \emptyset$ and $\mathcal{N}_n \neq \emptyset$. We recall excellent criteria of Koplatadze et al. [11]:

Theorem A. *If $\tau(t) \le t$ and*

$$\begin{aligned} \limsup_{t\to\infty} \Big\{ \tau(t) \int_t^\infty s^{n-3}\tau(s)p(s)\,\mathrm{d}s + \int_{\tau(t)}^t s(\tau(s))^{n-2}p(s)\,\mathrm{d}s \\ + \frac{1}{\tau(t)} \int_0^{\tau(t)} s^2(\tau(s))^{n-2}p(s)\,\mathrm{d}s \Big\} > 2(n-2)!, \end{aligned} \tag{1.2}$$

then(E) has property (B), i.e., $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$.

Theorem B. *If $\tau(t) \ge t$ and*

$$\begin{aligned} \limsup_{t\to\infty} \Big\{ \tau(t) \int_t^\infty s^{n-3}\tau(s)p(s)\,\mathrm{d}s + \int_t^{\tau(t)} s^{n-2}\tau(s)p(s)\,\mathrm{d}s \\ + \frac{1}{\tau(t)} \int_0^{t} s^{n-2}(\tau(s))^{2}p(s)\,\mathrm{d}s \Big\} > 2(n-2)!, \end{aligned} \tag{1.3}$$

then(E) has property (B), i.e., $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$.

The situation for (E) with $\tau(t) \not\equiv t$ is different. In fact, it may happen that $\mathcal{N}_0 = \emptyset$ or $\mathcal{N}_n = \emptyset$ when the deviation $|t-\tau(t)|$ is sufficiently large. This remarkable fact was first observed by Ladas et al. [15]. Later Koplatadze and Chanturia [12] contributed to the subject and formulated the following results:

Theorem C. *If $\tau(t) \le t$ and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^t (s-\tau(t))^{n-1}p(s)\,\mathrm{d}s > (n-1)!, \tag{1.4}$$

then(E) does not allow solutions of degree 0, i.e. $\mathcal{N}_0 = \emptyset$.

Theorem D. *If $\tau(t) \ge t$ and*

$$\limsup_{t\to\infty} \int_t^{\tau(t)} (\tau(t)-s)^{n-1}p(s)\,\mathrm{d}s > (n-1)!, \tag{1.5}$$

then(E) does not allow solutions of degree n, i.e. $\mathcal{N}_n = \emptyset$.

So combining Theorem A together with Theorem C we see that $\mathcal{N} = \mathcal{N}_n$ for (E), while joining Theorem B together with Theorem D we obtain that $\mathcal{N} = \mathcal{N}_0$.

Our aim in this work is to significantly improve the above mentioned results and the progress will be demonstrated via set of illustrative examples in which we shall compare our results with these presented in Theorems C and D.

2. Main results

Condition (H_2) implies that there exists the inverse function $\tau^{-1}(t)$ and we can introduce the auxiliary function $\xi(t) \in C^1([t_0,\infty))$ in this way

$$\xi(\xi(t)) = \tau^{-1}(t). \tag{2.6}$$

We present some basic properties of $\xi(t)$.

Lemma 2.1. *Let $\tau(t) < t$. Then*

$$\xi(t) > t, \quad \xi(\xi(\tau(t))) = t, \quad \xi(\tau(t)) = \tau(\xi(t)) = \xi^{-1}(t).$$

We are going to establish sufficient conditions for $\mathcal{N}_0 = \emptyset$ of (E). For our next consideration we will use the notation

$$\begin{aligned} P_1(t) &= \int_t^{\xi(t)} p(s)\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s,\\ P_2(t) &= \int_{\xi(t)}^{\tau^{-1}(t)} p(s)\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s,\\ P_3(t) &= \int_{\tau^{-1}(t)}^{\tau^{-1}(\xi(t))} p(s)\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

Theorem 2.1. *Assume that $\tau(t) < t$ and there exists a function $\xi(t) \in C^1([t_0,\infty))$ satisfying* (2.6)*. If*

$$\begin{aligned} \limsup_{t\to\infty} \Bigg[&\frac{P_1(t)P_1(\xi^{-1}(t)) + P_1(t)P_3(\xi^{-1}(t))}{(1-P_2(t))(1-P_2(\xi^{-1}(t)))}\\ &+ \frac{P_3(t)P_1(\xi(t))}{(1-P_2(t))(1-P_2(\xi(t)))}\Bigg] > 1, \end{aligned} \tag{2.8}$$

then $\mathcal{N}_0 = \emptyset$.

Proof. Assume on the contrary that $y(t)$ is an eventually positive solution of (E) such that $y(t) \in \mathcal{N}_0$. An integration of (E) from t to ∞ yields

$$-y^{(n-1)}(t) \ge \int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s.$$

Integrating again from t to ∞ and changing the order of integration, we have

$$y^{(n-2)}(t) \ge \int_t^\infty \int_u^\infty p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}u = \int_t^\infty p(s)y(\tau(s))(s-t)\,\mathrm{d}s.$$

Repeating this procedure, we are led to

$$y(t) \ge \int_t^\infty p(s)y(\tau(s))\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s. \tag{2.9}$$

Employing function $\xi(t)$ to the above inequality gives

$$\begin{aligned} y(t) \ge &\int_t^{\xi(t)} p(s)y(\tau(s))\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s + \int_{\xi(t)}^{\tau^{-1}(t)} p(s)y(\tau(s))\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s\\ &+ \int_{\tau^{-1}(t)}^{\tau^{-1}(\xi(t))} p(s)y(\tau(s))\frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

Since $y(t)$ is decreasing function, we are led to

$$y(t) \ge y(\xi^{-1}(t))P_1(t) + y(t)P_2(t) + y(\xi(t))P_3(t),$$

which is

$$y(t) \geq \frac{1}{1-P_2(t)} \left[y(\xi^{-1}(t))P_1(t) + y(\xi(t))P_3(t) \right]. \tag{2.11}$$

Setting $t = \xi^{-1}(t)$ and $t = \xi(t)$, we obtain

$$y(\xi^{-1}(t)) \geq \frac{1}{1-P_2(\xi^{-1}(t))} \left[y(\tau(t))P_1(\xi^{-1}(t)) + y(t)P_3(\xi^{-1}(t)) \right] \tag{2.12}$$

and

$$y(\xi(t)) \geq \frac{1}{1-P_2(\xi(t))} \left[y(t)P_1(\xi(t)) + y(\tau^{-1}(t))P_3(\xi(t)) \right] \geq \frac{P_1(\xi(t))}{1-P_2(\xi(t))} y(t), \tag{2.13}$$

respectively. Combining (2.12), (2.13) and (2.11), one gets

$$\begin{aligned} y(t) \geq \frac{1}{1-P_2(t)} \Bigg[&\frac{P_1(t)}{1-P_2(\xi^{-1}(t))} \left[y(\tau(t))P_1(\xi^{-1}(t)) + y(t)P_3(\xi^{-1}(t)) \right] \\ &+ \frac{P_3(t)}{1-P_2(\xi(t))} y(t)P_1(\xi(t)) \Bigg] \end{aligned}$$

which in view of $y(\tau(t)) \geq y(t)$, leads to

$$\begin{aligned} y(t) \geq y(t) \Bigg[&\frac{\left[P_1(t)P_1(\xi^{-1}(t)) + P_1(t)P_3(\xi^{-1}(t)) \right]}{(1-P_2(t))(1-P_2(\xi^{-1}(t)))} \\ &+ \frac{P_3(t)P_1(\xi(t))}{(1-P_2(t))(1-P_2(\xi(t)))} \Bigg]. \end{aligned}$$

This contradicts (2.8) and we conclude that $\mathcal{N}_0 = \emptyset$. □

The following criteria immediately result from the proof of Theorem 2.1.

Corollary 2.1. *If $P_2(t) > 1$, then $\mathcal{N}_0 = \emptyset$.*

Corollary 2.2. *Let $P_i(t) \geq P_i^*$, for $i = 1, 2, 3$, where P_i^* are positive constants. If*

$$\frac{1}{(1-P_2^*)^2} \left[(P_1^*)^2 + 2P_1^* P_3^* \right] > 1, \tag{2.14}$$

then $\mathcal{N}_0 = \emptyset$.

Remark 2.1. The assertion of Theorem 2.1 can be reformulated as every bounded solution is oscillatory.

Now we provide couple of illustrative examples to show the progress that Theorem 2.1 yields in regard of Theorem C.

Example 2.1. Consider even-order differential equation with delay argument of the form

$$y^{(n)}(t) = p_0^n y(t - n\tau_0), \tag{$Ex1$}$$

where $p_0 > 0$, $\tau_0 > 0$.

Since $\tau(t) = t - n\tau_0$, we have $\tau^{-1}(t) = t + n\tau_0$ and auxiliary function $\xi(t) = t + \frac{n}{2}\tau_0$. It is easy to verify that

$$P_1(t) = \int_t^{t+\frac{n}{2}\tau_0} p_0^n \frac{(s-t)^{n-1}}{(n-1)!} \, \mathrm{d}s = \frac{p_0^n n^n \tau_0^n}{2^n n!} = P_1^*,$$

$$P_2(t) = \int_{t+\frac{n}{2}\tau_0}^{t+n\tau_0} p_0^n \frac{(s-t)^{n-1}}{(n-1)!} \, \mathrm{d}s = \frac{p_0^n n^n \tau_0^n}{n!} \left(1 - \left(\frac{1}{2} \right)^n \right) = P_2^*,$$

$$P_3(t) = \int_{t+n\tau_0}^{t+\frac{3n}{2}\tau_0} p_0^n \frac{(s-t)^{n-1}}{(n-1)!}\,\mathrm{d}s = \frac{p_0^n n^n \tau_0^n}{n!}\left(\left(\frac{3}{2}\right)^n - 1\right) = P_3^*.$$

Consequently condition (2.14) takes the form

$$\left(\frac{p_0^n n^n \tau_0^n}{2^n n!}\right)^2 + 2\frac{p_0^n n^n \tau_0^n}{2^n n!}\frac{p_0^n n^n \tau_0^n}{n!}\left(\left(\frac{3}{2}\right)^n - 1\right) > \left[1 - \frac{p_0^n n^n \tau_0^n}{n!}\left(1 - \left(\frac{1}{2}\right)^n\right)\right]^2,$$

which implies that

$$p_0^n \tau_0^n > \frac{2^n \cdot n!}{(2\cdot 3^n - 2^{2n})n^n}\left(\sqrt{2\cdot 3^n - 2^{n+1} + 1} - 2^n + 1\right) \tag{2.15}$$

and by Corollary 2.2 condition (2.15) guarantees that $\mathcal{N}_0 = \emptyset$. On the other hand, condition (1.2) holds true, therefore we conclude that every nonoscillatory solution is of degree n, i.e., $\mathcal{N} = \mathcal{N}_n$.

To see the progress which our criteria bring, let us consider $n = 4$ ($n = 6$). The condition (2.15) is fulfilled for

$$p_0\tau_0 > 0.488 \quad (p_0\tau_0 > 0.4636)$$

while (1.4) requires $p_0\tau_0 > 0.5533$ ($p_0\tau_0 > 0.4990$).

Example 2.2. We consider the even-order Euler delayed differential equation

$$y^{(n)}(t) = \frac{p_0}{t^n} y(\lambda t), \quad p_0 > 0, \quad \lambda \in (0,1). \tag{E_{x2}}$$

First we observe that (1.2) reduces to

$$p_0\left(\lambda^2 - \lambda^{n-2}\ln\lambda + \lambda^{n-3}\right) > 2(n-2)!. \tag{2.16}$$

Since $\tau(t) = \lambda t$, then $\tau^{-1}(t) = \frac{t}{\lambda}$ and $\xi(t) = \frac{t}{\sqrt{\lambda}}$. It is easy to verify that

$$P_1(t) = \frac{p_0}{(n-1)!}\left[\ln\frac{1}{\sqrt{\lambda}} - \sum_{i=1}^{n-1}\frac{(1-\sqrt{\lambda})^{n-i}}{n-i}\right] = P_1^*,$$

$$P_2(t) = \frac{p_0}{(n-1)!}\left[\ln\frac{1}{\sqrt{\lambda}} - \sum_{i=1}^{n-1}\frac{(1-\lambda)^{n-i} - (1-\sqrt{\lambda})^{n-i}}{n-i}\right] = P_2^*,$$

$$P_3(t) = \frac{p_0}{(n-1)!}\left[\ln\frac{1}{\sqrt{\lambda}} - \sum_{i=1}^{n-1}\frac{(1-\lambda\sqrt{\lambda})^{n-i} - (1-\lambda)^{n-i}}{n-i}\right] = P_3^*.$$

So condition (2.14) with calculated constants P_1^*, P_2^*, P_3^* together with (2.16) implies that every nonoscillatory solution of (E_{x2}) is of degree n. To see quality of our criterion, we consider (E_{x2}) with $\lambda = 0.5$ and $n = 4$ ($n = 6$). The condition (2.14), which guaranties that $\mathcal{N}_0 = \emptyset$ is fulfilled for

$$p_0 > 143 \quad (p_0 > 17200)$$

while (1.4) requires $p_0 > 227$ ($p_0 > 26000$).

Now we offer sufficient conditions for $\mathcal{N}_n = \emptyset$. Condition ($H_2$) implies that there exists the inverse function $\tau^{-1}(t)$ and we can introduce the auxiliary function $\chi(t) \in C^1([t_0,\infty))$ in this way

$$\chi(\chi(t)) = \tau^{-1}(t). \tag{2.17}$$

We present some basic properties of $\chi(t)$.

Lemma 2.2. *Let $\tau(t) > t$. Then*

$$\chi(t) < t, \quad \chi(\chi(\tau(t))) = t, \quad \chi(\tau(t)) = \tau(\chi(t)) = \chi^{-1}(t).$$

We shall use the notation

$$\begin{aligned} Q_1(t) &= \int_{\tau^{-1}(\chi(t))}^{\tau^{-1}(t)} p(s)\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s, \\ Q_2(t) &= \int_{\tau^{-1}(t)}^{\chi(t)} p(s)\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s, \\ Q_3(t) &= \int_{\chi(t)}^{t} p(s)\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s. \end{aligned} \tag{2.18}$$

Theorem 2.2. *Assume that $\tau(t) \geq t$ and there exists a function $\chi(t) \in C^1([t_0,\infty))$ satisfying* (2.17). *If*

$$\begin{aligned} \limsup_{t\to\infty} &\left[\frac{Q_3(t)Q_1(\chi^{-1}(t)) + Q_3(t)Q_3(\chi^{-1}(t))}{(1-Q_2(t))(1-Q_2(\chi^{-1}(t)))}\right. \\ &\left. + \frac{Q_1(t)Q_3(\chi(t))}{(1-Q_2(t))(1-Q_2(\chi(t)))}\right] > 1, \end{aligned} \tag{2.19}$$

then $\mathcal{N}_n = \emptyset$.

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t) \in \mathcal{N}_n$. An integration of (E) from t_1 to t yields

$$y^{(n-1)}(t) \geq \int_{t_1}^{t} p(s)y(\tau(s))\,\mathrm{d}s.$$

Integrating the above inequality from t_1 to t and swapping order of integration, we obtain

$$y^{(n-2)}(t) \geq \int_{t_1}^{t} p(s)y(\tau(s))(t-s)\,\mathrm{d}s.$$

Repeating this approach $n-3$ times, we finally obtain

$$y(t) \geq \int_{t_1}^{t} p(s)y(\tau(s))\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

Employing auxiliary function $\chi(t)$, we get

$$\begin{aligned} y(t) \geq &\int_{\tau^{-1}(\chi(t))}^{\tau^{-1}(t)} p(s)y(\tau(s))\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s + \int_{\tau^{-1}(t)}^{\chi(t)} p(s)y(\tau(s))\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s \\ &+ \int_{\chi(t)}^{t} p(s)y(\tau(s))\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s. \end{aligned} \tag{2.20}$$

Taking into account that $y(t)$ is an increasing function, we have

$$y(t) \geq Q_1(t)y(\chi(t)) + Q_2(t)y(t) + Q_3(t)y(\chi^{-1}(t)). \tag{2.21}$$

Substituting in the last inequality $t = \chi(t)$ and thereafter $t = \chi^{-1}(t)$ and assuming that $1 - Q_2(t)$ is eventually positive, we obtain

$$\begin{aligned} y(\chi(t)) &\geq \frac{1}{1-Q_2(\chi(t))}\left[Q_1(\chi(t))y(\tau^{-1}(t)) + Q_3(\chi(t))y(t)\right] \\ &\geq \frac{Q_3(\chi(t))}{1-Q_2(\chi(t))}y(t) \end{aligned} \tag{2.22}$$

and

$$y(\chi^{-1}(t)) \geq \frac{1}{1-Q_2(\chi^{-1}(t))}\left[Q_1(\chi^{-1}(t))y(t)+Q_3(\chi^{-1}(t))y(\tau(t))\right], \tag{2.23}$$

respectively. Setting (2.22) and (2.23) into (2.21) and using that $y(t)$ is an increasing function, we get

$$\begin{aligned} y(t) \geq \frac{1}{1-Q_2(t)}&\left[\frac{Q_1(t)Q_3(\chi(t))}{1-Q_2(\chi(t))}y(t)\right.\\ &\left.+\frac{Q_3(t)}{1-Q_2(\chi^{-1}(t))}\left[y(t)Q_1(\chi^{-1}(t))+y(t)Q_3(\chi^{-1}(t))\right]\right], \end{aligned}$$

which contradicts to (2.19) and we conclude that $\mathcal{N}_n=\emptyset$. □

The following criteria ensue immediately from the proof of Theorem 2.2.

Corollary 2.3. *If $Q_2(t)>1$, then $\mathcal{N}_n=\emptyset$.*

Corollary 2.4. *Let $Q_i(t)>Q_i^*$, for $i=1,2,3$, where Q_i^* are positive constants. Assume that*

$$\frac{1}{(1-Q_2^*)^2}\left[(Q_3^*)^2+2Q_1^*Q_3^*\right]>1. \tag{2.24}$$

Then $\mathcal{N}_n=\emptyset$.

Theorem 2.3. *Assume that $\tau(t)\geq t$ and there exists a function $\chi(t)\in C^1([t_0,\infty))$ satisfying* (2.17) *. If* (2.19) *and* (1.3) *hold, then every unbounded solution is oscillatory.*

Example 2.3. Consider even-order differential equation with advanced argument of the form

$$y^{(n)}(t)=p_0^n y(t+n\tau_0), \tag{E_{x3}}$$

where $p_0>0$, $\tau_0>0$.

For equation (E_{x3}) we have $\tau(t)=t+n\tau_0$, and thus $\tau^{-1}(t)=t-n\tau_0$, $\chi(t)=t-\frac{n}{2}\tau_0$. $\chi^{-1}(t)=t+\frac{n}{2}\tau_0$. Some computation yields

$$Q_1(t)=\int_{t-\frac{3}{2}n\tau_0}^{t-n\tau_0} p_0^n\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s=\frac{p_0^n n^n\tau_0^n}{n!}\left(\left(\frac{3}{2}\right)^n-1\right)=Q_1^*,$$

$$Q_2(t)=\int_{t-n\tau_0}^{t-\frac{n}{2}\tau_0} p_0^n\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s=\frac{p_0^n n^n\tau_0^n}{n!}\left(1-\left(\frac{1}{2}\right)^n\right)=Q_2^*,$$

$$Q_3(t)=\int_{t-\frac{n}{2}\tau_0}^{t} p_0^n\frac{(t-s)^{n-1}}{(n-1)!}\,\mathrm{d}s=\frac{p_0^n n^n\tau_0^n}{n!2^n}=Q_3^*.$$

Settings Q_1^*,Q_2^*,Q_3^* into condition (2.24), we get

$$\left(\frac{p_0^n n^n\tau_0^n}{2^n n!}\right)^2+2\frac{p_0^n n^n\tau_0^n}{2^n n!}\frac{p_0^n n^n\tau_0^n}{n!}\left(\left(\frac{3}{2}\right)^n-1\right)>\left[1-\frac{p_0^n n^n\tau_0^n}{n!}\left(1-\left(\frac{1}{2}\right)^n\right)\right]^2,$$

which implies that

$$p_0^n\tau_0^n>\frac{2^n\cdot n!}{(2\cdot 3^n-2^{2n})n^n}\left(\sqrt{2\cdot 3^n-2^{n+1}+1}-2^n+1\right). \tag{2.25}$$

By Corollary 2.4 condition (2.25) guarantees that $\mathcal{N}_n=\emptyset$ and moreover it is easy see that (1.3) holds true and we conclude that every nonoscillatory solution of (E_{x3}) is of degree n, i.e., $\mathcal{N} = \mathcal{N}_n$. To see the improvement which our criteria brings, let us consider $n = 4$ $(n = 6)$. The condition (2.25) is fulfilled for

$$p_0\tau_0 > 0.4880 \quad (p_0\tau_0 > 0.4636)$$

while (1.5) requires $p_0\tau_0 > 0.5533$ $(p_0\tau_0 > 0.4990)$.

Example 2.4. We consider the even-order advanced Euler differential equation

$$(E_{x4}) \qquad y^{(n)}(t) = \frac{p_0}{t^n} y(\lambda t), \quad p_0 > 0, \quad \lambda > 1.$$

It is easy to see that (1.3) reduces to

$$p_0 \lambda \left(2 + \ln \lambda\right) > 2(n-2)!. \tag{2.26}$$

Now $\tau(t) = \lambda t$, so that $\tau^{-1}(t) = \frac{t}{\lambda}$, $\chi(t) = \frac{t}{\sqrt{\lambda}}$ and $\chi^{-1}(t) = \sqrt{\lambda}\, t$. Consequently,

$$Q_1(t) = \frac{p_0}{(n-1)!}\left[\ln \frac{1}{\sqrt{\lambda}} + \sum_{i=1}^{n-1} \frac{(-1)^i(\lambda-1)^{n-i} - (\lambda\sqrt{\lambda}-1)^{n-i}}{n-i}\right] = Q_1^*,$$

$$Q_2(t) = \frac{p_0}{(n-1)!}\left[\ln \frac{1}{\sqrt{\lambda}} + \sum_{i=1}^{n-1} \frac{(-1)^i(\sqrt{\lambda}-1)^{n-i} - (\lambda-1)^{n-i}}{n-i}\right] = Q_2^*,$$

$$Q_3(t) = \frac{p_0}{(n-1)!}\left[\ln \frac{1}{\sqrt{\lambda}} + \sum_{i=1}^{n-1} \frac{(-1)^{i+1}(\sqrt{\lambda}-1)^{n-i}}{n-i}\right] = Q_3^*.$$

By Theorem 3 condition (2.24) together with (2.26) implies that every nonoscillatory solution of (E_{x4}) is of degree 0 (every unbounded solution is oscillatory). To show the improvement of our criterion, we consider (E_{x4}) with $\lambda = 1.5$ and $n = 4$ $(n = 6)$. The condition (2.24), which guaranties $\mathcal{N}_n = \emptyset$ is fulfilled for

$$p_0 > 317 \quad (p_0 > 41750)$$

while (1.5) requires $p_0 > 536$ $(p_0 > 65700)$.

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