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Voronovskaja type theorem for some nonpositive Kantorovich type operators

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ABSTRACT. In this paper we will study a Voronovskaja type theorem and a simultaneous approximation result for a new class of generalized Bernstein operators. The new operators are obtained using a generalization of Kantorovich's method, namely, we will introduce a sequence of operators $K_n^l = D^l \circ B_{n+l} \circ I^l$, where B_{n+l} are Bernstein operators, $D^l f = f^{(l)} + a_{l-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f$ is a differential operator with constant coefficients $a_j,\ j \in \{0, \ldots, l-1\}$ and I^l a corresponding antiderivative operator such that $D^l \circ I^l = Id$.

1. INTRODUCTION

S. N. Bernstein introduced, in paper [2], the following sequence of positive and linear operators, in order to prove Weierstrass's approximation theorem [13]:

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ x \in [0,1],\ f \in C[0,1] \tag{1.1}$$

where $p_{n.k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, for $0 \le k \le n$, and $p_{n.k}(x) = 0$ for $k > n$. These operators have been extensively studied. Bernstein operators can be useful in the uniform approximation of all continuous functions on $[0,1]$. Also, these operators play an important role in simultaneous approximation, that is, for $f \in C^k[0,1]$ we have $(B_n f)^{(k)} \to f^{(k)}$, uniformly as $n \to \infty$, see [3, 11].

M. Floater, in paper [4], proved the following Voronovskaja type theorem regarding the derivatives of Bernstein operators:

Theorem 1.1. *If $f \in C^{k+2}[0,1]$, for some $k \ge 0$ then*

$$\lim_{n\to\infty} n\left[(B_n(f,x))^{(k)} - f^{(k)}(x)\right] = \frac{1}{2}\frac{d^k}{dx^k}\left[x(1-x)f''(x)\right],$$

uniformly for $x \in [0,1]$.

One of the many generalizations of Bernstein operators, which we will use throughout our paper, was given by D.D. Stancu in [10], who introduced a modification of Bernstein operators depending on two parameters $0 \le \alpha \le \beta$:

$$B_n^{\alpha,\beta}(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),\ x \in [0,1],\ f \in C[0,1], \tag{1.2}$$

and proved that $\left\|B_n^{\alpha,\beta} f - f\right\| \to 0$ as $n \to \infty$. Further, it was shown that for $f \in C^k[0,1]$, the simultaneous approximation $(B_n^{\alpha,\beta} f)^{(k)} \to f^{(k)}$ holds uniformly (see [1]).

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Another generalization of high importance was introduced by Kantorovich in paper [6]. The operators studied there can be constructed as follows: let $D=\frac{d}{dx}$ be a differential operator, and I a corresponding antiderivative operator, with respect to the composition $D\circ I=Id$, and $If(0)=0$. In this case, the antiderivative operator is given by $If(x)=\int_0^x f(t)\,dt$. Therefore Kantorovich operators are obtained as the following composition:

$$K_n = D\circ B_{n+1}\circ I, \tag{1.3}$$

where B_{n+1} are the Bernstein operators of order $n+1$. Explicitly, the operators constructed as in (1.3) have the following expression:

$$K_n(f,x)=(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ x\in[0,1],\ f\in L_1[0,1]. \tag{1.4}$$

Bernstein-Kantorovich operators from (1.4) can be used to uniformly approximate all functions in $L_1[0,1]$.

A notable generalization of Bernstein-Kantorovich operators using Kantorovich's method was studied in papers [5, 7, 9]. In these papers were analyzed the operators obtained by taking the derivative operators $D^k=\frac{d^k}{dx^k}$, and the antiderivative operators as $I^kf(x)=\int_0^x \frac{(x-t)^{k-1}}{(k-1)!}f(t)\,dt$, and considering the operators $\tilde{K}_n=D^k\circ B_n\circ I^k$ now called k-th order Kantorovich operators. They proved the uniform convergence $\tilde{K}_nf\to f$, for $f\in L_1[0,1]$.

In paper [8], Păltănea R. used Kantorovich's method to modify Bernstein operators, this time by taking a more general differential operator $D^cf=f'+cf$. In this paper it was proved that the operators constructed as $K_n^*=D^c\circ B_{n+1}\circ I^c$ can be used to approximate functions on $C[0,1]$ and these operators are linear but not positive. Here I^c is a corresponding antiderivative operator of the form $I^cf(x)=e^{-cx}\int_0^x e^{ct}f(t)dt$, $I^cf(0)=0$, such that $D^c\circ I^c=Id$.

In this paper, we will study a Voronovskaja type theorem and a simultaneous approximation property of the operators we introduced in paper [12]. These operators are constructed using a more general differential operator: let $l\in\{1,2,\dots\}$,

$$D^lg=g^{(l)}+a_{l-1}g^{(l-1)}+\cdots+a_1g'+a_0g, \tag{1.5}$$

$a_0,a_1,a_2,\dots,a_{l-1}\in\mathbb{R}$. By an antiderivative operator of D^l we mean an operator I^l which satisfies $D^l\circ I^l=Id$. This condition leads to:

$$\left(D^l\circ I^l\right)(f)=f,$$

which is equivalent with:

$$\left(I^lf\right)^{(l)}+a_{l-1}\left(I^lf\right)^{(l-1)}+\cdots+a_1\left(I^lf\right)'+a_0\left(I^lf\right)=f. \tag{1.6}$$

The equation (1.6) is a linear differential equation of order l with constant coefficients for which it is known that its solutions, $I^lf\in C^l[0,1]$, exist but are not unique. Since there is an infinity of such antiderivative operators, one can obtain a unique fixed one by imposing some initial values of I^lf and its derivatives up to order $l-1$ in a certain point x_0, but for the construction of the operators studied in this paper, the exact expression of the antiderivative operators does not play an important role. Choosing a different antiderivative operator, our operator will be different, but the approximation processes we study don't depend on the choice of the antiderivative operator.

The Kantorovich type operators for which we will study the approximation properties mentioned before, are defined as:

$$K_n^l = D^l \circ B_{n+l} \circ I^l, \tag{1.7}$$

where B_{n+l} are the Bernstein operators of order $n+l$, $n \geq 1,\ l \geq 0$.

Remark 1.1. We chose the order $n+l$ of Bernstein operators in (1.7) in order to not have vanishing terms after differentiation of order $l \geq 0$.

Remark 1.2. We mention that our operators are a generalization of the ones studied in papers [5, 7, 8, 9].

Further, we will present the steps we followed to obtain the expression of our operators.

In order to express the derivatives of Bernstein operators we will need the finite differences $\Delta_h f(x) = f(x+h) - f(x)$ which is the first finite difference of f with step h. The l-th iterate of Δ is denoted by Δ^l and is defined as follows

$$\Delta_h^l f(x) = \Delta_h \left[\Delta_h^{l-1} f(x)\right], \tag{1.8}$$

and has the following expression

$$\Delta_h^l f(x) = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} f(x+ih). \tag{1.9}$$

Now, it is known that the l-th derivative of Bernstein operators B_n can be written in terms of finite differences of order l as follows, see [3]:

$$B_n^{(l)}(f,x) = \frac{n!}{(n-l)!} \sum_{j=0}^{n-l} \Delta_{\frac{1}{n}}^l f\left(\frac{j}{n}\right) p_{n-l,j}(x),\ x \in [0,1]. \tag{1.10}$$

With all the above considerations, the operators which will be studied in this paper, $K_n^l(f,x) = \left(D^l \circ B_{n+l} \circ I^l\right)(f,x),\ x \in [0,1]$, $f \in C[0,1]$, can be written as

$$K_n^l(f,x) = D^l \left(B_{n+l} I^l f\right)(x),\ x \in [0,1],\ f \in C[0,1]. \tag{1.11}$$

To simplify the notations we will denote $I^l f := F$ and we will obtain the following explicit representation of the operators K_n^l:

$$\begin{aligned} &K_n^l(f,x) = \qquad\qquad (1.12)\\ &= \left[B_{n+l}(F,x)\right]^{(l)} + a_{l-1}\left[B_{n+l}(F,x)\right]^{(l-1)} + \cdots + a_1 \left[B_{n+l}(F,x)\right]' + a_0 B_{n+l}(F,x) \\ &= \frac{(n+l)!}{n!} \sum_{j=0}^{n} \Delta_{\frac{1}{n+l}}^{l} F\left(\frac{j}{n+l}\right) p_{n,j}(x) + \\ &+ a_{l-1} \frac{(n+l)!}{(n+1)!} \sum_{j=0}^{n+1} \Delta_{\frac{1}{n+l}}^{l-1} F\left(\frac{j}{n+l}\right) p_{n+1,j}(x) + \\ &+ \cdots + a_1 \frac{(n+l)!}{(n+l-1)!} \sum_{j=0}^{n+l-1} \Delta_{\frac{1}{n+l}} F\left(\frac{j}{n+l}\right) p_{n+l-1,j}(x) + a_0 B_{n+l}(F,x). \end{aligned}$$

Remark 1.3. The operators in (1.12) are linear operators.

We proved in paper [12] the following results concerning these operators:

Remark 1.4. Operators (1.12) are not positive.

Lemma 1.1. [12] *Let F be a C^k function on a compact interval I, then the following convergence holds:*

$$(n+l)^k \, \Delta^k_{\frac{1}{n+l}} F(x) \to F^{(k)}(x), \ uniformly\ as\ n \to \infty, \ x \in I. \tag{1.13}$$

Using the result in Lemma 1.1 we proved that the following approximation result holds even though the operators are not positive operators:

Theorem 1.2. [12] *Let $f \in C[0,1]$. The following convergence holds:*

$$K_n^l f \to f, \ uniformly\ as\ n \to \infty. \tag{1.14}$$

Further, the following remark states that, using operators K_n^l and Theorem 1.2, we can show that B_{n+l} can approximate differential operators D^l.

Remark 1.5. Let us denote

$$\mathcal{F} = \{g \in C[a,b], I^l D^l g = g\}.$$

If $g \in \mathcal{F}$, then $D^l B_{n+l} g = D^l B_{n+l} I^l D^l g = K_n^l D^l g$. Now, by applying Theorem 1.2 we have $K_n^l D^l g \to D^l g$. Consequently, we get

$$D^l B_{n+l} g \to D^l g, \ \textit{as } n \to \infty, \textit{ for } g \in \mathcal{F}.$$

2. Voronovskaja type theorem

In this section we will prove a Voronovskaja type theorem for the operators K_n^l.

Let I^l be the antiderivative operator corresponding to the differential operator D^l. For simplicity, we denote $I^l f(x) := F(x)$.

Let us introduce the following differential operator:

$$D_y^l g(x) = y^l g^{(l-1)}(x) + a_{l-1} y^{l-1} g^{(l-2)}(x) + a_{l-2} y^{l-2} g^{(l-3)}(x) + \cdots + a_1 y g(x). \tag{2.15}$$

Theorem 2.3. *Let $f \in C^2[0,1]$ and $F \in C^{l+2}[0,1]$, then*

$$\begin{aligned} \lim_{n\to\infty} n\left[K_n^l(f,x) - f(x)\right] &= \frac{1}{2} D^l \left\{x(1-x)\left[F(x)\right]''\right\} \\ = \frac{1}{2} x(1-x) f''(x) &+ \left(\frac{1}{2} - x\right) \frac{\partial D_y^l F''(x)}{\partial y}\Big|_{y=1} - \frac{1}{2} \frac{\partial^2 D_y^l F'(x)}{\partial y^2}\Big|_{y=1}. \end{aligned} \tag{2.16}$$

uniformly for $x \in [0,1]$.

Proof. We have that

$$K_n^l(f,x) = D^l(B_{n+l}(I^l f(x))),$$

and we can write $f(x) = D^l(F(x)) = F^{(l)}(x) + a_{l-1}F^{(l-1)}(x) + \cdots + a_1 F'(x) + a_0 F(x)$. Now, we compute:

$$\begin{aligned} n\left[K_n^l(f,x) - f(x)\right] &= n\Big\{\left[B_{n+l}(F,x)\right]^{(l)} + a_{l-1}\left[B_{n+l}(F,x)\right]^{(l-1)} + \\ &\cdots + a_1\left[B_{n+l}(F,x)\right]' + a_0\left[B_{n+l}(F,x)\right] - nD^l(F(x))\Big\} \\ &= n\Big\{\left[B_{n+l}(F,x)\right]^{(l)} - F^{(l)}(x)\Big\} + a_{l-1} n\Big\{\left[B_{n+l}(F,x)\right]^{(l-1)} - F^{(l-1)}(x)\Big\} + \\ &+ \cdots + a_0 n\left\{\left[B_{n+l}(F,x)\right] - F(x)\right\}. \end{aligned}$$

Passing to the limit, with $n \to \infty$, and using Theorem 1.1, we get:

$$\lim_{n\to\infty} n\left[K_n^l(f,x) - f(x)\right] = \tag{2.17}$$

$$= \frac{1}{2}\left\{\frac{d^l}{dx^l}\left[x(1-x)F''(x)\right] + a_{l-1}\frac{d^{l-1}}{dx^{l-1}}\left[x(1-x)F''(x)\right] + \dots\right.$$

$$\left. + a_1\frac{d}{dx}\left[x(1-x)F''(x)\right] + a_0 x(1-x)F''(x)\right\}$$

$$= \frac{1}{2}D^l\left[x(1-x)F''(x)\right].$$

For the last expression we will use Leibniz formula in order to obtain the derivative of $x(1-x)F''(x)$ of an arbitrary order k:

$$\left[x(1-x)F''(x)\right]^{(k)} = x(1-x)\left[F''(x)\right]^{(k)} + k(1-2x)\left[F''(x)\right]^{(k-1)} - k(k-1)\left[F''(x)\right]^{(k-2)}. \tag{2.18}$$

Using the expression of D^l and relation (2.18), we get:

$$D^l\left[x(1-x)F''(x)\right] = \tag{2.19}$$

$$= \left[x(1-x)F''(x)\right]^{(l)} + a_{l-1}\left[x(1-x)F''(x)\right]^{(l-1)} + \dots + a_0\left[x(1-x)F''(x)\right]$$

$$= x(1-x)\left[F''(x)\right]^{(l)} + l(1-2x)\left[F''(x)\right]^{(l-1)} - l(l-1)\left[F''(x)\right]^{(l-2)} +$$

$$+ a_{l-1}\left\{x(1-x)\left[F''(x)\right]^{(l-1)} + (l-1)(1-2x)\left[F''(x)\right]^{(l-2)} - (l-1)(l-2)\left[F''(x)\right]^{(l-3)}\right\} +$$

$$+ \dots + a_1 x(1-x)\left[F''(x)\right]' + a_1(1-2x)\left[F''(x)\right] + a_0 x(1-x)F''(x).$$

After some computations we get:

$$D^l\left[x(1-x)F''(x)\right] = \tag{2.20}$$

$$= x(1-x)\left\{\left[F''(x)\right]^{(l)} + a_{l-1}\left[F''(x)\right]^{(l-1)} + \dots + a_1\left[F''(x)\right]' + a_0F''(x)\right\} +$$

$$+ (1-2x)\left\{l\left[F''(x)\right]^{(l-1)} + a_{l-1}(l-1)\left[F''(x)\right]^{(l-2)} + \dots + 2a_2\left[F''(x)\right]' + a_1F''(x)\right\}$$

$$- \left\{l(l-1)\left[F''(x)\right]^{(l-2)} + a_{l-1}(l-1)(l-2)\left[F''(x)\right]^{(l-3)} + \dots + 2a_2\left[F''(x)\right]\right\}$$

$$= x(1-x)\left[D^lF(x)\right]'' + (1-2x)\frac{\partial D_y^l F''(x)}{\partial y}\Big|_{y=1} - \frac{\partial^2 D_y^l F'(x)}{\partial y^2}\Big|_{y=1}$$

$$= x(1-x)f''(x) + (1-2x)\frac{\partial D_y^l F''(x)}{\partial y}\Big|_{y=1} - \frac{\partial^2 D_y^l F'(x)}{\partial y^2}\Big|_{y=1},$$

which is the result announced in hypothesis. □

3. Simultaneous approximation

In the next part of our paper we will prove a simultaneous approximation result concerning operators K_n^l.

Theorem 3.4. *Let $f \in C^r[0,1]$ with $r \in \mathbb{N} \cup \{0\}$. Then:*

$$\lim_{n\to\infty}\left[K_n^l(f,x)\right]^{(r)} = f^{(r)}(x), \tag{3.21}$$

holds uniformly and $F \in C^{l+r}[0,1]$.

Proof. We will need the expression of $[K_n^l(f,x)]^{(r)}$, for $n>r$:

$$[K_n^l f\left(x\right)]^{(r)}=\frac{(n+l)!}{(n-r)!}\frac{1}{\left(n+l\right)^{l+r}}\sum_{j=0}^{n-r}\left(n+l\right)^{l+r}\Delta_{\frac{1}{n+l}}^{l+r}F\left(\frac{j}{n+l}\right)p_{n-r,j}\left(x\right)+ \tag{3.22}$$
$$+a_{l-1}\frac{(n+l)!}{(n-r+1)!}\frac{1}{\left(n+l\right)^{l+r-1}}\sum_{j=0}^{n-r+1}\left(n+l\right)^{l+r-1}\Delta_{\frac{1}{n+l}}^{l+r-1}F\left(\frac{j}{n+l}\right)p_{n-r+1,j}\left(x\right)$$
$$+\cdots+a_1\frac{(n+l)!}{(n+l-r-1)!}\frac{1}{(n+l)^{r+1}}\sum_{j=0}^{n+l-r-1}(n+l)^{r+1}\Delta_{\frac{1}{n+l}}^{r+1}F\left(\frac{j}{n+l}\right)p_{n+l-r-1,j}\left(x\right)$$
$$+a_0\left[B_{n+l}\left(F,x\right)\right]^{(r)}$$
$$=A_n^{-r}\sum_{j=0}^{n-r}\left(n+l\right)^{l+r}\Delta_{\frac{1}{n+l}}^{l+r}F\left(\frac{j}{n+l}\right)p_{n-r,j}\left(x\right)+$$
$$+a_{l-1}A_n^{-r+1}\sum_{j=0}^{n-r+1}\left(n+l\right)^{l+r-1}\Delta_{\frac{1}{n+l}}^{l+r-1}F\left(\frac{j}{n+l}\right)p_{n-r+1,j}\left(x\right)+$$
$$+\cdots+a_1A_n^{l-r-1}\sum_{j=0}^{n+l-r-1}(n+l)^{r+1}\Delta_{\frac{1}{n+l}}^{r+1}F\left(\frac{j}{n+l}\right)p_{n+l-r-1,j}\left(x\right)+a_0\left[B_{n+l}\left(F,x\right)\right]^{(r)},$$

where $A_n^k:=\frac{(n+l)!}{(n+k)!}\frac{1}{\left(n+l\right)^{l-k}}$, $k\in\{-r,-r+1,\dots,l-r-1\}$.

For all $\varepsilon>0$, there exists $n_\varepsilon^1\in\mathbb{N}$ such that $\left|A_n^k-1\right|<\varepsilon$ for $n\geq n_\varepsilon^1$, for $k\in\{-r,-r+1,\dots,l-r-1\}$, and, as we proved in Lemma 1.1 we also have that for all $\varepsilon_1>0$, there exists $n_{\varepsilon_1}\in\mathbb{N}$ such that $\left|(n+l)^k\Delta_{\frac{1}{n+l}}^kF\left(\frac{j}{n+l}\right)-F^{(k)}\left(\frac{j}{n+l}\right)\right|<\varepsilon_1$ for all $n\geq n_{\varepsilon_1}$. With these considerations we have:

$$\left|\left[K_n^lf\left(x\right)\right]^{(r)}-\left[\sum_{j=0}^{n-r}F^{(l+r)}\left(\frac{j}{n+l}\right)p_{n-r,j}\left(x\right)+a_{l-1}\sum_{j=0}^{n-r+1}F^{(l-1)}\left(\frac{j}{n+l}\right)p_{n-r+1,j}\left(x\right)+\right.\right.$$
$$\left.\left.+\cdots+a_1\sum_{j=0}^{n+l-r-1}F^{(r+1)}\left(\frac{j}{n+l}\right)p_{n+l-r-1,j}\left(x\right)+a_0\left[B_{n+l}\left(F,x\right)\right]^{(r)}\right]\right|\to0.$$

Now, we notice that the sums appearing above can be expressed in terms of Bernstein-Stancu operators as follows:

$$\sum_{j=0}^{n+k}F^{(p)}\left(\frac{j}{n+l}\right)p_{n+k,j}\left(x\right)=B_{n+k}^{0,l-k}\left(F^{(p)},x\right),\ -r\leq k\leq r-l-1,\ p=l-k\geq0.$$

For these operators it is known that $B_{n+k}^{0,l-k}\left(F^{(p)},x\right)\to F^{(p)}\left(x\right)$ as $n\to\infty$, uniformly with respect to x. Therefore, we obtain:

(3.23)
$$\lim_{n\to\infty}\left[K_n^lf\left(x\right)\right]^{(r)}=F^{(l+r)}\left(x\right)+a_{l-1}F^{(l-1+r)}\left(x\right)+\cdots+a_1F^{(r+1)}\left(x\right)+a_0\left[F\left(x\right)\right]^{(r)}$$
$$=\left[F^{(l)}\left(x\right)+a_{l-1}F^{(l-1)}\left(x\right)+\cdots+a_1F'\left(x\right)+a_0F\left(x\right)\right]^{(r)}$$
$$=\left[D^l\circ I^l(f(x))\right]^{(r)}=f^{(r)}(x),$$

which completes our proof.

□

4. Example

In this final section we will take a particular case of our operators and we will present some computations and graphics.

Let $D^*f = f'' - 3f' + 2f$ be a differential operator of order two and I^*, a fixed corresponding antiderivative operator, in the sense $D^* \circ I^* = Id$, with initial conditions $I^*f(0) = 0$ and $(I^*f)'(0) = 0$. The condition $(D^* \circ I^*) f = f$, $f \in C^2[0,1]$, leads to the following differential equation:

$$(I^*f)''(x) - 3(I^*f)'(x) + 2(I^*f)(x) = f(x). \tag{4.24}$$

In order to get the solution, we will first solve the homogeneous equation:

$$(I^*f)''(x) - 3(I^*f)'(x) + 2(I^*f)(x) = 0,$$

for which the solution, denoted by I_o is given by:

$$I_o(x) = C_1e^x + C_2e^{2x}, \tag{4.25}$$

with $C_1,\ C_2$ constants. In order to get the solution of the non-homogeneous equation we consider that the solution I^*f of the equation (4.24) is given by $I^*f(x) = C_1(x)e^x + C_2(x)e^{2x}$, where $C_1(x)$ and $C_2(x)$ are functions instead of constants. In order to obtain the missing functions $C_1(x)$ and $C_2(x)$, we will solve the following system:

$$\begin{cases} C_1'(x)e^x + C_2'(x)e^{2x} = 0 \\ C_1'(x)e^x + 2C_2'(x)e^{2x} = f(x) \end{cases}, \tag{4.26}$$

from which we obtain the exact solution $I^*f(x)$ after imposing the initial conditions $I^*f(0) = 0$ and $(I^*f)'(0) = 0$:

$$I^*f(x) = \int_0^x e^{x-t}\left(e^{x-t} - 1\right) f(t)dt,\ x \in [0,1]. \tag{4.27}$$

We denote $F(x) := I^*f(x)$.

Now, let us consider the operators:

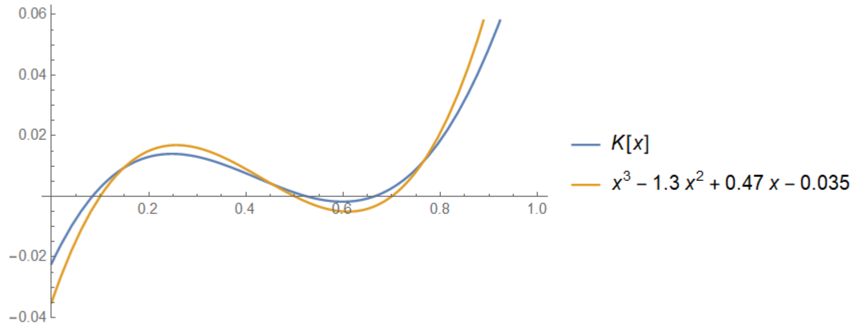
$$K_n^*(f,x) = (D^* \circ B_{n+2} \circ I^*f)(x) = D^*(B_{n+2}(F,x)),\ x \in [0,1]. \tag{4.28}$$

Then the operators $K_n^*(f,x)$ have the following expression:

(4.29)

$$\begin{aligned} K_n^*(f,x) = (n+1)(n+2)\sum_{k=0}^{n} p_{n,k}(x)\left(F\left(\frac{k+2}{n+2}\right) - 2F\left(\frac{k+1}{n+2}\right) + F\left(\frac{k}{n+2}\right)\right) + \\ -3(n+2)\sum_{k=0}^{n+1} p_{n+1,k}(x)\left(F\left(\frac{k+1}{n+2}\right) - F\left(\frac{k}{n+2}\right)\right) + \\ +2\sum_{k=0}^{n+2} p_{n+2,k}(x)F\left(\frac{k}{n+2}\right),\ x \in [0,1]. \end{aligned}$$

Now, we will consider the function $f(x) = x^3 - 1.3x^2 + 0.47x - 0.035$. For this function, for $n = 30$ we have obtained the following graphical process, using Wolfram Mathematica software:



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