CARPATHIAN J. MATH.
Volume **40** (2024), No. 1,
Pages 195 - 206

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.01.14

Approximating solutions of a split fixed point problem of demicontractive operators

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ABSTRACT. The concept of demicontractive operators introduced by Ştefan Măruşter is widely used in application and has been investigated by many scholars. The purpose of this paper is to continue to survey iterative methods for solving the split problem relevant to demicontractive operators. With the help of fixed point techniques, we construct an iterative sequence for solving the split fixed point problem in which three demicontractive operators are involved. Strong convergence result is obtained under several additional conditions. Finally, two numerical examples are given to illustrate the performance of the algorithm.

1. INTRODUCTION

Let H_1 and H_2 be two real Hilbert spaces with inner $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator and A^* be the adjoint operator of A. Recall that the split feasibility problem is to find a point u such that

$$u \in C \text{ and } Au \in Q, \tag{1.1}$$

where $C \subset H_1$ and $Q \subset H_2$ are two nonempty closed convex sets.

The split feasibility problem (1.1) was introduced by Censor and Elfving [4] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction ([2]). A special case of the split feasibility problem (1.1) is the convexly constrained linear inverse problem

$$u \in C \text{ and } Au = b$$

which has extensively been investigated in the literature using the projected Landweber iterative method [10].

To solve (1.1), a popular technique is to use projection method which generates a sequence $\{u_{n+1}\}$ by

$$u_{n+1} = P_C(u_n - \mu A^*(I - P_Q)Au_n), \ n \geq 1. \tag{1.2}$$

Censor et. al [5] noted that the intensity-modulated radiation therapy can mathematically be formulated as a multiple-sets split feasibility problem which is to find a point u with the property

$$u \in \cap_{i=1}^{s} C_i \text{ and } Au \in \cap_{j=1}^{t} Q_j, \tag{1.3}$$

where $C_i \subset H_1, i = 1, \cdots, s$ and $Q_j \subset H_2, j = 1, \cdots, t$ are closed convex sets.

The multiple-sets split feasibility problem (1.3) extends the well-known convex feasibility problem as well as the split feasibility problem. A nature idea is to use algorithm (1.2) to solve the multiple-sets split feasibility problem (1.3) by setting $C = \cap_{i=1}^{s} C_i$ and $Q = \cap_{j=1}^{t} Q_j$. However, the computation of $P_{\cap_{i=1}^{s} C_i}$ may be very difficult due to the complexity of $\cap_{i=1}^{s} C_i$. Note that calculating $P_{C_i}, i = 1, \cdots, s$ are easier than calculating

Received: 27.11.2022. In revised form: 22.07.2023. Accepted: 29.07.2023
2010 *Mathematics Subject Classification.* 47H09, 47H10, 47J25.
Key words and phrases. *split problem, fixed point, demicontractive operator, strong convergence.*
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$P_{\cap_{i=1}^s C_i}$. With the help of this fact, several valuable projection algorithms were proposed for solving the multiple-sets split feasibility problem (1.3), see [11, 22–24].

Observe that all projection algorithms have to compute the orthogonal projection P_C which is a special case of directed operators. Afterwards, Censor and Segal [6] proposed a general split fixed point problem of finding a point $u \in H_1$ such that

$$u \in Fix(f) \text{ and } Au \in Fix(g), \tag{1.4}$$

where $Fix(f) := \{x \in H_1 : f(x) = x\}$ and $Fix(g) := \{y \in H_2 : g(y) = y\}$ are the fixed point sets of two directed operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, respectively.

In algorithm (1.2), by replacing P_C and P_Q by f and g, respectively, Censor and Segal [6] obtained the following algorithm for solving the split fixed point problem (1.4):

$$u_{n+1} = f(u_n - \mu A^*(I - g)Au_n), \quad n \geq 1, \tag{1.5}$$

where f and g are two directed operators.

Moudafi [14] further extended f and g from directed operators to demicontractive operators and proposed the following iterate for finding a solution of (1.4): for an initial guess $u_0 \in H_1$,

$$\begin{cases} v_n = u_n - \mu A^*(I - g)Au_n, \\ u_{n+1} = (1 - \beta_n)v_n + \beta_n f(v_n), \quad n \geq 0, \end{cases} \tag{1.6}$$

where f and g are κ-demicontractive operators, $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\mu \in (0, \frac{1-\kappa}{\|A\|^2})$ is a constant.

In [20], Wang investigated the following iterate for solving (1.4): let $u_0 \in H_1$ be an initial point, for given u_n, if $\|(u_n - f(u_n)) + A^*(I - g)Au_n\| \neq 0$, compute

$$\begin{cases} \mu_n = \frac{\|\|u_n - f(u_n)\|^2 + \|(I-g)Au_n\|^2}{\|(u_n - f(u_n)) + A^*(I-g)Au_n\|^2}, \\ u_{n+1} = u_n - \mu_n[(u_n - f(u_n)) + A^*(I - g)Au_n], \end{cases} \tag{1.7}$$

if $\|(u_n - f(u_n)) + A^*(I - g)Au_n\| = 0$, then stop; where f and g are two directed operators.

Remark 1.1. There are some fixed point techniques applied in the iterates (1.5)-(1.7). In fact, solving (1.4) can be translated to solve the fixed point equation $x = f(x - \mu A^*(I - g)Ax)$ for all $\mu > 0$. Applying this fixed point equation, one can generate an iterate via the forms of (1.5) and (1.6) to solve the split problem (1.4).

On the other hand, finding a solution of (1.4) means to find a fixed point of the operator $I - \mu[(I - f) + A^*(I - g)A]$ for all $\mu > 0$. By using this relation, we can construct an iterate (1.7) for solving (1.4).

Further, according to the fixed point equation $x = f(x) - \mu A^*(I - g)Ax(\mu > 0)$, Zheng et. al. [25] proposed the following iterate for finding a solution of (1.4): for an initial guess $u_0 \in H_1$,

$$u_{n+1} = (1 - \sigma)u_n + \sigma[f(u_n) - \mu A^*(I - g)Au_n], \ n \geq 0, \tag{1.8}$$

where f and g are demicontractive operators.

Note that the directed operator is a special case of the demicontractive operator which was initially introduced by Ştefan Măruşter. The class of demicontractive operators is fundamental because many common types of operators arising in optimization belong to this class, see for example [13] and references therein. There are many iterative methods and widely applications relevant to the demicontractive operators, see for example [1, 3, 7–9, 12, 15–19].

Motivated by the related work of the multiple-sets split feasibility problem and the split fixed point problem, in this paper, we consider the following split fixed point problem of finding a point $u \in H_1$ such that

$$u \in Fix(f) \cap Fix(S) \text{ and } Au \in Fix(g), \tag{1.9}$$

where $f, S : H_1 \to H_1$ and $g : H_2 \to H_2$ are three demicontractive operators. Here, the solution set of (1.9) is denoted by Γ, namely, $\Gamma := \{x \in H_1 : x \in Fix(f) \cap Fix(S) \text{ and } Ax \in Fix(g)\}$.

The main purpose of this paper is to construct an iterative algorithm for finding a solution of the split fixed point problem (1.9). By utilizing fixed point technique, we suggest an iterative sequence for approximating a solution of (1.9). Strong convergence analysis of the proposed iterate is given provided some additional conditions are satisfied.

2. Preliminaries

In this section, we give some useful notation and lemmas. Let H be a real Hilbert space. Let $\{u_n\}$ be a sequence in H. Throughout, we use the following symbols: (i) $u_n \rightharpoonup u$ indicates that u_n converges weakly to u as $n \to \infty$; (ii) $u_n \to u$ indicates that u_n converges strongly to u as $n \to \infty$; (iii) $\omega_w(u_n)$ means the weak ω-limit set of the sequence $\{u_n\}$, namely, $\omega_w(u_n) := \{z :$ there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup z (i \to \infty)\}$.

Let $\varphi : H \to H$ be an operator. Recall that

- φ is said to be directed if

$$\|\varphi(x) - p\|^2 \le \|x - p\|^2 - \|x - \varphi(x)\|^2, \tag{2.10}$$

$\forall x \in H$ and $\forall p \in Fix(\varphi)$.

- φ is said to be κ-demicontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|\varphi(x) - p\|^2 \le \|x - p\|^2 + \kappa\|x - \varphi(x)\|^2, \tag{2.11}$$

$\forall x \in H$ and $\forall p \in Fix(\varphi)$.

- φ is said to be demiclosed if $u_n \rightharpoonup u$ and $\varphi(u_n) \to v$ implies that $\varphi(u) = v$.

Note that (2.10)$\Rightarrow$(2.11) implies that a directed operator must be a demicontractive operator. It is easy to verify that the inequality (2.11) is equivalent the following inequality

$$\langle x - \varphi(x), x - p\rangle \ge \frac{1-\kappa}{2}\|x - \varphi(x)\|^2, \kappa \in [0, 1), \tag{2.12}$$

$\forall x \in H$ and $\forall p \in Fix(\varphi)$.

Let Γ be a nonempty closed convex subset of H. Let P_Γ be the orthogonal projection from H onto Γ, namely,

$$P_\Gamma(u) := \arg\min_{x\in\Gamma} \|x - u\|, \; u \in H.$$

It is well known that P_Γ satisfies the following characteristic inequality, for $u \in H$,

$$\langle u - P_\Gamma(u), x - P_\Gamma(u)\rangle \le 0, \; \forall x \in \Gamma. \tag{2.13}$$

The following conclusion is well-known.

Lemma 2.1. *In a real Hilbert space H, we have*

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \; \forall x, y \in H, \tag{2.14}$$

and

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \; \forall x, y \in H. \tag{2.15}$$

Lemma 2.2 ([8]). *Let H be a real Hilbert space. Let $\varphi : H_1 \to H_1$ be a κ-demicontractive operator. Let $\delta \in (0, 1-\kappa)$ be a constant. For all $x \in H$ and $p \in Fix(\varphi)$, the following result holds*

$$\|(1-\delta)x + \delta\varphi(x) - p\|^2 \leq \|x-p\|^2 - \delta(1-\kappa-\delta)\|\varphi(x) - x\|^2.$$

Lemma 2.3 ([21]). *Let $\{b_n\}$, $\{\gamma_n\}$ and $\{t_n\}$ be three real number sequences. Suppose the following conditions are satisfied:*

(i) $b_n \geq 0$ *and* $\gamma_n \in [0,1]$ *for all* $n \geq 0$;
(ii) $\sum_{n=0}^{\infty} \gamma_n = +\infty$ *and* $\limsup_{n\to\infty} t_n \leq 0$;
(iii) $b_{n+1} \leq (1-\gamma_n)b_n + \gamma_n t_n$ *for all* $n \geq 0$.

Then, $\lim_{n\to\infty} b_n = 0$.

3. Main results

In this section, we state our main results.

Let H_1 and H_2 be two real Hilbert spaces. Let $f, S : H_1 \to H_1$ be two demicontractive operators with coefficients κ_1 and κ_2, respectively. Let $g : H_2 \to H_2$ be a κ_3-demicontractive operator. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator and A^* be the adjoint operator of A. Throughout, suppose $\Gamma \neq \emptyset$.

Let μ, τ and δ be three constants satisfying $\tau \in (0, \frac{1-\kappa_1}{2})$, $\mu \in (0, \frac{1-\kappa_3}{2\tau\|A\|^2})$ and $\delta \in (0, 1-\kappa_2)$. Let $\{\gamma_n\}$ be a sequence in $[0,1]$ satisfying $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_n \gamma_n = \infty$.

Next, we first present an iterative algorithm for solving the split fixed point problem (1.9).

Algorithm 3.1. Let $u \in H_1$ be a fixed point and u_0 be an initial point in H_1. Let the sequence $\{u_n\}$ be defined by the following way

$$\begin{cases} w_n = f(u_n) - \mu A^*(I-g)Au_n, & (3.16)\\ v_n = (1-\tau)u_n + \tau w_n, & (3.17)\\ z_n = (1-\delta)v_n + \delta S v_n, & (3.18)\\ u_{n+1} = \gamma_n u + (1-\gamma_n)z_n, \ n \geq 0. & (3.19)\end{cases}$$

To demonstrate the convergence of Algorithm 3.1, we need the following lemma which can be found in [25].

Lemma 3.4 ([25]). *$x \in Fix(f)$ and $Ax \in Fix(g)$ if and only if $x \in Fix(f - \mu A^*(I-g)A)$ for all $\mu > 0$.*

Lemma 3.5. *The sequence $\{u_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $p \in \Gamma$. Then, $p = f(p) = Sp$ and $Ap = g(Ap)$. Utilizing Lemma 3.4, we have $p = f(p) - \mu A^*(I-g)Ap$ for all $\mu > 0$. From (2.14) and (3.5), we have

$$\begin{aligned}\|v_n - p\|^2 &= \|u_n - p - \tau(u_n - w_n)\|^2\\ &= \|u_n - p\|^2 - 2\tau\langle u_n - p, u_n - w_n\rangle + \tau^2\|u_n - w_n\|^2.\end{aligned} \tag{3.20}$$

Next, we estimate $\|u_n - w_n\|^2$ and $\langle u_n - p, u_n - w_n\rangle$. By (3.16), we obtain

$$\begin{aligned}\|u_n - w_n\|^2 &= \|u_n - f(u_n) + \mu A^*(I-g)Au_n\|^2\\ &\leq (\|u_n - f(u_n)\| + \mu\|A^*(I-g)Au_n\|)^2\\ &\leq 2\|u_n - f(u_n)\|^2 + 2\mu^2\|A\|^2\|(I-g)Au_n\|^2.\end{aligned} \tag{3.21}$$

Since f is κ_1-demicontractive, from (2.12), we derive

$$\langle u_n - p, u_n - f(u_n)\rangle \geq \frac{1-\kappa_1}{2}\|u_n - f(u_n)\|^2. \tag{3.22}$$

Similarly, using the demicontraction of g, we receive

$$\langle Au_n - Ap, (I-g)Au_n\rangle \geq \frac{1-\kappa_3}{2}\|(I-g)Au_n\|^2. \tag{3.23}$$

Taking into account (3.16), (3.22) and (3.23), we have

$$\begin{aligned}\langle u_n - p, u_n - w_n\rangle &= \langle u_n - p, u_n - f(u_n)\rangle + \langle u_n - p, \mu A^*(I-g)Au_n\rangle \\ &= \langle u_n - p, u_n - f(u_n)\rangle + \mu\langle Au_n - Ap, (I-g)Au_n\rangle \\ &\geq \frac{1-\kappa_1}{2}\|u_n - f(u_n)\|^2 + \frac{\mu(1-\kappa_3)}{2}\|(I-g)Au_n\|^2.\end{aligned} \tag{3.24}$$

Substituting (3.21) and (3.24) into (3.20), we deduce

$$\begin{aligned}\|v_n - p\|^2 &\leq \|u_n - p\|^2 - \tau(1-\kappa_1)\|u_n - f(u_n)\|^2 - \tau\mu(1-\kappa_3)\|(I-g)Au_n\|^2 \\ &\quad + 2\tau^2\|u_n - f(u_n)\|^2 + 2\tau^2\mu^2\|A\|^2\|(I-g)Au_n\|^2 \\ &= \|u_n - p\|^2 - \tau(1-\kappa_1-2\tau)\|u_n - f(u_n)\|^2 \\ &\quad - \tau\mu(1-\kappa_3-2\tau\mu\|A\|^2)\|(I-g)Au_n\|^2 \\ &\leq \|u_n - p\|^2.\end{aligned} \tag{3.25}$$

Applying Lemma 2.2 to (3.18) to derive

$$\begin{aligned}\|z_n - p\|^2 &= \|(1-\delta)v_n + \delta Sv_n - p\|^2 \\ &\leq \|v_n - p\|^2 - \delta(1-\delta-\kappa_2)\|Sv_n - v_n\|^2 \\ &\leq \|v_n - p\|^2.\end{aligned} \tag{3.26}$$

By virtue of (3.19), (3.25) and (3.26), we have

$$\begin{aligned}\|u_{n+1} - p\| &= \|\gamma_n(u-p) + (1-\gamma_n)(z_n - p)\| \\ &\leq \gamma_n\|u-p\| + (1-\gamma_n)\|z_n - p\| \\ &\leq \gamma_n\|u-p\| + (1-\gamma_n)\|u_n - p\| \\ &\leq \cdots \\ &\leq \max\{\|u-p\|, \|u_0 - p\|\}.\end{aligned}$$

Thus, the sequences $\{u_n\}$, $\{v_n\}$, $\{z_n\}$ are bounded. □

Lemma 3.6. *Suppose that $I-f$, $I-S$ and $I-g$ are all demiclosed at zero. Then, $\omega_w(u_n) \subset \Gamma$.*

Proof. By (2.15) and (3.19), we obtain

$$\begin{aligned}\|u_{n+1} - p\|^2 &= \|\gamma_n(u-p) + (1-\gamma_n)(z_n - p)\|^2 \\ &\leq (1-\gamma_n)\|z_n - p\|^2 + 2\gamma_n\langle u-p, u_{n+1} - p\rangle.\end{aligned} \tag{3.27}$$

Combining (3.25) and (3.26), we attain

$$\begin{aligned}\|z_n - p\|^2 &\leq \|u_n - p\|^2 - \tau(1-\kappa_1-2\tau)\|u_n - f(u_n)\|^2 \\ &\quad - \tau\mu(1-\kappa_3-2\tau\mu\|A\|^2)\|(I-g)Au_n\|^2 \\ &\quad - \delta(1-\delta-\kappa_2)\|Sv_n - v_n\|^2.\end{aligned} \tag{3.28}$$

Substituting (3.28) into (3.27), we have

$$\begin{aligned}\|u_{n+1}-p\|^2 &\le (1-\gamma_n)\|u_n-p\|^2-(1-\gamma_n)\tau(1-\kappa_1-2\tau)\|u_n-f(u_n)\|^2\\&\quad-(1-\gamma_n)\tau\mu(1-\kappa_3-2\tau\mu\|A\|^2)\|(I-g)Au_n\|^2\\&\quad-(1-\gamma_n)\delta(1-\delta-\kappa_2)\|Sv_n-v_n\|^2+2\gamma_n\langle u-p,u_{n+1}-p\rangle\\&=(1-\gamma_n)\|u_n-p\|^2+\gamma_n\Big\{-(1-\gamma_n)\tau(1-\kappa_1-2\tau)\frac{\|u_n-f(u_n)\|^2}{\gamma_n}\\&\quad-(1-\gamma_n)\tau\mu(1-\kappa_3-2\tau\mu\|A\|^2)\frac{\|(I-g)Au_n\|^2}{\gamma_n}\\&\quad-(1-\gamma_n)\delta(1-\delta-\kappa_2)\frac{\|Sv_n-v_n\|^2}{\gamma_n}+2\langle u-p,u_{n+1}-p\rangle\Big\}.\end{aligned}\tag{3.29}$$

For all $n\ge 0$, write $b_n=\|u_n-p\|^2$ and

$$\begin{aligned}t_n&=2\langle u-p,u_{n+1}-p\rangle-(1-\gamma_n)\tau(1-\kappa_1-2\tau)\frac{\|u_n-f(u_n)\|^2}{\gamma_n}\\&\quad-(1-\gamma_n)\tau\mu(1-\kappa_3-2\tau\mu\|A\|^2)\frac{\|(I-g)Au_n\|^2}{\gamma_n}\\&\quad-(1-\gamma_n)\delta(1-\delta-\kappa_2)\frac{\|Sv_n-v_n\|^2}{\gamma_n}.\end{aligned}\tag{3.30}$$

According to (3.29), we have

$$b_{n+1}\le(1-\gamma_n)b_n+\gamma_n t_n.\tag{3.31}$$

Now, we show that $\limsup_{n\to\infty}t_n$ is bounded. First, we note that

$$t_n\le 2\langle u-p,u_{n+1}-p\rangle\le 2\|u-p\|\|u_{n+1}-p\|$$

which implies that $\limsup_{n\to\infty}t_n<+\infty$. Next, we prove $\limsup_{n\to\infty}t_n\ge -1$. If not so, there is a positive integer n_0 such that $t_n<-1$ for all $n\ge n_0$. Take into account of (3.31), we have $b_{n+1}\le b_n-\gamma_n$ when $n\ge n_0$. If follows that $b_{n+1}\le b_{n_0}-\sum_{i=n_0}^n\gamma_i$. So,

$$\limsup_{n\to\infty}b_{n+1}\le b_{n_0}-\limsup_{n\to\infty}\sum_{i=n_0}^{n}\gamma_i=-\infty,$$

which is impossible. Thus, $\limsup_{n\to\infty}t_n$ is bounded. At the same time, the sequence $\{u_n\}$ is bounded. Then, we can select a common subsequence $\{n_k\}\subset\{n\}$ such that $u_{n_k}\rightharpoonup u^*$ and

$$\begin{aligned}\limsup_{n\to\infty}t_n&=\lim_{k\to\infty}t_{n_k}\\&=\lim_{k\to\infty}\Big\{2\langle u-p,u_{n_k+1}-p\rangle-(1-\gamma_{n_k})\tau(1-\kappa_1-2\tau)\frac{\|u_{n_k}-f(u_{n_k})\|^2}{\gamma_{n_k}}\\&\quad-(1-\gamma_{n_k})\tau\mu(1-\kappa_3-2\tau\mu\|A\|^2)\frac{\|(I-g)Au_{n_k}\|^2}{\gamma_{n_k}}\\&\quad-(1-\gamma_{n_k})\delta(1-\delta-\kappa_2)\frac{\|Sv_{n_k}-v_{n_k}\|^2}{\gamma_{n_k}}\Big\}.\end{aligned}\tag{3.32}$$

Since the sequence $\{u_{n_k+1}\}$ is bounded, without loss of generality, suppose $\lim_{k\to\infty}\langle u - p, u_{n_k+1} - p\rangle$ exists. This together with (3.32) implies that

$$\lim_{k\to\infty}\left\{ -(1-\gamma_{n_k})\tau(1-\kappa_1-2\tau)\frac{\|u_{n_k}-f(u_{n_k})\|^2}{\gamma_{n_k}} - (1-\gamma_{n_k})\tau\mu(1-\kappa_3-2\tau\mu\|A\|^2) \times \frac{\|(I-g)Au_{n_k}\|^2}{\gamma_{n_k}} - (1-\gamma_{n_k})\delta(1-\delta-\kappa_2)\frac{\|Sv_{n_k}-v_{n_k}\|^2}{\gamma_{n_k}}\right\} \text{ exists.}$$

Therefore,

$$\begin{cases} \lim_{k\to\infty}\|u_{n_k}-f(u_{n_k})\| = 0, & (3.33)\\ \lim_{k\to\infty}\|(I-g)Au_{n_k}\| = 0, & (3.34)\\ \lim_{k\to\infty}\|Sv_{n_k}-v_{n_k}\| = 0. & (3.35)\end{cases}$$

Owing to $\|f(u_{n_k}) - w_{n_k}\| = \|\mu A^*(I-g)Au_{n_k}\|$, by (3.33) and (3.34), we deduce $\|u_{n_k} - w_{n_k}\| \to 0$ as $k\to\infty$. This together with (3.5) implies that $\|v_{n_k} - u_{n_k}\| \to 0$ as $k\to\infty$.

Since $I-f$, $I-S$ and $I-g$ are all demiclosed at zero, $u^* \in Fix(f)$ (by (3.33)), $u^* \in Fix(S)$ (by (3.35)) and $Au^* \in Fix(g)$ (by (3.34)). Hence, $u^* \in \Gamma$ and $\omega_w(u_n) \subset \Gamma$. □

Theorem 3.1. *Suppose that $I-f$, $I-S$ and $I-g$ are all demiclosed at zero. Then, the sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to $P_\Gamma(u)$.*

Proof. First, note that $\|u_{n_k+1} - u_{n_k}\| \to 0$ as $k\to\infty$. Then, $u_{n_k+1} \rightharpoonup u^* \in \Gamma$. It follows from (2.13) and (3.32) that

$$\limsup_{n\to\infty} t_n \le \lim_{k\to\infty} 2\langle u - P_\Gamma(u), u_{n_k+1} - P_\Gamma(u)\rangle = 2\langle u - P_\Gamma(u), u^* - P_\Gamma(u)\rangle \le 0.$$

Taking into account (3.29), we acquire

$$\|u_{n+1} - P_\Gamma(u)\|^2 \le (1-\gamma_n)\|u_n - P_\Gamma(u)\|^2 + 2\gamma_n\langle u - P_\Gamma(u), u_{n+1} - P_\Gamma(u)\rangle. \tag{3.36}$$

Combining (3.36) with Lemma 2.3, we conclude that $u_n \to P_\Gamma(u)$ as $n\to\infty$. □

Algorithm 3.2. Let $u \in H_1$ be a fixed point and u_0 be an initial point in H_1. Let the sequence $\{u_n\}$ be defined by the following way

$$\begin{cases} w_n = f(u_n) - \mu A^*(I-g)Au_n,\\ v_n = (1-\tau)u_n + \tau w_n,\\ u_{n+1} = \gamma_n u + (1-\gamma_n)v_n,\ n \ge 0.\end{cases}$$

Corollary 3.1. *Suppose that $I-f$ and $I-g$ are all demiclosed at zero. Then, the sequence $\{u_n\}$ generated by Algorithm 3.2 converges strongly to $P_{\Gamma_1}(u)$ where Γ_1 is the solution set of the split problem* (1.4).

Algorithm 3.3. Let $u \in H_1$ be a fixed point and u_0 be an initial point in H_1. Let the sequence $\{u_n\}$ be defined by the following way

$$\begin{cases} w_n = u_n - \mu A^*(I-g)Au_n,\\ v_n = (1-\tau)u_n + \tau w_n,\\ z_n = (1-\delta)v_n + \delta S v_n,\\ u_{n+1} = \gamma_n u + (1-\gamma_n)z_n,\ n \ge 0.\end{cases}$$

Corollary 3.2. *Suppose that $I-S$ and $I-g$ are all demiclosed at zero. Then, the sequence $\{u_n\}$ generated by Algorithm 3.3 converges strongly to $P_{\Gamma_2}(u)$ where $\Gamma_2 := \{x \in H_1 : x \in Fix(S)$ and $Ax \in Fix(g)\}$.*

4. Numerical examples

In this section, we give three numerical examples to illustrate the performance of our Algorithm 3.1. In all Example 4.1, Example 4.2 and Example 4.3, we take $\tau = 0.25, \mu = 0.005, \delta = 1/3$ and $\gamma_n = \frac{1}{n+3}, n \geq 0$.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^5$ and $f, g, S : \mathbb{R}^5 \to \mathbb{R}^5$ be three mappings. For every $u = (u_1, u_2, u_3, u_4, u_5)^T \in \mathbb{R}^5$, set

$$fu = \frac{1}{2}u, gu = \frac{1}{4}u, Su = (0, u_1, u_2, u_3, u_4)^T.$$

Clearly, those mappings f, g, S are 0-demicontractive. Suppose that

$$A = \begin{pmatrix} 7 & -3 & -5 & 2 & 1 \\ -2 & 4 & 2 & 4 & 2 \\ 6 & 3 & 2 & 5 & 4 \\ 2 & 1 & 3 & 1 & 2 \\ 5 & -3 & 2 & 1 & 2 \end{pmatrix}.$$

In this case, we see that $x = (0, 0, 0, 0, 0)^T$ is a solution to the problem (1.9). Let $\kappa(A)$ be the condition number of matrix A, then we have $\kappa(A) = 49.028$. For an initial point $x_0 = (-5, 1, 3, 2, 0)^T$, we take anchor $u = (0, 2, 5, 1, 4)^T$ and $u = (7, 4, 1, 3, 6)^T$ respectively. Now, we illustrate the results in TABLE 1 and FIGURE 1, FIGURE 2.

FIGURE 1. $u = (0, 2, 5, 1, 4)^T$

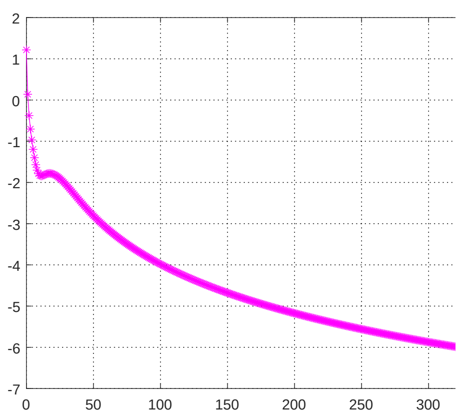

FIGURE 2. $u = (7, 4, 1, 3, 6)^T$

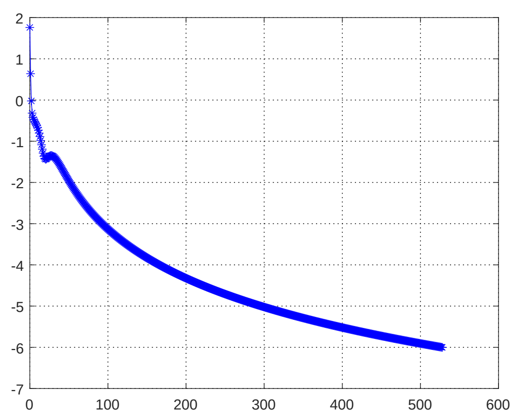

TABLE 1. Results of Example 4.1

error	$u = (0, 2, 5, 1, 4)^T$		$u = (7, 4, 1, 3, 6)^T$	
10^{-3}	CPU(s)	iter.	CPU(s)	iter.
10^{-3}	3.439	323	5.312	528

Example 4.2. In Example 4.1, let A be a fifth-order Hilbert matrix, i.e.,

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{pmatrix}.$$

In this case, our algorithm is still executable. Note that $x = (0,0,0,0,0)^T$ is a solution to the problem (1.9) and $\kappa(A) = 4.766 \times 10^5$. For an initial point $x_0 = (-5,1,3,2,0)^T$, we take anchor $u = (0,2,5,1,4)^T$ and $u = (7,4,1,3,6)^T$ respectively. Next, we illustrate the results in TABLE 2 and FIGURE 3, FIGURE 4.

FIGURE 3. $u = (0,2,5,1,4)^T$

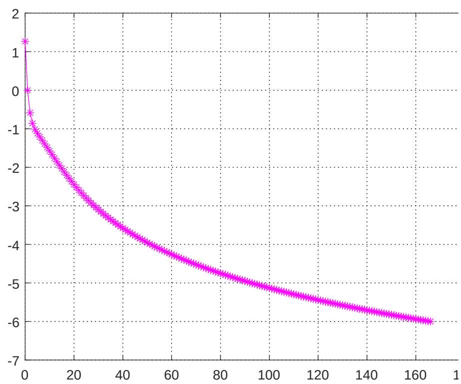

FIGURE 4. $u = (7,4,1,3,6)^T$

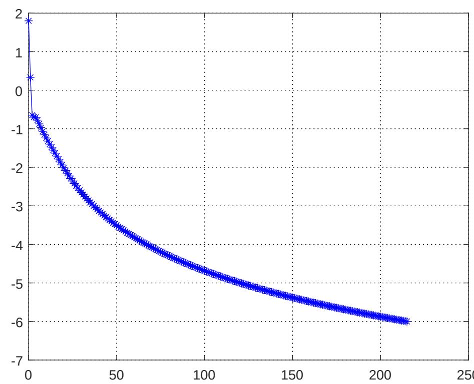

TABLE 2. Results of Example 4.2

$error$	$u = (0,2,5,1,4)^T$		$u = (7,4,1,3,6)^T$	
10^{-3}	CPU(s)	iter.	CPU(s)	iter.
10^{-3}	1.983	166	2.469	215

Example 4.3. Let $H_1 = H_2 = [-2,1]^5$ and $f, g, S : [-2,1]^5 \to [-2,1]^5$ be three mappings. For every $u = (u_1,u_2,u_3,u_4,u_5)^T \in [-2,1]^5$, set

$$fu = (-u_1^2 - u_1, -u_2^2 - u_2, -u_3^2 - u_3, -u_4^2 - u_4, -u_5^2 - u_5)^T,$$

$$gu = \frac{1}{4}u, Su = (0,u_1,u_2,u_3,u_4)^T.$$

We can check that the mapping f is $\frac{1}{3}$-demicontractive and the mappings g, S are 0-demicontractive. Suppose that

$$A = \begin{pmatrix} 7 & -3 & -5 & 2 & 1 \\ -2 & 4 & 2 & 4 & 2 \\ 6 & 3 & 2 & 5 & 4 \\ 2 & 1 & 3 & 1 & 2 \\ 5 & -3 & 2 & 1 & 2 \end{pmatrix}.$$

In this case, we see that $u = (0,0,0,0,0)^T$ is a solution to the problem (1.9). Let $\kappa(A)$ be the condition number of matrix A, then we have $\kappa(A) = 49.028$. For the initial point $x_0 = (-\frac{1}{2}, 1, \frac{3}{10}, \frac{1}{5}, 0)^T$, we take the anchor point $u = (0,2,5,1,4)^T$ and $u = (7,4,1,3,6)^T$, respectively. Now, we illustrate the result in TABLE 3 and FIGURE 5, FIGURE 6.

FIGURE 5. $u = (0,2,5,1,4)^T$

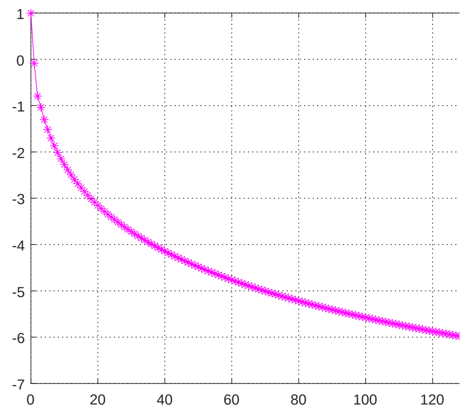

FIGURE 6. $u = (7,4,1,3,6)^T$

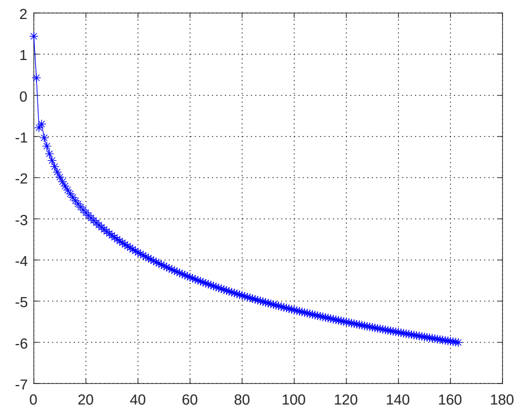

TABLE 3. Result of Example 4.3

$error$	$u = (0,2,5,1,4)^T$		$u = (7,4,1,3,6)^T$	
10^{-3}	CPU(s)	iter.	CPU(s)	iter.
10^{-3}	1.625	130	1.713	163

Acknowledgments. Li-Jun Zhu was supported in part by the National Natural Science Foundation of China [grant number 11861003], the Natural Science Foundation of Ningxia province [grant number NZ17015], the Major Research Projects of NingXia [grant numbers 2021BEG03049]and Major Scientific and Technological Innovation Projects of YinChuan [grant numbers 2022RKX03 and NXYLXK2017B09].

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