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# Weak convergence of inertial proximal point algorithm for a family of nonexpansive mappings in Hilbert spaces

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ABSTRACT. In this paper, we modified proximal point algorithm with some convex combination technique to approximate a minimizer, equilibrium point and a common fixed point of a family of nonexpansive mappings in Hilbert spaces. We establish a weak convergence theorem under some mild conditions. Moreover, we also provide a numerical example to illustrate the convergence behavior of the proposed iterative method.

## 1. INTRODUCTION

Let $H$ be a real Hilbert space and let $g : H \to (-\infty, \infty]$ be a proper and convex function. One of the major problems for optimization is to find a point $x \in H$ such that

$$g(x) = \min_{y \in H} g(y).$$

We denote the set of all minimizers of $g$ on $H$ by $\operatorname{argmin}_{y\in H} g(y)$.

The proximal point algorithm is an important tool in solving optimization problem which was initiated by Martinet [23] in 1970. Later, Rockafellar [32] studied the convergence of a proximal point algorithm for finding a solution of the unconstrained convex minimization problem in $H$ as follows. Let $g$ be a proper, convex and lower semi-continuous function on $H$. The proximal point algorithm is defined by $x_1 \in H$ and

$$x_{n+1} = \operatorname*{argmin}_{u \in H} \left[ g(y) + \frac{1}{2\lambda_n} \|u - x_n\|^2 \right], \quad \forall n \geq 1, \tag{1.1}$$

where $\lambda_n > 0$ for all $n \geq 1$. It was shown that if $g$ has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$ then the sequence $\{x_n\}$ converges weakly to a minimizer of $g$; see also [7]. However, the proximal point algorithm does not necessarily converges strongly in general; see [4]. Recently, several authors proposed modifications of Rockafellar's proximal point algorithm to have strong convergence, for example [20, 21].

In recent years, many convergence results by the proximal point algorithm for solving optimization problems have been extended in many directions, see [37, 29]. The minimizers of the objective convex functionals in the spaces with nonlinearlity play an important role in the branch of analysis and geometry. Several applications in machine learning, computer vision, system balancing and robot manipulation can be considered as solving optimization problems, see [37, 29, 33].

Let $C$ be a nonempty closed convex subset of $H$. Let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $R$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$

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The set of solutions of (1.2) is denoted by $EP(F)$. Given mapping $S : C \to H$, let $F(x, y) = \langle Sx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $F(z, y) = \langle Sz, y - z\rangle$ for all $y \in C$, i.e., $z$ is a solution of the variational inequality. Many problems arising in physics, engineering, economics, game theory, optimization, operation research, etc. can be reduced to find a solution of the EP (1.2), and then, the EP (1.2) has gained attention from many researchers (see [9, 8, 10, 12, 34, 28]).

The fixed point problem is a very important tool for study engineering, physics, chemistry, and economixs in different mathematical models. Furthermore, the fixed point problem has manny important applications, such as null problem, variational inequality problem, optimization problem, (see [2, 3, 36, 17]), and the references therein. The fixed point problem is a problem of finding a point $x \in H$ such that $Tx = x$. The set of fixed points of the mapping $T$ is denoted by $F(T)$.

In convex optimization, numerous problems in applied sciences (image reconstruction, radiation therapy, artificial intelligence and sensor networks) can be modelled as the minimization problem, fixed point problem and equalibrium problem. Moreover, several authors have studied common solutions in different ways. In 2014, T.T.V. Ngunten et al.[27] have obtained convergence results for finding a common solution of an equilibrium problem and an infinite number of fixed-point problems. In 2015, Cholamjiak et al. [11] introduced a new iterative algorithm for finding a common solution of convex minimization problem and fixed point problem. In 2020, Hanjing and Suantai[15] proposed a MWA-algorithm for finding a common fixed point of a countable family of nonexpansive operators in a real Hilbert space.

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-\gamma_n)w_n + \gamma_n Tnw_n, \\ y_n = (1-\beta_n)T_n w_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)T_n z_n + \alpha_n T_n y_n \end{cases} \tag{1.3}$$

Inspired by these works mentioned above, we presented a new iterative method for finding the solutions of the minimization problem, the equilibrium problem, and common fixed point of an infinite family of nonexpansive mappings in a Hilbert space. Then a weak convergence theorem is established under some control conditions. The presented results in ithis work also generalize some well-known results in the literature. This paper is organized as follows: In Section 2, we recall some definitions and the useful facts which will be used in the later sections. A weak convergence theorem will be proved in Section 3. Moreover, we give a numerical example to illustrate our main result.

## 2. Preliminaries

Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of $f$ in a real Hilbert space $H$ as follows:

$$J_\lambda x = \underset{u \in H}{\operatorname{argmin}} \left[ f(u) + \frac{1}{2\lambda}\|u - x\|^2 \right]$$

for all $x \in H$. It was shown in [14] that the set of fixed points of the resolvent associated with $f$ coincides with the set of minimizers of $g$. Also, the resolvent $J_\lambda$ of $f$ is nonexpansive for all $\lambda > 0$; see [18].

**Lemma 2.1** ([18]). *Let $H$ be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in H$ and $\lambda > \mu > 0$, we have the following identity* *holds:*

$$J_\lambda x = J_\mu\left(\frac{\lambda-\mu}{\lambda}J_\lambda x + \frac{\mu}{\lambda}x\right).$$

**Lemma 2.2** ([1]). *Let $H$ be a real Hilbert space and $g : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{2\lambda}\|J_\lambda x - y\|^2 - \frac{1}{2\lambda}\|x-y\|^2 + \frac{1}{2\lambda}\|x - J_\lambda x\|^2 \leq f(y) - f(J_\lambda x). \tag{2.4}$$

**Lemma 2.3** ([6]). *Let $H$ be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in $H$ such that $x_n \rightharpoonup x$ with $x_n - Tx_n \to 0$, then $x = Tx$.*

**Lemma 2.4** ([5]). *Let $K$ be a nonempty closed convex subset of $H$ and $F$ be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions:*

- $(A1)$ *$F(x, x) = 0$ for all $x \in K$;*
- $(A2)$ *is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$;*
- $(A3)$ *for each $x, y \in K$,*

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

- $(A4)$ *for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.*

*Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } x, y \in K.$$

**Lemma 2.5** ([16]). *Let $K$ be a nonempty closed convex subset of $H$ and let $F$ be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to K$ as follows:*

$$T_r^F(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in K \right\} \tag{2.5}$$

*for all $x \in H$. Then the following hold:*

1. *$T_r^F$ is single-valued;*
2. *$T_r^F$ is firmly-nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y\rangle;$$

3. *$F(T_r^F) = EP(F)$ for all $r > 0$;*
4. *$EP(F)$ is closed and convex.*

**Lemma 2.6** ([16]). *Let $K$ be a nonempty closed convex subset of $H$. For $x \in H$, let the mapping $T_r^F$ be the same as in Lemma 2.5. Then for $r, s > 0$ and $x, y \in H$,*

$$\|T_r^F(x) - T_r^F(y)\| \leq \|y - x\| + \frac{|s-r|}{s}\|T_s^F(y) - y\|.$$

**Definition 2.1.** A mapping $T : H \to H$ is said to be

- (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H$;
- (ii) *quasi-nonexpansive* if $\mathcal{F}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|, \quad \forall x \in H$ and $y \in \mathcal{F}(T)$, where $\mathcal{F}(T) := \{x \in H \ : \ Tx = x\}$.

**Remark 2.1.** It follows from Definition 2.1 that if $\mathcal{F}(T) \neq \emptyset$, then (i) $\Rightarrow$ (ii) but converse is not true in general.

**Definition 2.2.** Let $T : C \to C$ be a mapping. The mapping $T - I$ is said to be *demiclosed at zero* if for any sequence $\{x_k\}$ in $C$ which $x_k \rightharpoonup u$ and $Tx_k - x_k \to 0$, then $x \in \mathcal{F}(T)$.

In order to deal with a common fixed point problem, Takahashi et al. [25, 26] introduced a useful condition called NST-condition (I) (NST*-condition) defined as follows:

**Definition 2.3.** ([25, 26]) Let $\{T_n\}$ and $\Gamma$ be families of mappings of $\mathcal{H}$ into itself such that $\emptyset \neq \mathcal{F}(\Gamma) \subset \Omega := \cap_{k=1}^{\infty} \mathcal{F}(T_k)$, where $\mathcal{F}(\Gamma)$ is the set of all common fixed points of $\Gamma$, A sequence $\{T_k\}$ is said to satisfy

(i) the *NST-condition (I) with* $\Omega$ if for every bounded sequence $\{x_k\}$ in $H$,

$$\lim_{k\to\infty} \|x_k - T_k x_k\| = 0 \implies \lim_{k\to\infty} \|x_k - T x_k\| = 0 \ \ \forall\, T \in \Gamma.$$

(ii) If $\Gamma$ is singleton, i.e., $\Gamma = \{T\}$, then $\{T_k\}$ is said to satisfy the *NST-condition (I) with* $T$.

(iii) the *NST*-condition* if whenever $\{x_k\}$ is a bounded sequence in $H$ such that

$$\lim_{k\to\infty} \|x_k - T_k x_k\| = \lim_{k\to\infty} \|x_{k+1} - x_k\| = 0,$$

it follows that $\omega_w(x_k) \subseteq \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$, where $\omega_w(x_k)$ is the set of all weak cluster point of $\{x_k\}$.

**Lemma 2.7** ([35]). *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \forall n \in \mathbb{N}.$$

*If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

**Lemma 2.8** (Opial [24]). *Let $\{x_k\}$ be a sequence in $H$ such that there exists a nonempty set $\Omega \subset \mathcal{H}$ satisfying:*

*(i) For every $p \in \Omega$, $\lim_{k\to\infty} \|x_k - p\|$ exists;*

*(ii) $\omega_w(x_k) \subset \Omega$.*

*Then, $\{x_k\}$ converges weakly to a point in $\Omega$.*

## 3. Main results

In this section, we prove the weak convergence theorems for minimizers of convex lower semi-continuous functions , equilibrium problem and common fixed point of a countable family of quasi-nonexpansive mapping in a real Hilbert space.

We first present an inertial alternating projection algorithm (Algorithm 3.6) by assuming the following:

- $g : H \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function;
- $f : H \times H \to \mathbb{R}$ a bi-function satisfying $(A1) - (A4)$
- $T : H \to H$ is a nonexpansive mapping
- $\{T_n : H \to H\}_{n=1}^{\infty}$ is a family of nonexpansive mappings and satisfies the NST-condition$(I)$ with $T$;
- $\emptyset \neq F(T) \subset \bigcap_{n=1}^{\infty} F(T_n)$;
- $\Omega :=$ argmin$_{u\in H} g(u) \bigcap EP(f) \bigcap \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$.

Let the sequence $\{x_n\}$ be defined by

$$(3.6) \qquad \begin{cases} s_n = x_n + \theta_n(x_n - x_{n-1}), \\ w_n = (1-\gamma_n)s_n + \gamma_n T_n s_n \\ u_n = \operatorname*{argmin}_{u\in H} \left[ g(u) + \dfrac{1}{2\lambda_n}\|u - w_n\|^2 \right], \\ y_n = T^f_{r_n} u_n \\ z_n = (1-\beta_n)y_n + \beta_n T_n y_n \\ x_{n+1} = (1-\alpha_n)T_n z_n + \alpha_n T_n y_n, \ \ \forall n \ge 1, \end{cases}$$

where $x_0, x_1 \in H$ and $\{\theta_n\}, \{\beta_n\}, \{\lambda_n\}, \{\gamma_n\}, \{\alpha_n\}$ are sequences in $[0,1]$.

**Lemma 3.9.** *Let $\{x_n\}$ be a sequence generated by Algorithm (3.6) and assume that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$. Then $\{x_n\}$ is a bounded sequence.*

*Proof.* Let $x^* \in \Omega$ and let $\{x_n\}$ be a sequence in $H$ generated by (3.6). So, we get

$$\|s_n - x^*\| \leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|,$$

and

$$\|w_n - x^*\| \leq (1-\gamma_n)\|s_n - x^*\| + \gamma_n \|T_n s_n - x^*\| \leq \|s_n - x^*\|$$

Since $g(x^*) \leq g(u)$ for all $u \in H$. This implies that

$$g(x^*) + \frac{1}{2\lambda_n}\|x^* - x^*\|^2 \leq g(u) + \frac{1}{\lambda_n}\|u - x^*\|^2, \quad \forall u \in H,$$

and hence $x^* = J_{\lambda_n} x^*$ for all $n \geq 1$. Since $u_n = J_{\lambda_n} w_n$, it implies by nonexpansiveness of $J_{\lambda_n}$ that

$$\|u_n - x^*\| = \|J_{\lambda_n} w_n - J_{\lambda_n} x^*\| \leq \|w_n - x^*\|. \tag{3.7}$$

By Algorithm (3.6), we obtain

$$\|z_n - x^*\| \leq (1-\gamma_n)\|y_n - x^*\| + \gamma_n \|T_n y_n - x^*\| \leq \|y_n - x^*\|$$

and

$$\|y_n - x^*\| = \left\|T_{r_n}^f u_n - T_{r_n}^f x^*\right\| \leq \|u_n - x^*\|.$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n \|y_n - x^*\| \\
&\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n \|u_n - x^*\| \\
&\leq (1-\alpha_n)\|s_n - x^*\| + \alpha_n \|s_n - x^*\| \\
&\leq (1-\alpha_n)\|s_n - x^*\| + \alpha_n(\|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|) \\
&= \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$, by using Lemma 2.7, we obtain that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Hence, $\{x_n\}$ is bounded which implies that $\{s_n\}$ is also bounded. ☐

**Lemma 3.10.** *Let $\{x_n\}$ be a sequence generated by Algorithm (3.6) and assume that $0 < p < \beta_n < q < 1$ and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$. Then $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\lim_{n\to\infty} \|y_n - u_n\| = 0$*

*Proof.* Since $\{x_n\}$ is bounded. Therefore

$$\lim_{n\to\infty} \|x_n - x^*\| = d \quad \text{for some } d. \tag{3.8}$$

It follows by (3.7) that

$$\limsup_{n\to\infty} \|u_n - x^*\| \leq \limsup_{n\to\infty} \|x_n - x^*\| + \limsup_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = d. \tag{3.9}$$

It follows by Algorithm (3.6) that

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|u_n - x^*\|$$

By simplifying we have

$$\begin{aligned}
\|x_n - x^*\| &\leq \frac{1}{\alpha_n}(\|x_n - x^*\| - \|x_{n+1} - x^*\|) + \|u_n - x^*\| \\
&\leq \frac{1}{a}(\|x_n - x^*\| - \|x_{n+1} - x^*\|) + \|u_n - x^*\|.
\end{aligned}$$

This implies that

$$d = \liminf_{n\to\infty} \|x_n - x^*\| \leq \liminf_{n\to\infty} \|u_n - x^*\|. \tag{3.10}$$

From (3.9) and (3.10), we can conclude that

$$\lim_{n\to\infty} \|u_n - x^*\| = d. \tag{3.11}$$

By Lemma 2.2, we have

$$\frac{1}{\lambda_n}\|u_n - x^*\|^2 - \frac{1}{\lambda_n}\|x_n - x^*\|^2 + \frac{1}{\lambda_n}\|x_n - u_n\|^2 \leq g(x^*) - g(u_n).$$

Since $g(x^*) \leq g(u_n)$ for all $n \geq 1$, we obtain

$$\|x_n - u_n\|^2 \leq \|x_n - x^*\|^2 - \|u_n - x^*\|^2.$$

It implies by (3.8) and (3.11) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.12}$$

Consider

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n \|y_n - x^*\| \\
&\leq (1-\alpha_n)\|y_n - x^*\| + \alpha_n \|y_n - x^*\| \\
&= \|y_n - x^*\| \\
&\leq \|u_n - x^*\|
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|y_n - x^*\| = d. \tag{3.13}$$

Further, we estimate

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|T^f_{r_n} u_n - x^*\|^2 \\
&\leq \langle T^f_{r_n} u_n - T^f_{r_n} x^*, u_n - x^* \rangle \\
&= \langle y_n - x^*, u_n - x^* \rangle \\
&= \frac{1}{2}\left(\|y_n - x^*\|^2 + \|u_n - x^*\|^2 - \|y_n - u_n\|^2\right)
\end{aligned}$$

It follows that

$$\|y_n - u_n\|^2 \leq \|u_n - x^*\|^2 - \|y_n - x^*\|^2.$$

It implis by (3.11) and (3.13) that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.14}$$

□

We are ready to prove our main theorem.

**Theorem 3.1.** *Let $\{x_n\}$ be a sequence in $H$ generated by (3.6) such that*

(a). $x_0, x_1$ *are choosen randomly,*
(b). $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$,
(c). $0 < p < \beta_n < q < 1$,
(d). $r_n > 0$ *where* $\liminf_{n\to\infty} r_n > 0$

*Then, $\{x_n\}$ converges weakly to a point in $\Omega$.*

*Proof.* From (3.6), we have

$$\begin{aligned}\|s_n - x^*\|^2 &= \|x_n - x^* + \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - x^*, x_n - x_{n-1}\rangle \\ &\leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\|,\end{aligned}$$

and

$$\begin{aligned}\|w_n - x^*\|^2 &= \|(1-\gamma_n)(s_n - x^*) + \gamma_n(T_n s_n - x^*)\|^2 \\ &= (1-\gamma_n)\|s_n - x^*\|^2 + \gamma_n \|T_n s_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|s_n - T_n s_n\|^2 \\ &\leq \|s_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|s_n - T_n s_n\|^2, \\ &\leq \|s_n - x^*\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|z_n - x^*\|^2 &= \|(1-\beta_n)(y_n - x^*) + \beta_n(T_n y_n - x^*)\|^2 \\ &= (1-\beta_n)\|y_n - x^*\|^2 + \beta_n \|T_n y_n - x^*\|^2 - \beta_n(1-\beta_n)\|y_n - T_n y_n\|^2 \\ &\leq \|y_n - x^*\|^2 - \beta_n(1-\beta_n)\|y_n - T_n y_n\|^2,\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1-\alpha_n)\|T_n z_n - x^*\|^2 + \alpha_n \|T_n y_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T_n z_n - T_n y_n\|^2 \\ &\leq (1-\alpha_n)\|T_n z_n - x^*\|^2 + \alpha_n \|T_n y_n - x^*\|^2 \\ &\leq \|u_n - x^*\|^2 \\ &\leq \|w_n - x^*\|^2 \\ &\leq \|s_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|s_n - T_n s_n\|^2, \\ &\leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\| \\ &\quad - \gamma_n(1-\gamma_n)\|s_n - T_n s_n\|^2.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$ and $\lim_{n\to\infty} \|x_n - x^*\|$ exists, it follows that $\|s_n - T_n s_n\| \to 0$. Since $\{s_n\}$ is bounded and $\{T_n\}$ satisfies NST-conditon(I) with $T$, we get $\|s_n - T s_n\| \to 0$. From

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - s_n\| + \|s_n - Ts_n\| + \|Ts_n - Tx_n\| \\ &\leq 2\|x_n - s_n\| + \|s_n - Ts_n\| \\ &= 2\theta_n \|x_n - x_{n-1}\| + \|s_n - Ts_n\|\end{aligned}$$

we obtain $\|x_n - Tx_n\| \to 0$. Let $w$ be a weak cluster point of $\{x_n\}$. Then $w \in F(T)$ by demicloseness of $I - T$ at 0. Therefore, by using Opial lemma, we conclude that there exists $p \in F(T)$ such that $x_n \rightharpoonup p$.

Now we prove $p \in \Omega$, that is $p \in \operatorname{argmin}_{u\in C} g(u)$ and $p \in EP(f)$. Consider

$$\begin{aligned}\|w_n - u_n\| &\leq \|w_n - x_n\| + \|u_n - x_n\| \\ &\leq \|w_n - s_n\| + \|s_n - x_n\| \\ &\leq \|s_n - T_n s_n\| + \theta_n \|x_n - x_{n-1}\|.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$ and $\|s_n - T_n s_n\| \to 0$, it follows that $\|w_n - u_n\| \to 0$ By Lemma 2.1, nonexpansiveness of $J_\lambda$, and $\lambda_n \geq \lambda > 0$ that

$$\begin{aligned}
\|w_n - J_\lambda w_n\| &\leq \|w_n - u_n\| + \|u_n - J_\lambda w_n\| \\
&= \|w_n - u_n\| + \|J_\lambda w_n - J_\lambda w_n\| \\
&= \|w_n - u_n\| + \left\| J_\lambda \left( \frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} w_n + \frac{\lambda}{\lambda_n} \right) - J_\lambda w_n \right\| \\
&\leq \|w_n - u_n\| + \left\| \left( \frac{\lambda_n - \lambda}{\lambda_n} \right) J_{\lambda_n} w_n + \frac{\lambda}{\lambda_n} - w_n \right\| \\
&= \|w_n - u_n\| + \left( 1 - \frac{\lambda}{\lambda_n} \right) \|J_{\lambda_n} w_n - w_n\| \\
&= \|w_n - u_n\| + \left( 1 - \frac{\lambda}{\lambda_n} \right) \|u_n - w_n\| \\
&= \left( 2 - \frac{\lambda}{\lambda_n} \right) \|w_n - u_n\| .
\end{aligned}$$

This together with (3.12) shows that

$$\lim_{n\to\infty} \|w_n - J_\lambda w_n\| = 0. \tag{3.15}$$

Since $J_\lambda$ is a nonexpansive mapping, by (3.15) and Lemma 2.3, we get $p \in F(J_\lambda) = \text{argmin}_{u \in H} g(u)$.

Otherwise, if $T_r^f p \neq p$ for some $r > 0$, then by Opial's condition, Lemma 2.6 and (3.14), we have

$$\begin{aligned}
\liminf_{j\to\infty} \|u_{n_j} - p\| &< \liminf_{j\to\infty} \|u_{n_j} - T_{r_{n_j}}^f p\| \\
&\leq \liminf_{j\to\infty} \left\{ \|u_{n_j} - T_{r_{n_j}}^f u_{n_j}\| + \|T_{r_{n_j}}^f u_{n_j} - T_r^f p\| \right\} \\
&\leq \liminf_{j\to\infty} \left\{ \|u_{n_j} - y_{n_j}\| + \|T_{r_{n_j}}^f u_{n_j} - T_r^f p\| \right\} \\
&= \liminf_{j\to\infty} \|T_{r_{n_j}}^f u_{n_j} - T_r^f p\| \\
&\leq \liminf_{j\to\infty} \left( \|u_{n_j} - p\| + \frac{|r_{n_j} - r|}{r} \|T_r^f u_{n_j} - u_{n_j}\| \right) \\
&= \liminf_{j\to\infty} \|u_{n_j} - p\|,
\end{aligned}$$

which is a contradiction. Therefore $T_r^f p = p$ for all $r > 0, i.e., p \in EP(f)$. The proof is completed. □

**Remark 3.2.**

(i) Theorem 3.1 generalize and improve the results of Kaewkhao *et al.* [19] to a minimization problem and equilibrium problem.

(ii) Theorem 3.1 is an improvement and generalization of the main result in Rockafellar [32] and Güler [14].

If $g(x) = x \quad \forall x \in H$ and $f(x, y) = 0 \quad \forall x, y \in H$, then the following result can be obtained from Theorem 3.1 immediately.

**Corollary 3.1** ([19])**.** *Let a family of nonexpansive mappings $\{T_n\}$ on a Hilbert space $H$ and a nonexpansive mapping $T$ on $H$ be such that $\{T_n\}$ satisfies NST-condition$(I)$ with $T$. Suppose* *that* $\emptyset \neq F(T) \subset \bigcap_{n=1}^{\infty} F(T_n)$*. Let* $\{x_n\}$ *be a sequence in* $H$ *generated by*

$$\begin{cases} s_n = x_n + \theta_n(x_n - x_{n-1}), \\ w_n = (1-\gamma_n)s_n + \gamma_n T_n s_n \\ z_n = (1-\beta_n)y_n + \beta_n T_n w_n \\ x_{n+1} = (1-\alpha_n)T_n z_n + \alpha_n T_n y_n, \quad \forall n \geq 1, \end{cases} \tag{3.16}$$

*such that*

(a). $x_0, x_1$ *are choosen randomly,*
(b). $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$,
(c). $0 < p < \beta_n < q < 1$.

*Then* $\{x_n\}$ *converges weakly to a point in* $F(T)$*.*

## 4. Numerical Examples

We now present a numerical example to demonstrate the performance and convergence of our theoretical results.

**Example 4.1.** Let $H = [-1, 1]$ with the usual norm. For each $x \in H$, we define $g : H \to (-\infty, \infty]$ by

$$g(x) = \frac{1}{2}\|x\|^2,$$

and define $T : H \to H$ by $Tx = \sin x$, for all $x \in H$. We define the folllowing sequence of nonexpansive mapping for $\{T_n\}_{n=1}^{\infty}$ as

$$\begin{cases} T_{3n-2} = x^n \sin x \\ T_{3n-1} = x \sin(\dfrac{x}{n}) \\ T_{3n} = \dfrac{\sin^n x}{2} \end{cases} \tag{4.17}$$

It is obvious that $\{T_n\}$ satisfies the NST-condition (I) with $T$. Moreover $g$ is proper convex and lower semi-continuous. For each $x, y \in H$, define bifunction $f : H \times H \to \mathbb{R}$ by

$$f(x, y) = x(y - x).$$

It is easy to check that $g, f, T, T_n$, satisfy all conditions in Theorem 3.1 with $\Omega = \{0\}$. Using the proximity operator [13], we know that

$$\underset{u \in H}{\text{argmin}} \left[ g(u) + \frac{1}{2}\|u - x\|^2 \right] = \text{prox}_g x = \frac{x}{2}.$$

For each $u \in H$, we compute $T_r^f u$. Find $z$ such that

$$\begin{aligned} 0 &\leq f(z, y) + \frac{1}{r}\langle y - z, z - u\rangle \\ &= z(y - z + 1) + \frac{1}{r}(y - z)(z - u) \\ &= zy - z^2 + z + \frac{1}{r}(y - z)(z - u) \\ &\Longleftrightarrow \\ 0 &\leq rzy - rz^2 + rz + yz - yu - z^2 + zu \\ &= (-r - 1)z^2 + (ry + t + y + u)z - yu. \end{aligned}$$

Let $J(z) = (-r-1)z^2 + (ry+r+y+u)z - yu$. Then $J(z)$ is a quadratic function of $z$ with coefficient $a = -r-1, b = ry+r+y+u$ and $c = -yu$. Determine the discriminant $\Delta$ of $J$ as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (ry+r+y+u)^2 - 4(-r-1)(-yu) \\
&= r^2y^2 + 2r^2y + r^2 + 2ruy + 2ru + 2ry^2 + 2ry + u^2 + 2uy + y^2 - 4ury - 4uy \\
&= (r^2+2r+1)y^2 + (2r^2 + 2ru + 2r + 2u - 4ru - 4u)y + (r^2 + 2ru + u^2) \\
&= (r+1)^2y^2 + 2(r^2 + ru + r + u - 2ru - 2u)y + (r+u)^2 \\
&= (r+1)^2y^2 + 2(r+1)(r-u)y + (r+u)^2 \\
&= ((r+1)y + (r-u))^2 + 4ru.
\end{aligned}$$

We know that $J(z) \geq 0 \quad \forall z \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we have $y = \dfrac{-\sqrt{4ru} - (r-u)}{r+1}$. This implies that $T_r^f u = \dfrac{u - r - \sqrt{4ru}}{r+1}$

Then, the algorithm (3.6) becomes:

$$\text{(4.18)} \qquad \begin{cases}
s_n = x_n + \theta_n(x_n - x_{n-1}), \\
w_n = (1-\gamma_n)s_n + T_n s_n \\
u_n = \dfrac{w_n}{2} \\
y_n = \dfrac{u_n - r_n - \sqrt{4u_n r_n}}{r_n + 1} \\
z_n = (1-\beta_n)y_n + \beta_n T_n y_n \\
x_{n+1} = (1-\alpha_n)T_n z_n + \alpha_n T_n y_n, \quad \forall n \geq 1,
\end{cases}$$

In this example, we set the parameter on (4.18) by $\alpha_n = \frac{2n}{5n+1}\beta_n = \frac{2n}{3n+1}, \gamma_n = \frac{1}{3n+1}$, $r_n = \frac{n}{n+1}, \theta_n = \frac{1}{2^n}$. It can be observed that all the assumptions of Theorem 3.1 are satisfied. In the experiment, we choose the stopping criterion $E_n := \|x_n - p\| < 10^{-40}$ where $p = 0$ or the maximum iteration exceeds $10,000$ iterations. The proposed algorithm is coded in MATLAB2014b, and run on MacBook Air (1.4 GHz Intel Core i5 and 4 GB 1600 MHz DDR3).

To study the behaviour of obtained solution by Algorithm 4.18, we set the initial solutions in 3 different cases: both initial solutions are less than exact solution ($x_0 = -0.9, x_1 = -0.5$), both initials are greater than the exact solution ($x_0 = 0.85, x_1 = 0.75$), one of initial is less than exact and the other one is greater than the exact solution($x_0 = 0.85, x_1 = -0.75$). The sequences of solution obtained by Algorithm 4.18 with different initial solutions are presented in Figure 1. As shown in Figure 1, the solution obtained by the proposed algorithm rapidly approaches to an exact solution. Moreover, the obtained solutions in all cases are oscillating convergent sequences.

Since the algorithm presented in this research is to find a solution of three problems ($\Omega$), while Kranoskii-Mann [22] and Colao [9] algorithms are to find the solution of two problems
$(EP(f) \bigcap \cap_{n=1}^{\infty} F(T_n))$ . Thus, we have defined function $g := I$, so that the interesting problem reduce to only $(EP(f) \bigcap \cap_{n=1}^{\infty} F(T_n))$. Then, we compare our algorithm with Krasnoselskii-Mann and Colao's algorithms to see the performance of the algorithm presented in this work. From Figure 2, it can be seen that the proposed algorithm converges to an exact solution faster than the other two algorithms.

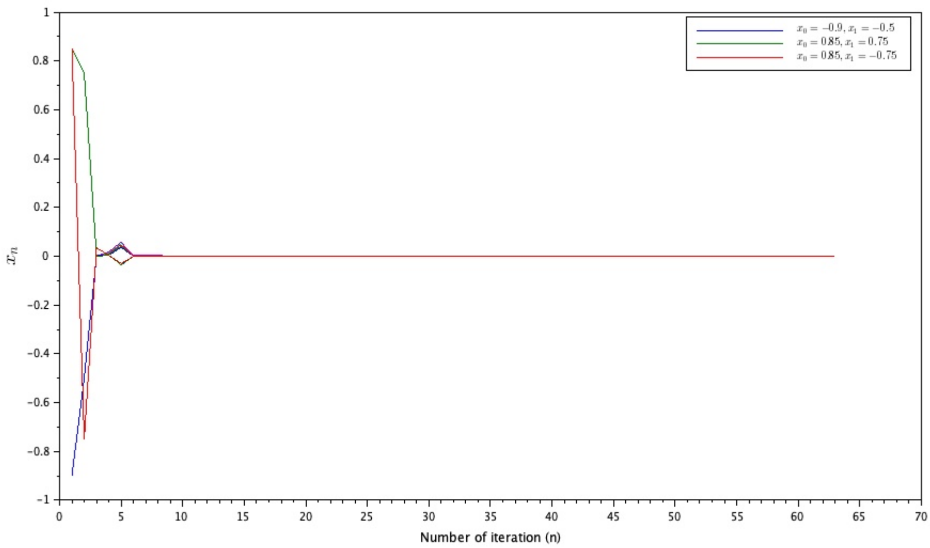


FIGURE 1. The sequence $\{x_n\}$ in different initial points $x_0, x_1$.

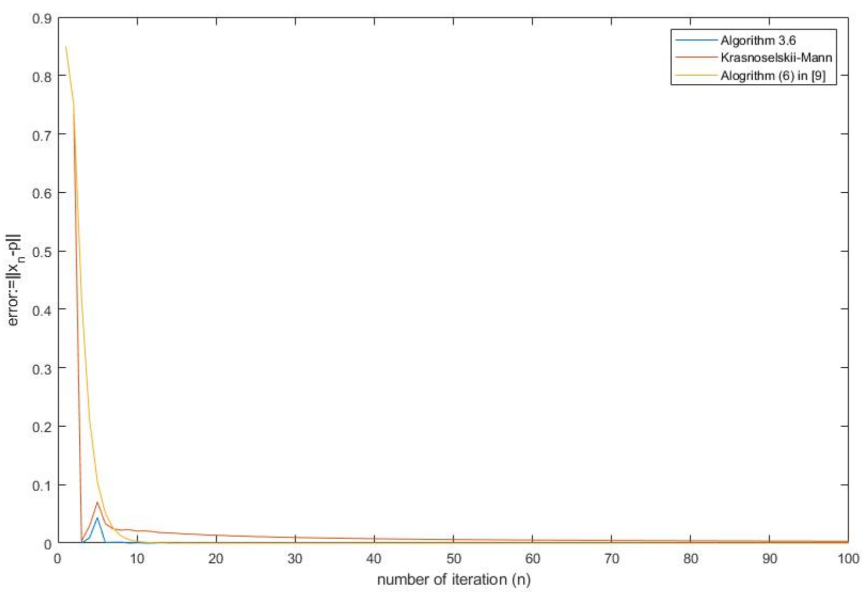


FIGURE 2. Comparison of Algorithm (3.6), Kranoskii-Mann algorithm and Algorithm (6) in [9] with $\{T_n\}$ defined in (4.17).

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