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A Family of Sequences which Converge to the Euler-Mascheroni Constant

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ABSTRACT. This article considers the Euler-Mascheroni constant (γ) as the limit of a continuous function. Graphs of the function shed light on some familiar approximations to γ, and lead to a family of sequences which converge to γ. It is shown that this family is an extension of convergent sequences suggested by Mortici.

1. INTRODUCTION

The Euler-Mascheroni constant γ is typically presented as

$$\lim_{n\to\infty}(H(n)-\log(n)),$$

where

$$H(n)=\sum_{k=1}^{n}\frac{1}{k}.$$

This representation then naturally leads to an infinite sequence $\{\gamma_n\}$ of approximations to γ, given by

$$\gamma_n=H(n)-\log(n)$$

for $n=1,2,\ldots$. The γ_n converge slowly to γ, and in fact, Euler showed that

$$\gamma_n=\gamma+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{1}{252n^6}+O(n^{-8}),$$

and thus $\{\gamma_n\}$ is a first-order convergent sequence. Two well-known simple improvements to γ_n are

$$\gamma_{n,a}=\gamma_n-\frac{1}{2n}=H(n)-\frac{1}{2n}-\log(n),$$

due to Vernescu [14], and often cited, and

$$\gamma_{n,b}=H(n)-\log(n+\frac{1}{2}),$$

due to DeTemple [5]. It is not difficult to show that these converge to γ with error $O(n^{-2})$, and so $\{\gamma_{n,a}\}$ and $\{\gamma_{n,b}\}$ are second-order convergent sequences.

The last 12 years has seen an explosion in the literature of alternative convergent sequences, with notable contributions by Mortici [9, 10, 11, 12], Cao et. al. [1], Chen and Mortici [2], Chen [3], Feng et. al. [6], Lu [7], Lu et. al. [8], Mortici and Chen [13], Yang [15], and You and Chen [16]. We do not have space to consider all of these, but Mortici's work is of particular relevance to this paper. Mortici [9] discusses

$$\mu_n(a,b,c)=\sum_{k=0}^{n-2}\frac{1}{a+k}+\frac{b}{a+n-1}-\log(\frac{a+n-1}{a}+c),$$

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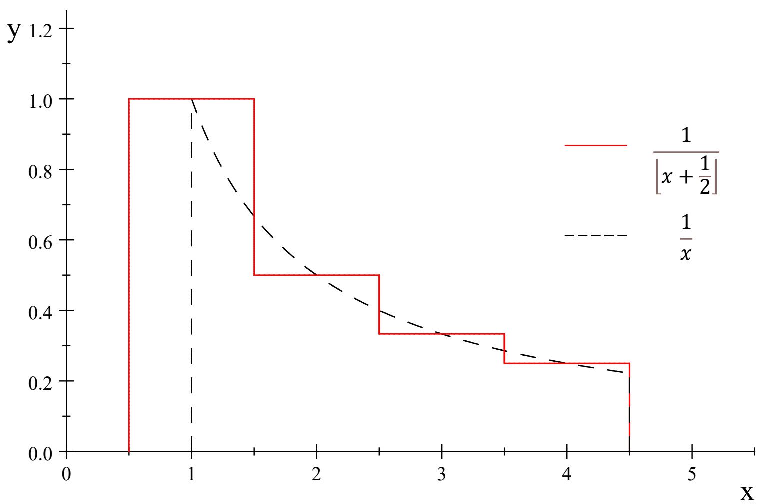
and shows that for $a = 1$ this converges to γ.

In this work, we investigate one of the many continuous function representations of γ. We show graphically that the familiar second-order convergent sequences of Vernescu and DeTemple are members of a family of second-order convergent sequences described by Mortici [9], and, interestingly, are locally the least optimal members of this family. We consider other members of the family, and derive the two most optimal members, which are shown to be sequences suggested by Mortici [9]. We then extend the Mortici sequences to a family of higher-order convergent sequences, obtaining some novel approximations to γ.

2. The family of convergent sequences.

Consider Figure 1, with the familiar steps of the harmonic series and the reciprocal function $1/x$ superimposed on top of the steps. We "shift" the steps slightly to the left, so that the step of height $1/1$ is centered on 1.

Figure 1: The reciprocal function and the shifted harmonic series



Now define the function

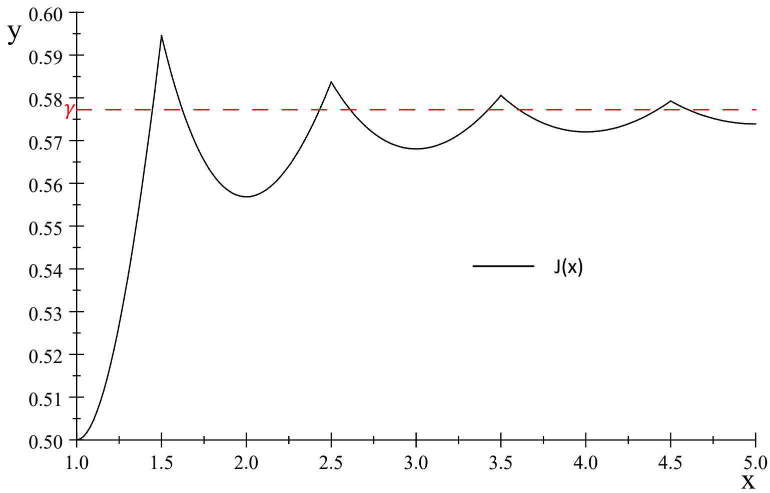
$$J(x) = \int_{1/2}^{x} \frac{1}{\lfloor t + 1/2 \rfloor} dt - \log x,$$

which for $x \geq 1$ represents the integral from $1/2$ to x of the step function Figure 1, minus the integral from 1 to x of $1/x$. Note we can also easily show that

$$J(x) = H(n-1) + \frac{x - n + 1/2}{n} - \log x,$$

for $x \in (n - 1/2, n + 1/2], n = 2, 3, \ldots$. The graph of J is shown in Figure 2.

Figure 2: The function $J(x)$



Clearly J oscillates around γ. It is convenient to focus attention on a single cycle of J by defining

$$J_n(x) = J(n+x) = H(n-1) + \frac{x+1/2}{n} - \log(n+x),$$

where $n = 2, 3, ...$ and $x \in (-1/2, 1/2]$. We can now use J_n to define a family of approximations to γ.

Definition 2.1. Let $\mathcal{X}$ be the set of all real functions $x(n)$ where $n \in \{2, 3, ...\}$ and $-1/2 < x(n) \leq 1/2$.

Definition 2.2. For any $x \in \mathcal{X}$ we can define a sequence $\{J_n(x(n))\}$. Let $\mathcal{J}$ be the set of such sequences.

Note that if we let $x(n) = k$ for a fixed k the resulting sequences are special cases of Mortici's [9] suggested family of approximations to γ, with $a = 1$, given by

$$\mu_n(1, b, c) = H(n-1) + \frac{b}{n} - \log(n+c),$$

where we have $c = x, b = x + 1/2$. A key difference between Mortici's formulation and ours is that we insist that the convergent sequences must be on the line $J(x)$, so this forces $b = c - 1/2$.

We can now show some simple results concerning J, and $\mathcal{J}$.

Theorem 2.1. *The two simple approximations $\gamma_{n,a}, \gamma_{n,b}$ are members of $\mathcal{J}$.*

Proof. Let $x(n) = 0$, $\forall n$, then we have $J_n(0) = \gamma_{n,a} = H(n) - \frac{1}{2n} - \log(n)$. Similarly, choosing $x(n) = 1/2$, $\forall n$, $J_n(1/2) = \gamma_{n,b} = H(n) - \log(n + \frac{1}{2})$. □

(Note that this theorem is implied by Mortici [9] in the definition of his μ_n.)

Theorem 2.2. *The local maxima of $J(x)$ are located at $x = 1.5, 2.5, 3.5, ...$ and the local minima of $J(x)$ are located at $x = 2, 3, ...$*

Proof. First, note that the theorem is equivalent to stating that the local maxima of $J_n(x)$, occur at $x = 1/2$, and that the local minima of $J_n(x)$ occur at $x = 0$. Note that $J_n(x)$ is differentiable $\forall x \in (-1/2, 1/2)$, with

$$J_n^{'}(x) = \frac{1}{\lfloor n + x + 1/2 \rfloor} - \frac{1}{n+x} = \frac{1}{n} - \frac{1}{n+x}.$$

So for $x \in (-1/2, 0)$, $J_n^{'}(x) < 0$, and for $x \in (0, 1/2)$, $J_n^{'}(x) > 0$. This establishes the result for the local minima. Also $\lim_{x \to 1/2^-} J_n^{'}(x) > 0$, and $\lim_{x \to 1/2^+} J_n^{'}(x) < 0$, and thus the local maxima result is established. ' □

Note that the above theorem justifies the remark in the introduction that the well known second-order convergent sequences are locally least optimal.

Now define $\gamma_{n,t} = J_n(t)$, where $t \in (-1/2, 1/2]$, that is we are choosing $x(n) = t$.

Theorem 2.3. *All sequences $\{\gamma_{n,t}\}$ are second-order convergent sequences apart from $t = \pm 6^{-1/2}$, which are third-order convergent sequences.*

Proof. This theorem is a special case of Theorem 2.1 of Mortici [9], specifically parts (ii) and (iii) of Mortici's theorem. □

The third-order convergent sequences from above are

$$\gamma_{n,t_1} = H(n-1) + \frac{1/2 - 6^{-1/2}}{n} - \log(n - 6^{-1/2}),$$

and

$$\gamma_{n,t_2} = H(n-1) + \frac{1/2 + 6^{-1/2}}{n} - \log(n + 6^{-1/2}).$$

These are of course precisely the same sequences as the ones proposed by Mortici [9]. Since these have opposite signs on the third-order term, and the same magnitude, their average

$$\gamma_{n,A} = H(n) - \frac{1}{2n} - \frac{1}{2}\log(n^2 - \frac{1}{6})$$

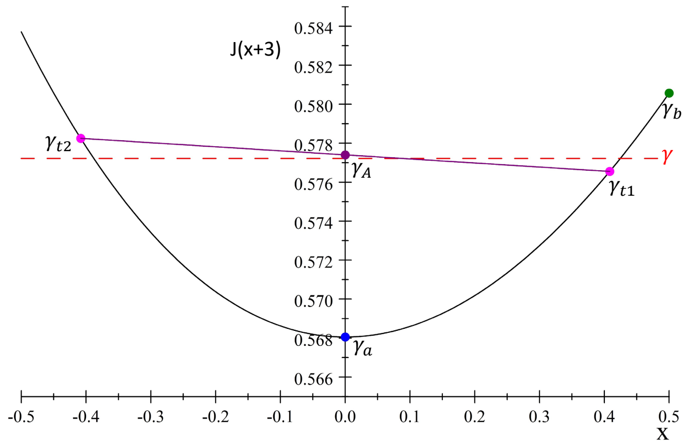
satisfies

$$\gamma_{n,A} = \gamma + O(n^{-4}),$$

and thus is fourth-order. The sequence $\{\gamma_{n,A}\}$ was also proposed by Mortici [9]. This last sequence is not however a member of the family $\mathcal{J}$.

Figure 3 below shows J_3 together with marked values showing $\gamma_{3,a}, \gamma_{3,b}, \gamma_{3,t_1}, \gamma_{3,t_2}$ and $\gamma_{3,A}$. The superiority of the third- and fourth-order convergent sequences is visually demonstrated in the plot.

Figure 3: The second, third and fourth-order convergent sequences for $n = 3$



3. AN EXTENSION OF THE FAMILY.

In this section we investigate members of the family $\mathcal{J}$ where the function $x(n)$ is not constant. Specifically, we choose

$$x(n) = f_M(n, \mathbf{t}) = \sum_{m=0}^{M} \frac{t_m}{n^m},$$

where $\mathbf{t} = (t_0, t_1, ..., t_M)$ and thus we can define a convergent sequence with members

$$\gamma_{n,M}(\mathbf{t}) = J_n(n + f_M(n, \mathbf{t})) = H(n-1) + \frac{f_M(n, \mathbf{t}) + 1/2}{n} - \log(n + f_M(n, \mathbf{t})).$$

We can view the convergent sequences produced by this approach as an extension of the third-order convergent sequences of Mortici to higher orders. Now one can choose the number of terms M and the particular values of $\mathbf{t}$ in order to remove different orders in the approximation. This is best seen by writing

$$\gamma_{n,M}(\mathbf{t}) = \gamma_n - \frac{1}{2n} + \sum_{m=0}^{M} \frac{t_m}{n^{m+1}} - \log(1 + \sum_{k=0}^{M} \frac{t_m}{n^{m+1}}).$$

Now use

$$\gamma_n - \frac{1}{2n} = \gamma - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}},$$

where B_{2k} are the Bernoulli numbers (e.g., Dence and Dence, [4]), and expand the log terms to get

$$\gamma_{n,M}(\mathbf{t}) - \gamma = -\sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}} + \sum_{r=2}^{\infty} (-1)^r \frac{(\sum_{m=0}^{M} \frac{t_m}{n^{m+1}})^r}{r}.$$

Now we can write

$$\gamma_{n,M}(\mathbf{t}) - \gamma = \sum_{r=1}^{\infty} \frac{g_{r,M}(\mathbf{t})}{n^r},$$

where the $g_{r,M}(\mathbf{t})$ are polynomials in $\mathbf{t} = (t_0, ..., t_M)$. Some conventional algebra can be used to derive the $g_{r,M}(\mathbf{t})$. For $M \geq 3$, the first few of these are

$$g_{2,M}(\mathbf{t}) = \frac{t_0^2}{2} - \frac{1}{12},$$

$$g_{3,M}(\mathbf{t}) = t_0 t_1 - \frac{t_0^3}{3},$$

$$g_{4,M}(\mathbf{t}) = \frac{t_0^4}{4} - t_0^2 t_1 + t_2 t_0 + \frac{t_1^2}{2} + \frac{1}{120},$$

$$g_{5,M}(\mathbf{t}) = t_0^3 t_1 - \frac{t_0^5}{5} - t_2 t_0^2 - t_0 t_1^2 + t_3 t_0 + t_2 t_1.$$

For a given M we can choose the $\mathbf{t}$ so as to make the $g_{r,M}(\mathbf{t}) = 0$, for $r = 2, ...M-2$, solving the equations sequentially and so produce a convergent sequence of order $n^{-(M+3)}$. The solutions come in pairs. As an example, for $M = 3$ the two solutions for $\mathbf{t} = (t_0, t_1, t_2, t_3)$ are $\mathbf{t} = (\frac{1}{\sqrt{6}}, \frac{1}{18}, -\frac{49\sqrt{6}}{6480}, -\frac{11}{1944})$ and $\mathbf{t} = (-\frac{1}{\sqrt{6}}, \frac{1}{18}, \frac{49\sqrt{6}}{6480}, -\frac{11}{1944})$.

For a given M, we denote the $f_M(n, \mathbf{t})$ with these substituted values for $\mathbf{t}$ by $f_{M,1}$ and $f_{M,2}$. For $0 \leq M \leq 3$, the functions are

$$f_{0,1}(n) = \frac{1}{\sqrt{6}},$$

$$f_{0,2}(n) = -\frac{1}{\sqrt{6}},$$

$$f_{1,1}(n) = \frac{1}{\sqrt{6}} + \frac{1}{18n},$$

$$f_{1,2}(n) = -\frac{1}{\sqrt{6}} + \frac{1}{18n},$$

$$f_{2,1}(n) = \frac{1}{\sqrt{6}} + \frac{1}{18n} - \frac{49\sqrt{6}}{6480n^2},$$

$$f_{2,2}(n) = -\frac{1}{\sqrt{6}} + \frac{1}{18n} + \frac{49\sqrt{6}}{6480n^2},$$

$$f_{3,1}(n) = \frac{1}{\sqrt{6}} + \frac{1}{18n} - \frac{49\sqrt{6}}{6480n^2} - \frac{11}{1944n^3},$$

and

$$f_{3,2}(n) = -\frac{1}{\sqrt{6}} + \frac{1}{18n} + \frac{49\sqrt{6}}{6480n^2} - \frac{11}{1944n^3}.$$

Now we can substitute these into $\gamma_{n,M}$ to give convergent sequences,

$$\gamma^*_{n,M,i} = H(n-1) + \frac{f_{M,i}(n) + 1/2}{n} - \log(n + f_{M,i}(n)),$$

where $i = 1, 2$. For $M = 0$ the two sequences are Mortici's third-order convergent sequences, $\gamma_{n,t_1}, \gamma_{n,t_2}$. For $M > 0$ these are novel sequences, and can be viewed as extensions of Mortici's third-order convergent sequences.

It is straightforward to find the leading coefficient in the error expansion for these sequences, and to show that

$$\lim_{n\to\infty} n^3(\gamma - \gamma^*_{n,0,1}) = -\frac{\sqrt{6}}{108},$$

$$\lim_{n\to\infty} n^3(\gamma - \gamma^*_{n,0,2}) = \frac{\sqrt{6}}{108},$$

(both of the above are shown by Mortici [9])

$$\lim_{n\to\infty} n^4(\gamma - \gamma^*_{n,1,1}) = \lim_{n\to\infty} n^4(\gamma - \gamma^*_{n,1,2}) = \frac{49}{6480},$$

$$\lim_{n\to\infty} n^5(\gamma - \gamma^*_{n,2,1}) = \frac{11\sqrt{6}}{11664},$$

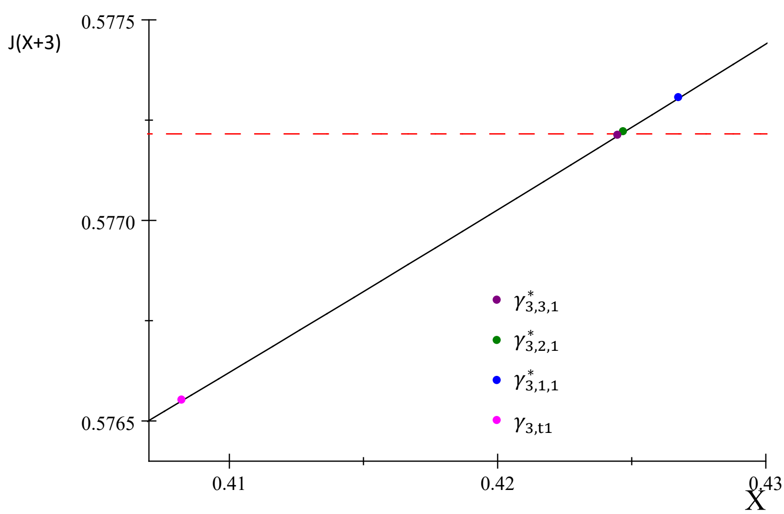
$$\lim_{n\to\infty} n^5(\gamma - \gamma^*_{n,2,2}) = -\frac{11\sqrt{6}}{11664},$$

and

$$\lim_{n\to\infty} n^6(\gamma - \gamma^*_{n,3,1}) = \lim_{n\to\infty} n^6(\gamma - \gamma^*_{n,3,2}) = -(\frac{\sqrt{6}}{1296} + \frac{144251}{32659200}).$$

Figure 4 shows, for $n = 3$, third, fourth, fifth, and sixth order coefficients, plotted on the line $y = J_3(x)$. Note the sixth-order coefficient ($\gamma^*_{3,3,1}$) is almost indistinguishable from the fifth-order coefficient ($\gamma^*_{3,2,1}$).

Figure 4: Third, fourth, fifth and sixth-order coefficients for $n = 3$



For the cases with even M, the leading error terms are of opposite sign, so that the average of the two convergent sequences produces a convergent sequence of $O(n^{-(M+4)})$. The case of $M = 0$ leads to the coefficient $\gamma_{n,A}$ proposed by Mortici, and discussed in the previous section. The averages of the other even M cases can then be viewed as extension of

Mortici's $\gamma_{n,A}$. The first of these is

$$\gamma_{n,B} = H(n-1) + \frac{\frac{1}{2} + \frac{1}{18n}}{n} - \frac{1}{2}\log(n^2 + \frac{59}{3240n^2} - \frac{2401}{6998400n^4} - \frac{1}{18}),$$

which is sixth-order.
As an illustration of the numerical results, consider the case for $n = 3$, shown in table 1.

Table 1: Convergent sequence values for the case $n = 3$

Convergent sequence	order	value	error $\times 10^6$
γ_3 (Euler)	1st	$0.734\,721$	158000
$\gamma_{3,a}$ (Vernescu, [14])	2nd	$0.568\,054$	-9160
$\gamma_{3,b}$ (DeTemple, [5])	2nd	$0.580\,570$	3350
$\gamma_{3,t_1}, \gamma_{3,t_2}$ (Mortici, [9])	3rd	$0.576\,551, 0.578\,250$	$-665, 1030$
$\gamma_{3,A}$ (Mortici, [9])	4th	$0.577\,400$	185
$\gamma^*_{3,1,1}, \gamma^*_{3,1,2}$	4th	$0.577\,305, 0.577\,303$	$89, 87$
$\gamma^*_{3,2,1}, \gamma^*_{3,2,2}$	5th	$0.577\,201, 0.577\,219\,819$	$-14.9, 4.2$
$\gamma^*_{3,3,1}, \gamma^*_{3,3,2}$	6th	$0.577\,211\,15, 0.577\,211\,24$	$-4.5, -4.4$
$\gamma_{3,B}$	6th	$0.577\,210$	-5.3

The basic 1st order convergent sequence gives 0.734, not even correct to one decimal place. The two second order convergent sequences give values of 0.568 and 0.5806, each correct in the first decimal place. The two third order convergent sequences, due to Mortici, give 0.57825 and 0.57655, both correct to two decimal places. The 4th order convergent sequence, the average of the two above is 0.577400, correct to three decimal places. The 5th order convergent sequences give 0.577200875, and 0.577219819,, correct to 4 and 5 decimal places respectively, and finally the 6th order convergent sequence values are all correct to the 5th decimal place.

4. Concluding remarks.

This paper gives a graphical demonstration of how the familiar second-order convergent sequences of Vernescu [14] and DeTemple [5], and the third-order convergent sequence of Mortici [9] belong to a family of convergent sequences for γ. In addition, it is shown in the paper how the convergent sequences of Mortici can be extended to a family of higher-order convergent sequences for γ.

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