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Equilibrium problems and proximal algorithm using tangent space products

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ABSTRACT. This paper presents equilibrium problems and their regularized problems in the framework of Hadamard spaces. Using the concept of tangent space products, we introduce a resolvent operator and deduce essential properties in relation to equilibrium problem. Furthermore, we analyze the regularized problems involving resolvent operators. Finally, we establish two convergence results concerning proximal algorithms.

1. INTRODUCTION

A substantial generalization of convex optimization and variational inequality problems is known as *equilibrium problem*. This problem can be traced to [3] and is to find a point

$$\widetilde{x} \in K \text{ such that } F(\widetilde{x}, y) \geq 0 \text{ for every } y \in K, \tag{1.1}$$

where F is a bifunction $F : K \times K \to \mathbb{R}$. In the sequel we shall, denote the problem in (1.1) by $EP(K,F)$ and its solution set $E(K,F)$. Several researchers have considered $EP(K,F)$ in various setting, mostly Hilbert and Banach spaces.

One of the fundamental approaches of approximating the solution of (1.1) is the proximal algorithm, which is triggered from convex optimization problem ranging from finite dimentional spaces to infinite spaces. The proximal algorithm mainly regularizes (or perturbs) the objective function to generate another version that is consistent under some desired assumption. This type of algorithm can be traced to Martinet [21] and Rockafellar [28] mainly in the setting of linear spaces. However, it is known that Hadamard spaces are suitable in handling most optimization problems since many non-convex and non-smooth problems can be viewed as convex and smooth problems [30].

In [1], Bačák introduced proximal algorithm for approximating a minimizer of a convex functional in the setting of Hadamard spaces. Later on in [13], the proximal algorithm has been extended to estimating solution of problems of variational inequalities. Thereafter the method is considered for monotone operators in [14, 20] in the same setting. Although equilibrium problems have been studied in Hadamard manifolds in [10] and later on extended to equilibrium problems for bifunctions defined on proximal pairs in [8], the results in [10] and [8] rely on different variants of the KKM lemma (see [18]). Relevant tools used in establishing such results can be found in [24, 25, 26, 12].

In [16, 17], Kimura and Kishi studied the equilibrium problem using KKM principle together with certain conditions. The authors established that the resolvent operator associated to a bifunction is well-defined and firmly non-spreading. Furthermore, they established fixed points characterization of the resolvent operator to the equilibrium point and obtained some convergence results. Later on, existence and approximation algorithm for equilibrium problems in Hadamard spaces has been analysed using quasilinearization

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with standard assumptions in [9] and the KKM principle has been obtain with weaker assumptions incomparison with the existing one.

On the other hands, the concept of monotone vector fields and generalized gradient flows was considered in the framework of Hadamard spaces for better geometric and linear description of the spaces [7]. This concept is substantial and its resulted to further studies up to optimization problems of finite mappings (see, e.g., [29] and the references therein).

The main purpose of this work is to incorporate the notion of monotone vector fields, developed in [7], into the results of Chaipunya et. al. [9]. This directly yield that the results hold in all Hadamard spaces whether from geometrical point of view or from the geodesic linearity properties. In addition to that, we establish strong convergence of a sequence generated by shrinking projection to a solution of (1.1). The algorithm is fashioned after that of [19] for a common fixed point of a finite family of quasi-nonexpansive mappings in an Hadamard spaces whose half spaces are convex.

The paper is organized as follows: the next section contains basic knowledge that will be essential throughout the rest of this work and auxiliary results, followed by the main results.

2. Preliminaries

The following definitions and lemmas will be needed in the proof of our main theorem.

A metric space (X, d) is said to be geodesic if for every pair of points, say, $x, y \in X$, there exists a mapping $\gamma : [0, l] \to X$ (with $l \geq 0$) and a constant $K \geq 0$ such that $\gamma(0) = x$, $\gamma(l) = y$, and $d(\gamma(t), \gamma(t')) = K\,|t - t'|$ for all $t, t' \in [0, l]$. The curve γ is called a geodesic joining x, y. The notations $\gamma_{x,x}$, $\gamma_{x,y}$ will be used to denote zero normalized geodesic at x and the nonzero geodesic joining x and y with $x \neq y$ respectively. In this paper we will use $\gamma_{x,y}(t)$ to denote the point $(1-t)x \oplus ty$ on the geodesic $\gamma_{x,y}$ where $t \in [0, 1]$. Also we used $[x, y]$ to denote the image of $\gamma_{x,y}$ over the interval $[0, 1]$. A subset $C \subset X$ is said to be convex if $[x, y] \subset C$ for all $x, y \in C$.

If we let $(E^2, \langle \cdot, \cdot \rangle)$ to be the Euclidean plane with usual inner product $\langle u, v \rangle := u^\top v$ and the Euclidean norm $\|u\|^2 := \langle u, u \rangle$, for $u, v \in E^2$. Then, for $p, q, r \in X$, the geodesic triangle $\Delta \subset X$ is defined by $\Delta(p, q, r) := [p, q] \cup [q, r] \cup [r, p]$., The triangle defined by $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r}) := \Delta(\bar{p}, \bar{q}, \bar{r})$ with $\bar{p}, \bar{q}, \bar{r} \in E^2$ is said to be a Euclidean comparison (or simply comparison) triangle of Δ, if $\|\bar{p} - \bar{q}\| = d(p, q)$, $\|\bar{q} - \bar{r}\| = d(q, r)$, and $\|\bar{r} - \bar{p}\| = d(r, p)$. If $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$ is the comparison triangle of $\Delta(p, q, r)$, the point $\bar{u} \in [\bar{p}, \bar{q}]$ is called the comparison point of $u \in [p, q]$ if $\|\bar{p} - \bar{u}\| = d(p, u)$.

Definition 2.1. A geodesic metric space (X, d) is said to be a $CAT(0)$ if for each geodesic triangle $\Delta \subset X$, and two points $u, v \in \Delta$ the following $CAT(0)$ inequality holds.

$$d(u, v) \leq \|\bar{u} - \bar{v}\|, \tag{2.2}$$

where $\bar{u}, \bar{v} \in \bar{\Delta}$ are the comparison points of u and v respectively, and $\bar{\Delta} \in E^2$ is a comparison triangle of Δ. A complete $CAT(0)$ space is called an Hadamard Space.

Definition 2.2. A function $F : X \to (-\infty, \infty]$ is called convex if

$$F((1-t)x \oplus ty) \leq (1-t)F(x) + tF(y) \tag{2.3}$$

for any $x, y \in X$ and $t \in (0, 1)$.

The following propositon gives us an important characterisation of $CAT(0)$ spaces

Proposition 2.1. *Suppose that (X, d) is a geodesic metric space. Then the following conditions are equivalent:*

(1) X is a $CAT(0)$ space.

(2) For all $v \in X$ and a normalized geodesic $\gamma : [0,1] \to X$, the following (CN) inequality holds for any $t \in [0,1]$:

$$d^2(\gamma(t), v) \leq (1-t)d^2(\gamma(0), v) + td^2(\gamma(1), v) - t(1-1)d^2(\gamma(0), \gamma(1)). \tag{2.4}$$

(3) for all $x, y, u, v \in X$, the following inequality holds;

$$d^2(x, v) + d^2(y, u) \geq d^2(x, u) + d^2(y, v) + 2d(x, y)d(u, v). \tag{2.5}$$

In the sequel, we will always assume that (X, d) is an Hadamard space and recall that X is uniquely geodesic. Although tangent spaces to a given $CAT(0)$ space were introduced in [27] (see also [5, 4]), in [7] some slight modifications were made on their representations for technical conveniences.

Let $p, q, r \in X$, then the comparison angle between q and r at p, denoted by $\bar{\measuredangle}_p(q, r)$, is given for $q, r \in X \setminus \{p\}$ by, set

$$\cos \bar{\measuredangle}_p(q, r) := \frac{\langle \bar{q} - \bar{p}, \bar{r} - \bar{p} \rangle}{\|\bar{q} - \bar{p}\| \|\bar{r} - \bar{p}\|}, \tag{2.6}$$

where $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$ is the comparison triangle of $\Delta(p, q, r)$, we let $\bar{\measuredangle}_p(p, q) := 0$, and $\bar{\measuredangle}_p(p, r) = \bar{\measuredangle}_p(r, p) := \frac{\pi}{2}$ for $r \in X \setminus \{p\}$.

Let γ_1, γ_2 be two geodesics directing from a common point $p \in X$, then the Alexandrov angle between the goedesics γ_1 and γ_2 is given by

$$\alpha_p(\gamma_1, \gamma_2) := \lim_{s,t \to 0^+} \cos \bar{\measuredangle}_p(\gamma_1(s), \gamma_2(t)). \tag{2.7}$$

Alternatively, one can compute the Alexandrov angle by using the First Variation Formula. which is given in the following proposition

Lemma 2.1. *(First Variation Formula). Suppose that $p \in H, u \in H \setminus \{p\}$, and γ is a nonzero unit-speed geodesic issuing from p. Then the following identity holds:*

$$\lim_{s \to 0^+} \frac{d(u, p) - d(u, \gamma(s))}{s} = \cos \alpha_p(\gamma_{p,u}, \gamma). \tag{2.8}$$

Recall that, if $(\widetilde{M}, \widetilde{\rho})$ is a pseudometric space, then its metric identification is the metric space (M, ρ) where M consist of equivalence classes $[u] := \{v \in \widetilde{M} \mid \widetilde{\rho}(u, v) = 0\}$ of $u \in \widetilde{M}$ and $\rho([u], [v]) := \widetilde{\rho}(u, v) \forall [u], [v] \in M$. Let $\widetilde{S}_p$ be the set of all geodesic issuing from a point $p \in X$, then $(\widetilde{S}_p, \widetilde{\measuredangle}_p)$ where $\widetilde{\measuredangle}_p := \alpha_p$ is a pseudometric metric space, the metric identification of $(\widetilde{S}_p, \widetilde{\measuredangle}_p)$ which is denoted by $(S_p, \measuredangle_p)$ is called the space of directions. For the remainder of this paper we will denote the elements of S_p by $\gamma := [\gamma]$.

Now, let us define an equivalence relation $\sim$ on $[0, \infty) \times S_p$ by $(t_1, \gamma_1) \sim (t_2, \gamma_2)$ if and only if one of the followings is satisfied:

(T1): $t_1 \zeta(\gamma_1) = t_2 \zeta(\gamma_2) = 0$ or
(T2): $t_1 \zeta(\gamma_1) = t_2 \zeta(\gamma_2) > 0$ and $\gamma_1 = \gamma_2$.

Let $T_pX := ([0, \infty) \times S_p)/\sim$ and the elements of T_pX be denoted by $\dot{\gamma} \equiv [(t, \gamma)]$. Then the function d_p defined by

$$d_p(\dot{\gamma}_1, \dot{\gamma}_2) = \sqrt{t_1^2 \zeta(\gamma_1) + t_2^2 \zeta(\gamma_2) - 2t_1 t_2 \zeta(\gamma_1)\zeta(\gamma_2) \cos \measuredangle_p(\gamma_1, \gamma_2)}$$

is a metric on T_pX (Proposition 2.12 [7]). Also let $\mathbb{T}_pX := ([0,\infty) \times S'_p)/\approx$, where $S'_p = \{\gamma \in S_p \mid \zeta(\gamma) = 1\}$ and $\approx := \sim\restriction \mathbb{T}_pX$. Now, let

$$\begin{cases} X_+ := \{[(t,\gamma)]_\sim \mid t > 0, \zeta(\gamma) = 1\}; \\ X_0 := \{[(0,\gamma)]_\sim\} = \{\{(t,\gamma) \mid t = 0 \vee \zeta(\gamma) = 0\}\}; \\ X'_0 := \{[(0,\gamma)]_\approx\} = \{\{(t,\gamma) \mid t = 0, \zeta(\gamma) = 1\}\}. \end{cases}$$

Then T_pX and $\mathbb{T}_pX$ can be represented as

$$T_pX = X_+ \cup X_0 \text{ and } \mathbb{T}_pX = X_+ \cup X'_0 \tag{2.9}$$

It was shown in [5] and [4] that $\mathbb{T}_pX$ is a metric space with respect to the metric D_p given by

$$D_p([t_1,\gamma_1]_\approx, [t_2,\gamma_2]_\approx) = \sqrt{t_1^2 + t_2^2 - 2t_1t_2\cos\measuredangle_p(\gamma_1,\gamma_2)}$$

for all $[t_1,\gamma_1]_\approx, [t_2,\gamma_2]_\approx \in \mathbb{T}_pX$. The metric space T_pX is shown to be isometric to $\mathbb{T}_pX$ in [7] (Proposition 2.13). In fact if X is a Hilbert space (Hadamard manifold and Tan_pX is the Riemannian tangent space) then T_pX is isometric to X (Tan_pX) (see [7] Proposition 2.14 and Corollary 2.15).

The metric space (T_pX, d_p) is called the tangent space of X at p. The tangent bundle of X is defined by $TX := \bigcup_{p\in X} T_pX$. On T_pX we write $0_p := \dot\gamma_{p,p}$ and $\|\dot\gamma\|_p := d_p(0p, \dot\gamma) = t\zeta(\gamma)$. We will use $\mathbf{0} := \{0_p \mid p \in X\}$ to denote the zero section of TX. Moreover, we let

$$G_p[\dot\gamma_1, \dot\gamma_2] := \frac{1}{2}\left[\|\dot\gamma_1\|_p^2 + \|\dot\gamma_2\|_p^2 - d_p^2(\dot\gamma_1,\dot\gamma_2)\right]$$

for any $\dot\gamma_1, \dot\gamma_2 \in T_pX$. An analogue of Cauchy Schwarz inequality can be deduce from G_p by direct calculation. That is,

$$G_p[\dot\gamma_1,\dot\gamma_2] = t_1t_2\zeta(\gamma_1)\zeta(\gamma_2)\cos\measuredangle_p(\gamma_1,\gamma_2) \le \|\dot\gamma_1\|_p\|\dot\gamma_2\|_p.$$

Next, we consider two types of convergence that we will use in this paper. the first one is Δ-convergence. Let $\{x^n\}$ be a bounded sequence in X. For $x \in X$, let the function $r(\cdot;\{x^n\}) : X \to [0,\infty]$

$$r(x,\{x^n\}) = \limsup_{n\to\infty} d(x,x^n)$$

The functional $r(\cdot,\{x^n\})$ is shown to have a unique minimizer (see [11]). The unique minimizer here is called the asymptotic center of $\{x^n\}$. Moreover, a point $\bar x \in X$ is called the Δ-limit of $\{x^n\}$, written $x^n \xrightarrow{\Delta} \bar x$, if it is the asymptotic center of $\{x^n\}$ as well as of all its subsequences. Likewise, a point $\bar x$ is called a Δ-accumulation point of $\{x^n\}$ if $\{x^n\}$ contains a subsequence which is Δ-convergent to $\bar x$. A functional $h : X \to \mathbb{R}$ is said to be Δ-upper semicontinuous (briefly, Δ-usc) if $\limsup_{n\to\infty} h(x^n) \le h(\bar x)$ whenever the sequence $\{x^n\}$ in X satisfies $x^n \xrightarrow{\Delta} \bar x$. The following fundamental results regarding Δ-convergent sequences are required for our main results.

Theorem 2.1 ([11]). *Let K be a closed convex subset of an Hadamard space X and $\{x^n\}$ be a bounded sequence in K. Then, the asymptotic center of $\{x^n\}$ is contained in K.*

Definition 2.3. A sequence $\{x^n\}$ in X is said to be Fejér monotone with respect to a nonempty set $K \subset X$ if for each $x \in K$, we have $d(x^{n+1}, x) \le d(x^n, x)$ for all $k \in \mathbb{N}$.

Proposition 2.2 ([9]). *Suppose that $\{x^n\}$ in X is a Fejér monotone with respect to a nonempty set $K \subset X$. Then the following conditions are equivalent:*

(1) $\{x^n\}$ is bounded.

(2) $\{d(x, x^n)\}$ converges for any $x \in K$.

(3) *If every $\Delta-$accumulation point of $\{x^n\}$ lie within K, then $\{x^n\}$ is $\Delta-$convergent to an element in K.*

Given a nonempty closed convex subset K of an Hadamard space X, the metric projection onto K, denoted by P_n, is defined by $P_n x := \mathrm{argmin}_{y\in K} d(x,y)$. This projection map is a nonexpensive mapping (see [4]).

Another type of convergence that we will use is the $\Delta-$Mosco convergence introduced by Kimura [15]. Let $\{K_n\}$ be a sequence of closed convex subsets of an Hadamard space X. Then we define $d - Li_n K_n$ and $\Delta - Ls_n K_n$ by

- $x \in d - Li_n K_n$ if and only if there exists $\{x^n\} \subset X$ such that $\{d(x^n, x)\}$ converges to 0 and that $x^n \in K_n$ for all $n \in \mathbb{N}$;
- $y \in \Delta - Ls_n K_n$ if and only if there exist a sequence $\{y^i\} \subset X$ and a sub-sequence $\{n_i\}$ of N such that $\{y^i\}$ has an asymptotic center $\{y\}$ and that $y^i \in K_{n_i}$ for all $i \in \mathbb{N}$.

If $K_0 = d - Li_n K_n = \Delta - Ls_n K_n$ we say that $\{K_n\}$ converges to K_0 in the sense of $\Delta-$Mosco, and we write $K_0 = \Delta M - \lim_{n\to\infty} K_n$.

It is shown in ([15]) that if $\{K_n\}$ is a sequence of closed and convex subsets of X, then the set $\Delta M - \lim_{n\to\infty} K_n$ is also closed and convex. If $\{K_n\}$ is a decreasing sequence of nonempty closed convex subsets with respect to set inclusion then $\Delta M - \lim_{n\to\infty} K_n = \bigcap_{n=1}^{\infty} K_n$.

Theorem 2.2 (see [15], Theorem 3.2). *Let X be an Hadamard space and K^* be a nonempty closed convex subset of X. Then, for a sequence $\{K_n\}$ of nonempty closed convex subsets in X, the following statements are equivalent:*

(1) *$\{K_n\}$ converges to K^* in the sense of $\Delta-$Mosco;*
(2) *$\{P_{K_n} x\}$ converges to $P_{K^*} x \in X$ for every $x \in X$.*

We say that the Hadamard space X has property $(*)$ ([19]), if, the half space $\{z \in X : d(z,x) \le d(z,y)\}$ is convex for any given $x, y \in X$. A metric space X is called an $\mathbb{R}$-tree if the following conditions hold:

- for each $x, y \in X$ there is a unique metric segment $[x,y]$;
- if $[x,y] \cap [y,z] = \{y\}$, then $[x,z] = [x,y] \cup [y,z]$.

These $\mathbb{R}$-trees are examples of spaces that possesses property $(*)$ and are neither a Hilbert spaces nor a real Hilbert balls.

3. Main results

Let us begin this section with the following fundamental inequality. Here we use the following notation

$$\langle t\vec{pq}, s\vec{pz}\rangle = \frac{ts}{2}\left[d^2(p,q) + d^2(p,z) - d^2(q,z)\right]$$

for every $p, x, y \in X$ and $t, s \ge 0$.

Proposition 3.3. [6] *For each $s, t \ge 0$ and $p, q, z \in H$, the following inequality holds:*

$$G_p[\dot\gamma_{p,q}, \dot\gamma_{p,z}] \ge \langle t\vec{pq}, s\vec{pz}\rangle$$

Lemma 3.2. *Let $p, q, z \in X$ such that $s, t \ge 0$, then*

$$G_p[\dot\gamma_{p,q}, \dot\gamma_{p,z}] + G_q[\dot\gamma_{q,p}, \dot\gamma_{q,z}] \ge 0$$

Proof. Let $p, q, z \in X$ $s, t \geq 0$, then using proposition 3.3 we have

$$\begin{aligned} G_p[\dot{\gamma}_{p,q}, \dot{\gamma}_{p,z}] + G_q[\dot{\gamma}_{q,p}, \dot{\gamma}_{q,z}] &\geq \langle t\vec{pq}, s\vec{pz}\rangle + \langle t\vec{qp}, s\vec{qz}\rangle \\ &= \frac{ts}{2}\left[d^2(p,q) + d^2(p,z) - d^2(q,z)\right] \\ &\quad + \frac{ts}{2}\left[d^2(q,p) + d^2(q,z) - d^2(p,z)\right] \\ &= \frac{ts}{2}\left[2d^2(p,q)\right] \\ &= tsd^2(p,q) \\ &\geq 0 \end{aligned}$$

□

Let us define the perturbation $\widetilde{F}_{\bar{x}} : K \times K \to \mathbb{R}$ of a bifunction $F : K \times K \to \mathbb{R}$, at a point $\bar{x}$ by

$$\widetilde{F}_{\bar{x}}(x, y) := F(x, y) - G_x[\dot{\gamma}_{x,\bar{x}}, \dot{\gamma}_{x,y}]$$

Definition 3.4. Suppose that $K \subset X$ is closed convex, and $F : K \times K \to \mathbb{R}$. The resolvent of F is the mapping $J_F : X \rightrightarrows K$ defined by

$$J_F(x) := E(K, \widetilde{F}_x) = \{z \in K | F(z, y) - G_z[\dot{\gamma}_{z,x}, \dot{\gamma}_{z,y}] \geq 0, \forall y \in K\} \qquad \forall x \in X.$$

The following Proposition is central to the proof of our main theorems.

Proposition 3.4. *Suppose that F is monotone and $dom(J_F) \neq \emptyset$. Then, the following properties hold.*

(1) J_F is single-valued.
(2) If $dom(J_F) \supset K$, then J_F is non-expansive restricted to K.
(3) If $dom(J_{\mu f}) \supset K$ for any $\mu > 0$, then $Fix(J_F) = E(K, F)$.

Proof. (1) Let $x \in dom(J_F)$ and let that $z, z' \in J_F(x)$. So, we get

$$\begin{cases} F(z, z') & \geq G_z[\dot{\gamma}_{z,x}, \dot{\gamma}_{z,z'}]; \\ F(z', z) & \geq G_{z'}[\dot{\gamma}_{z',x}, \dot{\gamma}_{z',z}]. \end{cases}$$

Summing the above equation and using the monotonicity of F give us that,

$$\begin{aligned} 0 &\geq F(z, z') + F(z', z) \\ &\geq G_z[\dot{\gamma}_{z,x}, \dot{\gamma}_{z,z'}] + G_{z'}[\dot{\gamma}_{z',x}, \dot{\gamma}_{z',z}] \\ &= tsd^2(z, z'). \end{aligned}$$

This implies that,

$$z = z'.$$

(2) Let $x, y \in K$, let $u = J_F(x), v = J_F(y)$, By the definition of J_F , we have

$$\begin{cases} F(u, v) - G_u[\dot{\gamma}_{u,x}, \dot{\gamma}_{u,v}] & \geq 0; \\ F(v, u) - G_v[\dot{\gamma}_{v,y}, \dot{\gamma}_{v,u}] & \geq 0. \end{cases}$$

Adding the above two inequalities gives us

$$\begin{aligned} 0 &\geq 2(G_u[\dot{\gamma}_{u,x}, \dot{\gamma}_{u,v}] + G_v[\dot{\gamma}_{v,y}, \dot{\gamma}_{v,u}]) \\ &\geq 2\langle t\bar{ux}, s\bar{uv}\rangle + 2\langle t\bar{vy}, s\bar{vu}\rangle \\ &= ts\left[d^2(u,x) + d^2(u,v) - d^2(x,v)\right] + ts\left[d^2(v,y) + d^2(v,u) - d^2(y,u)\right]. \end{aligned}$$

This implies that,

$$0 \geq d^2(u,x) - d^2(x,v) + d^2(v,y) - d^2(y,u) + 2d^2(u,v).$$

Thus we have,

$$2d^2(u,v) \leq d^2(x,v) + d^2(y,u) - d^2(u,x) - d^2(v,y).$$

Since X is CAT(0) space then the above inequality implies that

$$2d^2(u,v) \leq d^2(x,u) + d^2(y,v) + 2d(x,y)d(u,v) - d^2(u,x) - d^2(v,y).$$

This implies,

$$d^2(u,v) \leq d(x,y)d(u,v).$$

Hence,

$$d(u,v) \leq d(x,y).$$

That is,

$$d(J_F(x), J_F(y)) \leq d(x,y).$$

(3) Let $x \in K$. Observe that the above two inequalities gives us

$$\begin{aligned} x \in Fix(J_F) &\Leftrightarrow x = J_F(x) \\ &\Leftrightarrow F(x,y) - G_x[\dot\gamma_{x,x}, \dot\gamma_{x,y}] \geq 0 \quad \forall y \in K \\ &\Leftrightarrow F(x,y) - G_x[\dot 0_x, \dot\gamma_{x,y}] \geq 0 \quad \forall y \in K \\ &\Leftrightarrow F(x,y) \geq 0 \quad \forall y \in K \\ &\Leftrightarrow x \in E(K,F). \end{aligned}$$

□

Lemma 3.3. *Given $w, z \in X$, then $\dot\gamma_{z,w} \in \partial g(z) \Longleftrightarrow z = prox_g(w)$.*

Proof. Follows from Proposition 3.8 of [7] with $\lambda = 1$. □

Lemma 3.4. *Let $F_g : X \times X \to \mathbb{R}$ by*

(3.10) $$F_g := g(y) - g(x), \qquad \forall x, y \in X.$$

Then we have $E(X, F_g) = \arg\min_X g$ and $J_{F_g} = prox_g$. Moreover we have $dom(prox_g) = X$.

Proof. From Lemma 3.3, we have

$$\begin{aligned} z \in Fix(J_{F_g}) &\Leftrightarrow F_g(z,y) - G_z[\dot\gamma_{z,x}, \dot\gamma_{z,y}] \geq 0 \quad \forall y \in X \\ &\Leftrightarrow g(y) \geq g(z) + G_z[\dot\gamma_{z,x}, \dot\gamma_{z,y}] \quad \forall y \in X \\ &\Leftrightarrow \dot\gamma_{z,x} \in \partial g(z) \\ &\Leftrightarrow z = \arg\min_{y \in X} \left\{ g(y) + \frac{1}{2} d^2(x,y) \right\} = prox_g(x). \end{aligned}$$

□

Lemma 3.5. *Assume that F is monotone. Let $\bar x \in X$, $\mu > 0$, $\widetilde{x} \in E(K, \widetilde{\mu F}_{\bar x})$ and $x^\star \in E(K,F)$, then $G_{\widetilde x}[\dot\gamma_{\widetilde x, \bar x}, \dot\gamma_{\widetilde x, x^\star}] \leq 0$.*

Proof. Let $\widetilde x \in E(K, \widetilde{F}_{\widetilde x})$, then

$$0 \leq \widetilde{\mu F}_{\bar x}(\widetilde x, x^\star) = \mu F(\widetilde x, x^\star) - G_{\widetilde x}[\dot\gamma_{\widetilde x, \bar x}, \dot\gamma_{\widetilde x, x^\star}],$$

which implies that $G_{\widetilde x}[\dot\gamma_{\widetilde x, \bar x}, \dot\gamma_{\widetilde x, x^\star}] \leq \mu F(\widetilde x, x^\star)$. Now, using the monotonicity of F is and the fact that $x^\star \in E(K,F)$ we obtain $F(y, x^\star) \leq 0, \forall y \in K$, and particularly $F(\widetilde x, x^\star) \leq 0$. Thus we have $G_{\widetilde x}[\dot\gamma_{\widetilde x, \bar x}, \dot\gamma_{\widetilde x, x^\star}] \leq 0$. □

Now, let $F : K \times K \to \mathbb{R}$ with $dom(\mu F) \supset K$ for all $\mu > 0$. Let $\{\lambda_n\} \subset (0, \infty)$ and $x^0 \in K$. Then the proximal algorithm is defined as follows

$$x^n := J_{\lambda_n F}(x^{n-1}), \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Theorem 3.3. *Suppose that F is monotone with $E(K, F) \neq \emptyset$, $\Delta-$usc in the first variable, and that $dom(J_{\mu F}) \supset K$ for all $\mu > 0$. Let $\{\lambda_n\}$ be bounded away from 0. Then the proximal algorithm* (3.11) *is $\Delta-$convergent to an element in $E(K, F)$ for any initial start $x^0 \in K$.*

Proof. Let $x^0 \in K$, and $x^\star \in E(K, F)$, then

$$d(x^\star, x^{n+1}) = d(J_{\lambda_n F}(x^\star), J_{\lambda_n F}(x^n)) \leq d(x^n, x^\star).$$

The above inequality implies that the sequence $\{x^n\}$ is Fejer convergent with respect to $E(K, F)$. Proposition 2.2 implies that the real sequence $d(x^n, x^\star)$ is bounded, and thus is convergent to some $\xi \geq 0$. Using Lemma 3.5 we will have that

$$d^2(x^{n+1}, x^n) \leq d^2(x^n, x^\star) - d^2(x^{n+1}, x^\star).$$

Letting $n \to \infty$, we obtain that $\lim_{n\to\infty} d(x^{n+1}, x^n) = 0$.

Now, Suppose that $\hat{x} \in K$ is a $\Delta-$accumulation point of the sequence $\{x^n\}$, also let $\{x^{n_j}\} \subset \{x^n\}$ be a subsequence with $x^{n_j} \xrightarrow{\Delta} \hat{x}$. Let $y \in K$, then by using (3.11), we obtain the following inequalities for any index $j \in \mathbb{N}$

$$F(x^{n_j}, y) \geq \frac{1}{\lambda_{n_j}} G_{x^{n_j}}[\dot{\gamma}_{x^{n_j}, x^{n_j-1}}, \dot{\gamma}_{x^{n_j}, y}] \geq -\frac{1}{\lambda_{n_j}} d(x^{n_j}, x^{n_j-1}) d(x^{n_j}, y). \tag{3.12}$$

Since $\{x^n\}$ is bounded (Proposition 2.2) and $\{\lambda_n\}$ is bounded away from 0, then (3.12) implies that there exists $M > 0$ such that

$$F(x^{n_j}, y) \geq -M d(x^{n_j}, x^{n_j-1}), \tag{3.13}$$

Letting $j \to \infty$ in (3.13) and using the $\Delta-$upper semi-continuity of $F(\cdot, y)$ we have

$$F(\hat{x}, y) \geq \limsup_{j\to\infty} F(x^{n_j}, y) \geq -M \lim_{j\to\infty} d(x^{n_j}, x^{n_j-1}) = 0.$$

As $y \in K$ is arbitrarily chosen, we conclude that $\hat{x} \in E(K, F)$. Thus, any $\Delta-$accumulation point of $\{x^n\}$ is a solution of $EP(K, F)$. Hence, the sequence $\{x^n\}$ is $\Delta-$convergent to an element of $E(K, F)$, in view of Proposition 2.2. □

The following result, which is is crucial for obtaining the next corollary, can be found in [2].

Proposition 3.5. [2] *A convex set $K \subset X$ is closed if and only if it is Δ-closed.*

Corollary 3.1. *Let $f : X \to \mathbb{R}$ be a lsc and convex function with* $\arg\min f \neq \emptyset$. *Suppose $\{\lambda_n\}$ is bounded away from 0. Then for $x^0 \in X$ the sequence generated by*

$$x^n := J_{\lambda_n f}(x^{n-1}) \quad \forall n \in \mathbb{N} \tag{3.14}$$

is $\Delta-$convergent to a minimizer of f.

Proof. Defined $F : X \times X \to \mathbb{R}$ by $F(x, y) = f(y) - f(x)$. Then it is clear that F is monotone. The Δ-usc follows from Proposition 3.5 by considering the $F(\cdot, y)$ for $y \in X$. Finally, Lemma 3.4 and Theorem 3.3 yield the desired result. □

Example 3.1. *Consider the Hadamard space (X, d) where, $X = \mathbb{R}^2$ and*

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2},$$

where $x = (x_1, x_2), y = (y_1, y_2)$ *(see for example* [30]*). Let* $K = X$ *and* $F : K \times K \to \mathbb{R}$ *be defined by* $F(x, y) = \Lambda\left[((y_2 + 1) - (y_1 + 1)^2)^2 - ((x_2 + 1) - (x_1 + 1)^2)^2\right] + y_1^2 - x_2^2$*, where* $\Lambda > 0$*. It is not difficult to see that* F *is monotone, continuous in the first variable and that* $E(K, F) = \{(0, 0)\}$*. Indeed, setting* $f(x) = \Lambda((x_2+1)-(x_1+1)^2)^2+x_1^2$*, the desired properties of* F *are guaranteed by Lemma 3.4 and Corollary 3.1.*

In Table 1 we report the convergence of the (3.14) to the equilibrium point $(0, 0)$ by using Matlab with $\Lambda = 24$.

TABLE 1. Few values of the sequence $\{x^n\}$ from Example 3.1

	$x^0 = (50, -1001)$	$x^0 = (-75, -100)$	$x^0 = (-27, 5)$	$x^0 = (12, 8)$
k	x^n	x^n	x^n	x^n
0	(50, -1001)	(-75, -100)	(-27, 5)	(12, 8)
1	(-330.296, 107856.4)	(-344.992, 117276.6)	(-166.158, 27202.4)	(-33.1613, 1019.413)
2	(-243.648, 58724.25)	(-228.854, 51676.44)	(-128.962, 16342.91)	(-26.0604, 621.3938)
3	(-172.651, 29392.52)	(-168.449, 27948.57)	(-95.2151, 8855.893)	(-19.106, 322.9214)
4	(-134.738, 17856.26)	(-133.482, 17518.59)	(-69.5548, 4684.581)	(-13.9511, 163.8879)
5	(-97.7053, 9330.131)	(-98.6561, 9515.408)	(-50.7774, 2466.446)	(-10.1845, 81.28041)
6	(-71.553, 4962.113)	(-72.073, 5035.673)	(-37.0679, 1292.337)	(-7.43477, 38.89128)
7	(-52.2441, 2614.312)	(-52.616, 2652.488)	(-27.0597, 672.5949)	(-5.42742, 17.49609)
8	(-38.1389, 1370.526)	(-38.41, 1390.683)	(-19.7537, 346.6768)	(-3.96205, 6.966399
9	(-27.8416, 713.7979)	(-28.0395, 724.4209)	(-14.4203, 176.1663)	(-2.89229, 1.991403)
⋮	⋮	⋮	⋮	⋮
39	(-0.00221, -0.00486)	(-0.00222, -0.00488)	(-0.00109, -0.00238)	(-0.00024, -0.00052)
40	(-0.00164, -0.00362)	(-0.00161, -0.00352)	(-0.00081, -0.00177)	(-0.00016, -0.00036)
41	(-0.00118, -0.00262)	(-0.00117, -0.0026)	(-0.00058, -0.0013)	(-0.00014, -0.0003)
42	(-0.00085, -0.00188)	(-0.00086, -0.00192)	(-0.00041, -0.00091)	(-9.92E-05, -0.00025)
43	(-0.00061, -0.00134)	(-0.00064, -0.00146)	(-0.00031, -0.00068)	(-7.99E-05, -0.00015)
44	(-0.00044, -0.00095)	(-0.0005, -0.0011)	(-0.00023, -0.00053)	(-3.02E-05, -8.92E-05
45	(-0.00034, -0.00075)	(-0.00038, -0.00083)	(-0.00016, -0.00036)	(-2.00E-05, -3.94E-05)
46	(-0.00024, -0.00052)	(-0.00029, -0.00064)	(-0.00014, -0.0003)	(-2.42E-05, -2.29E-05
47	(-0.00016, -0.00036)	(-0.00023, -0.00051)	(-9.92E-05, -0.00025)	(-2.42E-05, -2.29E-05)
48	(-0.00014, -0.0003)	(-0.00019, -0.00039)	(-7.99E-05, -0.00015)	(-2.42E-05, -2.29E-05
49	(-9.92E-05,-0.00025)	(-0.00014, -0.00031)	(-3.02E-05, -8.92E-05)	(-2.42E-05, -2.29E-05)

Finally, if the Hadamard space X posess property $(*)$, then we have the following strong convergence result.

Theorem 3.4. *Let* X *be an Hadamard satisfying property* $(*)$ *and* $K \subset X$ *be a nonempty closed convex set.* $F : K \times K \to \mathbb{R}$ *be a monotone mapping with* $E(K, F) \neq \emptyset$ *and that* $dom(J_F) \supset K$*. The sequence generated for each* $n \in \mathbb{N}$ *by*

$$\begin{cases} x^0 \in K_0 := K; \\ y^n = J_F(x^n); \\ K_n = \{z \in K | d(z, y^n) \leq d(z, x^n)\} \cap K_{n-1}; \\ x^{n+1} = P_{K_n} x^n, \end{cases}$$

strongly converges to an element of $E(K, F)$*.*

Proof. Let $z \in E(K, F)$, then

$$d(z, y^n) = d(J_F(z), J_F(x^n)) \leq d(z, x^n).$$

This implies that $E(K,F) \subset K_n$ for all $n \in \mathbb{N}$. From our assumption we have that K_n is closed and convex. Thus P_{K_n} is well-defined for each $n \in \mathbb{N}$. Also, the sequence $\{K_n\}$ is convergent to $K^* := \bigcap_{n=0}^{\infty} K_n$ in the sense of $\Delta-$Mosco. Now using Theorem 2.2 we get that $\{x_n\}$ converges to $x^* := P_{K^*}x$. As $x^* \in K_n$ for all $n \in \mathbb{N}$, we obtain

$$d(x^*, y^n) \leq d(x^*, x^n) \text{ for all } n \in \mathbb{N}. \tag{3.15}$$

Letting $n \to \infty$ in (3.15) we get

$$\lim_{n\to\infty} y^n = \lim_{n\to\infty} J_F(x^n) = x^*. \tag{3.16}$$

Using Proposition 3.4 (2) we have

$$J_F(x^*) = \lim_{n\to\infty} J_F(x^n). \tag{3.17}$$

Combining (3.16) and (3.17) we have

$$x^* = J_F(x^*). \tag{3.18}$$

The equation (3.18) and Proposition 3.4 (3) imply that

$$x^* \in E(K,F),$$

as required. □

Remark 3.1. It is worth noting that the result of Theorem 3.4 holds for all $\mathbb{R}$-trees, Hadamard manifolds and Hilbert spaces as they are all special Hadamard spaces with property $(*)$.

Corollary 3.2. *Let X be an Hadamard satisfying property $(*)$ and $K \subset X$ be a nonempty closed convex set. Suppose $f : K \to \mathbb{R}$ is a lsc and convex function with $\arg\min f \neq \emptyset$. Then the sequence generated for each $n \in \mathbb{N}$ by*

$$\begin{cases} x^0 \in K_0 := K \\ y^n = J_f(x^n) \\ K_n = \{z \in K | d(z, y^n) \leq d(z, x^n)\} \cap K_{n-1} \\ x^{n+1} = P_{K_n}(x^n). \end{cases}$$

strongly converges to a minimizer of f.

Proof. The proof follows from Theorem 3.4 based on the arguments presented in the proof lines of Corollary 3.1. □

Conclusion and remarks

In this work, a resolvent operator is introduced using the concept of monotone vector field thereby making the operator meaningful in both linear and Hadamard manifolds. Under suitable conditions, it is shown that this operator is a nonexpansive single-valued mapping and its fixed points set coincide with the solution of the equilibrium problem. Furthermore, using the proximal operator, we established Δ-convergent and strong convergence results for approximating a solution of equilibrium problem in the setting of Hadamard spaces, which include Hilbert space, $\mathbb{R}$-trees, Hilbert balls, Hadamard manifolds and complete CAT(κ) spaces for $\kappa \leq 0$.

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