CARPATHIAN J. MATH.
Volume **40** (2024), No. 1,
Pages 99 - 120

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.01.08

Inertial split projection and contraction method for pseudomonotone variational inequality problem in Banach spaces

ROSE MALULEKA[1], G. C. UGWUNNADI[2], M. APHANE[3], H. A. ABASS[4] and A. R. KHAN[5]

ABSTRACT. In this article, we introduce a Halpern iterative method together with an inertial projection and contraction method for finding an approximate solution of variational inequality problem involving pseudomonotone mapping which also solves split common fixed point problem of Bregman demigeneralized mapping and Bregman strongly nonexpansive mapping in the framework of p-uniformly convex and uniformly smooth real Banach spaces. Using our iterative method, we establish a strong convergence result for approximating the solution of the aforementioned problems and state some consequences of our main result. The result discussed in this article extends and complements many related results in the literature.

1. INTRODUCTION

Let C be a nonempty, closed and convex subset of a real Banach space E and let E^* denote the dual space of E. The Variational Inequality Problem (in short VIP) is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall \ y \in C, \tag{1.1}$$

where $A : C \to E^*$ is a nonlinear mapping. We denote by VI(C,A) the set of solutions of (1.1).

Variational inequality theory introduced by Stampacchia and Fichera [24, 36] independently, in early sixties in mechanics and potential theory respectively provides the natural, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems arising in elasticity, economics, transportation, optimization, control theory and engineering sciences (see [9, 10]).

The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. The first aspect reveals the fundamental facts on the qualitative behavior of solutions to important classes of problems. On the other hand, it allows us to develop highly efficient and powerful numerical methods to solve, for instance, obstacle, unilateral, free and moving boundary value problems.

In 1985, Pang [34] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a VIP. It is well-known that $VI(C, A)$ is equivalent to the fixed point problem:

$$\text{find } x^* \in C \text{ such that } x^* = P_C(x^* - \tau Ax^*),$$

Received: 26.01.2023. In revised form: 22.09.2023. Accepted: 29.09.2023
2020 *Mathematics Subject Classification.* 47H06, 47H09, 47J05, 47J25.
Key words and phrases. *Inertial Projection and contraction methods, Pseudomonotone variational inequality problem, p-uniformly convex Banach spaces, fixed point problem.*
Corresponding author: G. C. Ugwunnadi; ugwunnadi4u@yahoo.com

where τ is any positive real number and P_C is a metric projection onto C. Recently, many methods have been applied for finding $VI(C, A)$, (see [2, 5, 7, 12, 13]). The simplest one is the projection method, which is called the Gradient Projection Method (in short, GPM). The basic idea of extending the GPM for solving problem of minimizing $f(x)$ subject to $x \in C$ is given by

$$x_{n+1} = P_C(x_n - \alpha_n \bigtriangledown f(x_n)),\ n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain condition and ∇f is the gradient function. An immediate extension of the method in (1.2) is the GPM which requires substituting the gradient function with operator F so that we can generate a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = P_C(x_n - \alpha_n F x_n),\ \forall\, n \geq 0.$$

However, the convergence of this method requires a slightly strong assumption that the operators are inverse strongly monotone or strongly monotone. In order to relax this condition, Korpelvich [29] and Antipin [6] proposed the Extragradient Method (in short EM) in finite dimensional Euclidean spaces for a monotone and L-Lipschitz continuous mapping A as follows:

$$\begin{cases} x_0 \in C,\ \tau > 0, \\ y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n),\ \forall\, n \geq 1, \end{cases} \tag{1.3}$$

where $\tau \in (0, \frac{1}{L})$. The sequence $\{x_n\}$ generated by EM (1.3) converges to an element of $VI(C, A)$ provided $VI(C, A)$ is nonempty. It should be noted that in EM, one needs to calculate two projections onto the feasible set C in each iteration. If the set C is not so simple, then the EM becomes very difficult and its implementation is costly. In addition, the convergence of the method (1.3) requires prior estimate of the Lipschitz constant which is often difficult to estimate and we emphasize that the stepsize defined by the process is too small and reduces the convergence rate of the method. To the best of our knowledge, there are some methods to overcome these drawbacks. The first one is the subgradient extragradient method (SEGM) proposed by *Censor et al.* [13], in which the second projection onto C is replaced by a projection onto a specific constructible half-space. Their method is of the form:

$$\begin{cases} y_n = P_C(x_n - \tau F x_n), \\ T_n = \{w \in H : \langle x_n - \tau F x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau F y_n),\ \forall\, n \geq 0, \end{cases} \tag{1.4}$$

where $\tau \in (0, \frac{1}{L})$.
The second one is the method proposed by Tseng in [38]. His method is of the form:

$$\begin{cases} y_n = P_C(x_n - \tau F x_n), \\ x_{n+1} = y_n - \lambda(F y_n - F x_n),\ \forall\, n \geq 0, \end{cases} \tag{1.5}$$

where $\tau \in (0, \frac{1}{L})$.
The third one is the projection and contraction method (PCM) proposed by He [25] for

solving VIP. The PCM can be summarized as follows:

$$(1.6)\qquad \begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \tau F x_n), \\ d(x_n, y_n) := (x_n - y_n) - \tau(F x_n - F y_n), \\ x_{n+1} = x_n - \gamma \eta_n d(x_n, y_n), \ \forall\, n \geq 0, \end{cases}$$

where $\gamma \in (0,2), \tau \in (0, \frac{1}{L})$ and

$$\eta_n := \frac{\phi(w_n, y_n)}{\|d(w_n, y_n)\|^2}, \ \phi(w_n, y_n) := \langle w_n - y_n, d(w_n, y_n)\rangle, \ \forall\, n \geq 0.$$

It is worth mentioning that the SEGM, TM and PCM described above need only to calculate one projection onto C in each iteration which may improve the performance of the algorithms. Also, the SEGM, TM and PCM have received great attention by many authors, who improved it in various ways (see [2, 7, 13, 14, 18, 21, 22, 23, 27, 28, 38] and the references contained in). We emphasize that the first two methods (SEGM and TM) have been considered by authors extensively in the settings of real Hilbert and Banach spaces,but no result on PCM in the literature can be found in the framework of Banach spaces.

Question: Can we employ the PCM for solving VIP in the setting of Banach spaces?

One of the best ways to speed up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. This term which is represented by $\theta_n(x_n - x_{n-1})$, is a remarkable tool for improving the performance of algorithms and it is known to have some nice convergence characteristics. Thus, there are growing interests by authors working in this direction (see [1, 2, 3, 4, 17, 26]). The idea of inertial extrapolation method was first introduced by Polyak [35] and was inspired by an implicit discretization of a second-order-in-time dissipative dynamical system, so-called "Heavy Ball with Friction".

$$(1.7)\qquad v''(t) + \gamma v'(t) + \bigtriangledown f(v(t)) = 0,$$

where $\gamma > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. System (1.7) is discretized so that, having the terms x_{n-1} and x_n, the next term x_{n+1} can be determined using

$$(1.8)\qquad \frac{x_{n-1} - 2x_n + x_{n-1}}{j^2} + \gamma \frac{x_n - x_{n-1}}{j} + \bigtriangledown f(x_n) = 0, \ n \geq 1,$$

where j is the step-size. Equation (1.8) yields the following iterative algorithm:

$$(1.9)\qquad x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha \bigtriangledown f(x_n), \ n \geq 1,$$

where $\beta = 1 - \gamma_j, \alpha = j^2$ and $\beta(x_n - x_{n-1})$ is called the inertial extrapolation term which is intended to speed up the convergence of the sequence generated by (1.9). Alvarez and Attouch [8] also employed the inertial extrapolation method to the setting of a general maximal monotone operator using the proximal point algorithm (PPA), which is called the inertial PPA, and is of the form:

$$(1.10)\qquad \begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n B)^{-1} y_n, n > 1. \end{cases}$$

They proved that if $\{r_n\}$ is non-decreasing and $\{\theta_n\} \subset [0,1)$ with

$$(1.11)\qquad \sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}||^2 < \infty,$$

then the iterative Algorithm (1.10) converges weakly to a zero of B. More precisely, condition (1.11) is true for $\theta_n < \frac{1}{3}$. Here θ_n is an extrapolation factor.
Very recently, *Dong et al.* [23] proposed an inertial projection and contraction method (IPCM) of the form:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau F w_n), \\ d(w_n, y_n) = (w_n - y_n) - \tau(F w_n - F y_n), \\ x_{n+1} = w_n - \gamma \eta_n d(w_n, y_n), \ \forall\, n \geq 0, \end{cases} \tag{1.12}$$

where $\gamma \in (0, 2),\ \tau \in (0, \frac{1}{L})$ and

$$\eta_n := \begin{cases} \frac{\phi(w_n, y_n)}{\|d(w_n, y_n\|^2)}, \text{ if } d(w_n, y_n) \neq 0, \\ 0, \text{ if } d(w_n, y_n) = 0, \end{cases} \tag{1.13}$$

where $\phi(w_n, y_n) := \langle w_n - y_n, d(w_n, y_n)\rangle$. Under suitable conditions, they established that the sequence $\{x_n\}$ generated by (1.12) converges weakly to an element of $VI(C, F)$. In the setting of Banach space, several authors have modified the inertial extrapolation method without the computation of the difference between the norm of the two adjacent iterates x_n and x_{n-1}. Approximating solutions of different optimization problems with inertial extrapolation method using either viscosity and Halpern method in the setting of Banach space requires the modification of inertial term due to the geometry of the space (see [3, 7, 33, 37] and the references contained in). The only case where the inertial term is not modified is when the hybrid and shrinking methods are employed in the setting of Banach space, (see [2, 17, 26]). To the best of our knowledge, there is no result on inertial extrapolation method without modification using Halpern method in the setting of Banach spaces.
Question 2: Can we introduce an inertial Halpern method without the computation of the difference between the norm of the two adjacent iterates x_n and x_{n-1} for finding the solution of VIP in the setting of p-uniformly convex real Banach space which are also uniformly smooth?.

Motivated by the results of [6, 14, 21, 22, 28, 38] and other related results in literature, we proposed a Halpern inertial iterative algorithm with projection and contraction method for approximating the solution of split variational inequality problem involving a pseudomonotone operator, fixed points of Bregman demigeneralized mapping and Bregman strongly nonexpansive mapping in the setting of p-uniformly convex real Banach space which is also uniformly smooth. Using our iterative method, we establish a strong convergence result for approximating the solution of the aforementioned problems. We emphasize that our iterative method is design in such a way that it does not require prior knowledge of the operator norm. We present some numerical examples to show the efficiency of our result. The result green discussed in this paper extends and complements many related results in the literature.

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$". respectively.
Let E be a Banach space with the dual E^*. An operator $A : E \to E^*$ is said to be $p-L$-Lipschitz, if

$$||Ax - Ay|| \leq L||x - y||^p \ \forall\, x,\ y \in E,$$

where $L \geq 0$ and $p \in [1, \infty)$ are two constants. If $p = 1$, the operator A is said to be L -Lipschitz.

Let $C \subseteq E$ be a nonempty set. Then a mapping $A : C \to E^*$ is called

(a) monotone on C, if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$;

(b) pseudomonotone on C, if for all $x, y \in E$, $\langle Ax, y - x\rangle \geq 0 \implies \langle Ay, y - x\rangle \geq 0$;

(c) weakly sequentially continuous if for any $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Let E be a real Banach space and $f : E \to \mathbb{R}$. Then f is called:

(i) G$\hat{a}$teaux differentiable at $x \in E$, denoted by $f'(x)$ or $\nabla f(x)$, if there exists an element y of E, such that

$$\lim_{t\to 0} \frac{f(x + ty) - f(x)}{t} = \langle y, f'(x)\rangle, \ \forall\, y \in E.$$

f is G$\hat{a}$teaux differentiable on E if f is G$\hat{a}$teaux differentiable at every $x \in E$;

(ii) weakly lower semicontinuous at $x \in E$, if $x_k \rightharpoonup x$ implies $f(x) \leq \liminf_{k\to\infty} f(x_k)$. f is weakly lower semicontinuous on E, if f is weakly lower semicontinuous at every $x \in E$.

Let $K(E) := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The modulus of convexity is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in K(E),\ \|x - y\| \geq \epsilon\right\}.$$

The space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$. Then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of E is the function $\rho_X : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in K(E)\right\}.$$

The space E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$. Then a Banach space E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [19]).

Let $p > 1$ be a real number, the generalized duality mapping $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{\overline{x} \in E^* : \langle x, \overline{x}\rangle = \|x\|^p, \|\overline{x}\| = \|x\|^{p-1}\},$$

where $\langle ., .\rangle$ denotes the duality pairing between elements of E and E^*. In particular, if $p = 2$, then J_2^E is called the normalized duality mapping. If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth, then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E, (see [20] for more details).

If $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted by $f^* : E^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E,\ x^* \in E^*\}.$$

Let the domain of f be denoted by $dom f = \{x \in E : f(x) < +\infty\}$. For any $x \in int(dom f)$ and $y \in E$, we denote and define the right-hand derivative of f at x in the direction of y by

$$f^0(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

Definition 1.1. [11]Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : E \times E \to [0, +\infty)$ defined by

$$\Delta_f(x, y) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{1.14}$$

is called the Bregman distance with respect to f, where $\langle \nabla f(x), y\rangle = f^0(x, y)$.

It is well-known that Bregman distance function Δ_f does not satisfy the properties of a metric function, because Bregman function Δ_f fails to satisfy the symmetric and triangular inequality properties. Moreover, it is well- known that the duality mapping J_p^E is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$, (see [16]). Using (1.14), one can show that the following equality called three-point identity is satisfied:

$$\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) = \langle J_p^E(z) - J_p^E(y), x - y\rangle,\ \forall\, x, y, z \in E. \tag{1.15}$$

In addition, if $f(x) = \frac{1}{p}||x||^p$, where $\frac{1}{p} + \frac{1}{q} = 1$, then we obtain

$$\begin{aligned}
\Delta_f(x, y) = \Delta_p(x, y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y - x, J_p^E(x)\rangle \\
&= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y, J_p^E(x)\rangle + \langle x, J_p^E(x)\rangle \\
&= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y, J_p^E(x)\rangle + \|x\|^p \\
&= \frac{1}{p}\|y\|^p + \frac{1}{q}\|x\|^p - \langle y, J_p^E(x)\rangle. \qquad (1.16)
\end{aligned}$$

Let $T : C \to C$ be a nonlinear mapping,

(i) a point $p \in C$ is called an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T;

(ii) T is said to be Bregman relatively nonexpansive, if

$$\hat{F}(T) = F(T) \neq \emptyset \text{ and } \Delta_p(u, Tx) \leq \Delta_p(u, x),\ \forall\, x \in C,\ u \in F(T).$$

(iii) T is said to be Bregman firmly nonexpansive mapping (BFNE) if

$$\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \leq \langle J_p^E(x) - J_p^E(y), Tx - Ty\rangle,\ \forall\, x, y \in C,$$

(iv) T is said to be Bregman strongly nonexpansive mapping (BSNE) with $\hat{F}(T) \neq \emptyset$ if

$$\Delta_p(y, Tx) \leq \Delta_p(y, x),\ \forall\, y \in \hat{F}(T)$$

and for any bounded sequence $\{x_n\}_{n\geq 1} \subset C$,

$$\lim_{n\to\infty} (\Delta_p(y, x_n) - \Delta_p(y, Tx_n)) = 0$$

implies

$$\lim_{n\to\infty} \Delta_p(Tx_n, x_n) = 0.$$

Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x := arg \min_{y\in E} ||x-y||, \ x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \le 0, \ \forall\, z \in C. \tag{1.17}$$

Also, the Bregman projection from E onto C denoted by Π_C satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y\in C} \Delta_p(x, y), \ \forall\, x \in E. \tag{1.18}$$

[19] Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x \in E$. Then the following assertions hold:,
$z = \Pi_C x$ if and only if

$$\langle J_E^p(x) - J_E^p(z),\ y - z\rangle \le 0, \ \forall\, y \in C; \tag{1.19}$$

and

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \le \Delta_p(x, y), \ \forall\, y \in C. \tag{1.20}$$

We now give some results that will help us in the proof of our main theorem.

Lemma 1.1. [15] *If E is a p-uniformly convex Banach space with $p \ge 2$, then there exists $K > 0$ such that for all $x, y \in E$, the following inequalities hold:*

$$\langle J_E^p(x) - J_E^p(y), x - y\rangle \ge K||x-y||^p, \tag{1.21}$$

and

$$||x - y|| \le \left(\frac{1}{K}\right)^{\frac{1}{(p-1)}} ||J_E^p(x) - J_E^p(y)||^{1/(p-1)}. \tag{1.22}$$

Lemma 1.2. [16] *Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$\|x - y\|^q \le \|x\|^q - q\langle J_E^q(x), y\rangle + C_q\|y\|^q.$$

Lemma 1.3. [39] *Let E be a p-uniformly convex Banach space, the metric and Bregman distance have the following relation for all $x, y \in E$*

$$\pi_p\|x-y\|^p \le \Delta_p(x, y) \le \langle x - y, J_E^p(x) - J_E^p(y)\rangle, \tag{1.23}$$

where $\pi_p > 0$ is a fixed number and for any $q > 1$, if $\frac{1}{p} + \frac{1}{q} = 1$, by Young's inequality, we have

$$\begin{aligned} \langle J_E^p(x), y\rangle \le ||J_E^p(x)||||y|| \quad &\le \quad \frac{1}{q}||J_E^p(x)||^q + \frac{1}{p}||y||^p \\ &= \quad \frac{1}{q}(||x||^{p-1})^q + \frac{1}{p}||y||^p \\ &= \quad \frac{1}{q}||x||^p + \frac{1}{p}||y||^p. \end{aligned} \tag{1.24}$$

Lemma 1.4. [16] *Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant C_q such that for all $x, y \in E$, we have*

$$\left\|\frac{x+y}{2}\right\|^q \geq \frac{1}{2}||x||^q + \frac{1}{2}||y||^q - 2^{-q}C_q||x-y||^q. \tag{1.25}$$

Putting $x = u - v$ and $y = u + v$ in (1.25), *we get for all $u, v \in E$*

$$||u-v||^q \leq 2(||u||^q + C_q||v||^q) - ||u+v||^q \leq 2(||u||^q + C_q||v||^q). \tag{1.26}$$

Lemma 1.5. [40] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \ \forall\, x \in E, x^* \in E^*.$$

Then the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \ \forall\, x \in E, \ x^* \in E^*$.
(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \forall\, x \in E, \ x^*, y^* \in E^*$.

Lemma 1.6. [19] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ implies $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Definition 1.2. [42] Let C be a nonempty subset of a p-uniformly convex ($0 < p < \infty$) and uniformly smooth real Banach space E. A mapping $T : C \to E$ is called $\theta-$Bregman demigeneralized type with respect to p, if $F(T) \neq \emptyset$ and there exists a real number θ such that

$$\Delta_p(x, Tx) \leq \theta\langle J_E^p(x) - J_E^p(Tx), x - x^*\rangle \ \forall\, x \in E, \ x^* \in F(T). \tag{1.27}$$

Lemma 1.7. [42] *Let C be a nonempty subset of a p-uniformly convex and uniformly smooth real Banach space E. Let $T : C \to E$ be a $\theta-$Bregman demigeneralized type mapping with $\theta \in \mathbb{R}$. Then $F(T)$ is closed and convex.*

Lemma 1.8. [30] *Let E be a real reflexive Banach space and C a nonempty, closed and convex subset of E Let A be a continuous pseudomonotone mapping from C into E^*. Then, $VI(C, A)$ is closed and convex. Furthermore, $x^* \in VI(C, A)$ if and only if $\langle Ax, x - x^*\rangle \geq 0$ for all $x \in C$.*

Lemma 1.9. [41] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with condition:*

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and $\{b_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

In the sequel, we assume the following hold.

Assumption 1.1. (C1) *E_1 and E_2 are $p-$uniformly convex real Banach spaces which are also uniformly smooth and C is a nonempty closed and convex subset of E_1.*

(C2) $A : C \to E_1^*$ *is pseudomonotone and* $(p-1) - L-$*Lipschitz continuous on* E_1.

(C3) *A is weakly sequentially continuous, that is for any* $\{x_n\} \subset E_1$, *we have* $x_n \rightharpoonup x^*$ *implies* $Ax_n \rightharpoonup Ax^*$.

(C4) $B : E_1 \to E_2$ *is a bounded linear operator with adjoint* $B^* : E_2^* \to E_1^*$ *and* $S : E_2 \to E_2$ *is a* $\xi-$ *Bregman demigeneralized type mapping which is demiclosed at* 0 *such that* $F(S) \neq \emptyset$, *where* $\xi \in (0,\infty)$ *with* $\frac{1}{\xi} \geq 1-\eta$, $\eta \in (-\infty, 0]$ *and* $T : E_1 \to E_1$ *is a Bregman strongly nonexpansive mapping.*

(C5) $\{\mu_n\}$ *is a positive sequence in* $\left(0, \frac{p\pi_p}{2^{p-1}}\right)$, *where* π_p *is defined in* (1.23), $\mu_n = \circ(\alpha_n)$, *where* α_n *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n \in (a,b) \subset (0,1)$ *and* $\gamma_n \in (c,d) \subset (0,1)$ *for all* $n \geq 1$.

(C6) *Denote the set of solution by* $\Gamma := SOl(A,C) \cap B^{-1}(F(S)) \cap F(T)$ *and is assumed to be nonempty. Then* Γ *is closed and convex.*

Next, we introduce an inertial extrapolation with projection and contraction method for finding common solution of fixed point problem and pseudomonotone variational inequality problem:

Algorithm 1.2. Initialization: *Choose* $x_0, x_1 \in E_1$ *to be arbitrary and* $\theta \in (0,\pi_p)$, $K^* > 0$.
Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step 1. *Given the iterates* x_{n-1} *and* x_n *for each* $n \geq 1$, $\theta > 0$, *choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta_n}$, *where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1})\|}\}, & if\ \ x_n \neq x_{n-1}, \\ \\ \theta, \quad otherwise. \end{cases} \tag{1.28}$$

Step 2. Compute

$$\begin{cases} y_n = J_{E_1^*}^q[J_{E_1}^p(x_n) + \theta_n(J_{E_1}^p(x_n) - J_{E_1}^p(x_{n-1}))], \\ w_n = \Pi_C(J_{E_1^*}^q[J_{E_1}^p(y_n) - \tau A(y_n)]). \end{cases} \tag{1.29}$$

If $w_n = y_n$ *for some* $n \geq 1$, *then stop and is a solution of the problem (VIP). Otherwise go to step 3.*

Step 3. Compute

$$\begin{cases} v_n = J_{E_1^*}^q[J_{E_1}^p(y_n) - \rho_n J_{E_1}^p(d_n)], \\ d_n = J_{E_1^*}^q(J_{E_1}^p(y_n) - J_{E_1}^p(w_n) - \tau(A(y_n) - A(w_n))). \end{cases} \tag{1.30}$$

where

$$\rho_n^{q-1} = K^* \frac{||J_{E_1}^p y_n - J_{E_1}^p w_n||^q}{||J_{E_1}^p d_n||^q}, \ \ if\ d_n \neq 0;\ otherwise\ \rho_n = 0 \tag{1.31}$$

$$and\ \ \langle J_{E_1}^p d_n, y_n - w_n\rangle \geq K^*||J_{E_1}^p y_n - J_{E_1}^p w_n||^q. \tag{1.32}$$

Step 4. Compute

$$\begin{cases} z_n = J_{E_1^*}^q(J_{E_1}^p(v_n) - \lambda_n B^*(J_{E_2}^p(Bv_n) - J_{E_2}^p(S(Bv_n)))) \\ x_{n+1} = J_{E_1}^q(\alpha_n J_{E_1}^p(u) + (1-\alpha_n)J_{E_1}^p(Tz_n)), \end{cases} \tag{1.33}$$

where, for any fixed $\epsilon > 0$, the step λ_n is chosen as follows:

$$0 < \epsilon \leq \lambda_n \leq \Big(\frac{\pi_p q||Bv_n - SBv_n||^p}{C_q\xi||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q} - \epsilon\Big)^{\frac{1}{p-1}}, \tag{1.34}$$

if $Bv_n \neq SBv_n$, otherwise $\lambda_n = (\lambda > 0)$. Set $n := n+1$ and return to **Step 1**.

Lemma 1.10. *If $y_n = w_n$ or $d_n = 0$ in Algorithm 1.2, then $y_n \in VI(C, A)$.*

Proof. Since A is $(p-1)-L-$ Lipschitz continuous with constant $L > 0$, from (1.30), (1.22) and (1.26), we get

$$\begin{aligned}
||J^p_{E_1} d_n||^q &= ||J^p_{E_1} y_n - J^p_{E_1} w_n - \tau[Ay_n - Aw_n]||^q \\
&\leq 2\Big(||J^p_{E_1} y_n - J^p_{E_1} w_n||^q + C_q\tau^q||Ay_n - Aw_n||^q\Big) \\
&\leq 2||J^p_{E_1} y_n - J^p_{E_1} w_n||^q + 2C_q\tau^q[L||y_n - w_n||^{(p-1)}]^q \\
&\leq 2||J^p_{E_1} y_n - J^p_{E_1} w_n||^q + 2C_q\tau^q L^q\Big(\frac{1}{K}\Big)^q||J^p_{E_1} y_n - J^p_{E_1} w_n||^q \\
&= 2\Big(1 + C_q\Big(\frac{\tau L}{K}\Big)^q\Big)||J^p_{E_1} y_n - J^p_{E_1} w_n||^q. \qquad (1.35)
\end{aligned}$$

Also

$$\begin{aligned}
||J^p_{E_1} d_n||^q &= ||J^p_{E_1} y_n - J^p_{E_1} w_n - \tau[Ay_n - Aw_n]||^q \\
&\geq 2^{-1}||J^p_{E_1} y_n - J^p_{E_1} w_n||^q - C_q\tau^q||Ay_n - Aw_n||^q \\
&\geq 2^{-1}||J^p_{E_1} y_n - J^p_{E_1} w_n||^q - C_q\tau^q[L||y_n - w_n||^{(p-1)}]^q \\
&\geq 2^{-1}||J^p_{E_1} y_n - J^p_{E_1} w_n||^q - C_q\tau^q L^q\Big(\frac{p}{c^2}\Big)^q||J^p_{E_1} y_n - J^p_{E_1} w_n||^q \\
&= \Big(\frac{1}{2} - C_q\Big(\frac{\tau L}{K}\Big)^q\Big)||J^p_{E_1} y_n - J^p_{E_1} w_n||^q. \qquad (1.36)
\end{aligned}$$

Combining (1.35) and (1.36), we obtain

(1.37)

$$\Big(\frac{1}{2} - C_q\tau^q L^q\Big(\frac{1}{K}\Big)^q\Big)^{\frac{1}{q}}||J^p_{E_1} y_n - J^p_{E_1} w_n|| \leq ||J^p_{E_1} d_n|| \leq 2\Big(1 + C_q\tau^q L^q\Big(\frac{1}{K}\Big)^q\Big)^{\frac{1}{q}}||J^p_{E_1} y_n - J^p_{E_1} w_n||$$

thus, $J^p_{E_1} y_n = J^p_{E_1} w_n \Rightarrow y_n = w_n$ if and only if $J^p_{E_1} d_n = 0$. Hence, if $y_n = w_n$ or $d_n = 0$, we get

$$y_n = P_C(y_n - \tau Ay_n).$$

□

Lemma 1.11. *Let $\{x_n\}$ be the sequence generated by Algorithm 1.2 under Assumption 1.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \Gamma$, since $w_n \in C$, it follows that $\langle Ax^*, w_n - x^*\rangle \leq 0$, thus by pseudomonotoncity of A, we get

$$\langle Aw_n, w_n - x^*\rangle \leq 0. \tag{1.38}$$

Also, by definition of w_n, i.e. $w_n := \Pi_C[J^q_{E_1}(J^p_{E_1} y_n - \tau Ay_n)]$, letting $J^p_{E_1} b_n := J^p_{E_1} y_n - \tau Ay_n$, then $w_n = \Pi_C z_n$, by (1.19), we get

$$\langle J^p_{E_1} z_n - J^p_{E_1} w_n, w_n - x^*\rangle \geq 0$$

which implies

$$\langle J^p_{E_1} y_n - \tau Ay_n - J^p_{E_1} w_n, w_n - x^*\rangle \geq 0. \tag{1.39}$$

Combining (1.38) and (1.40), we obtain

$$\langle J^p_{E_1} y_n - J^p_{E_1} w_n - \tau(Ay_n - Aw_n), w_n - x^* \rangle \geq 0.$$

Hence

$$\langle J^p_E d_n, w_n - x^* \rangle \geq 0. \tag{1.40}$$

Now, by Lemma 1.2 and 1.5, and relations (1.31), (1.32), (1.40), (1.22) and (1.23), we get

$$\begin{aligned}
\Delta_p(v_n, x^*) &= \Delta_p(J^q_{E_1^*}(J^p_{E_1} y_n - \rho_n J^p_{E_1} d_n), x^*) \\
&= V_p(J^p_{E_1} y_n - \rho_n J^p_{E_1} d_n, x^*) \\
&= \frac{1}{q}||J^p_{E_1} y_n - \rho_n J^p_{E_1}(d_n)||^q - \langle J^p_{E_1} y_n - \rho_n J^p_{E_1}(d_n), x^* \rangle + \frac{1}{p}||x^*||^p \\
&\leq \frac{1}{q}||J^p_{E_1} y_n||^q - \rho_n \langle J^p_{E_1} d_n, y_n \rangle + C_q q^{-1}||\rho_n J^p_{E_1} d_n||^q \\
&\quad - \langle J^p_{E_1} y_n, x^* \rangle + \rho_n \langle J^p_{E_1} d_n, x^* \rangle + \frac{1}{p}||x^*||^p \\
&= V_p(J^p_{E_1} y_n, x^*) - \rho_n \langle J^p_{E_1}(d_n), y_n - x^* \rangle + C_q q^{-1}||\rho_n J^p_{E_1}(d_n)||^q \\
&= \Delta_p(y_n, x^*) - \rho_n \langle J^p_{E_1}(d_n), y_n - w_n \rangle - \rho_n \langle J^p_{E_1}(d_n), w_n - x^* \rangle \\
&\quad + C_q q^{-1}||\rho_n J^p_{E_1}(d_n)||^q \\
&\leq \Delta_p(y_n, x^*) - \rho_n K||J^p_{E_1} y_n - J^p_{E_1} w_n||^q + C_q q^{-1}||\rho_n J^p_{E_1}(d_n)||^q \\
&\leq \Delta_p(y_n, x^*) - ||\rho_n J^p_{E_1}(d_n)||^q + C_q q^{-1}||\rho_n J^p_{E_1}(d_n)||^q \\
&= \Delta_p(y_n, x^*) - (1 - C_q q^{-1})||\rho_n J^p_{E_1}(d_n)||^q \\
&= \Delta_p(y_n, x^*) - (1 - C_q q^{-1})||J^p_{E_1} y_n - J^p_{E_1} v_n||^q. \qquad (1.41)
\end{aligned}$$

From Bregman identity (1.15), we get

$$\Delta_p(y_n, x^*) = \Delta_p(x_n, x^*) - \Delta_p(x_n, y_n) + \langle J^p_{E_1} y_n - J^p_{E_1} x_n, y_n - x^* \rangle. \tag{1.42}$$

Since we have $y_n = J^q_{E^*_{E_1}}(J^p_{E_1} x_n + \theta_n(J^p_{E_1} x_n - J^p_{E_1} x_{n-1}))$, it follows from (1.23), (1.24) and (1.28) that

$$\begin{aligned}
\langle J^p_{E_1} y_n - J^p_{E_1} x_n, y_n - x^* \rangle &\leq ||J^p_{E_1} y_n - J^p_{E_1} x_n||||y_n - x^*|| \\
&= \theta_n ||J^p_{E_1} x_n - J^p_{E_1} x_{n-1}||||y_n - x^*|| \\
&\leq \theta_n ||J^p_{E_1} x_n - J^p_{E_1} x_{n-1}||\Big[\frac{1}{p}||y_n - x^*||^p + \frac{1}{q}\Big] \\
&\leq \frac{\theta_n}{p}||J^p_{E_1} x_n - J^p_{E_1} x_{n-1}||[2^{p-1}(||x_n - y_n||^p + ||x_n - x^*||^p)] \\
&\quad + \frac{\theta_n}{q}||J^p_{E_1} x_n - J^p_{E_1} x_{n-1}|| \\
&\leq \frac{2^{p-1}\mu_n}{p\pi_p}\Big(\Delta_p(x_n, y_n) + \Delta_p(x_n, x^*)\Big) + \frac{\mu_n}{q}. \qquad (1.43)
\end{aligned}$$

Combining (1.42) and (1.43), we get

$$\Delta_p(y_n, x^*) \leq \Big(1 + \frac{2^{p-1}\mu_n}{p\pi_p}\Big)\Delta_p(x_n, x^*) - \Big(1 - \frac{2^{p-1}\mu_n}{p\pi_p}\Big)\Delta_p(x_n, y_n) + \frac{\mu_n}{q}. \tag{1.44}$$

Indeed from the definition of (z_n), Lemma 1.2 and 1.5, and relations (1.23) and (1.2), we get

$$\begin{aligned}
\Delta_p(z_n, x^*) &= \Delta_p(J^q_{E_1}(J^p_{E_1}(v_n) - \lambda_n B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))), x^*) \\
&= V_p(J^p_{E_1}(v_n) - \lambda_n B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n)), x^*) \\
&= \frac{1}{p}||x^*||^p - \langle J^p_{E_1}(v_n) - \lambda_n B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n)), x^*\rangle \\
&\quad + \frac{1}{q}||J^p_{E_1}(v_n) - \lambda_n B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q \\
&\leq \frac{1}{p}||x^*||^p - \langle J^p_{E_1}(v_n), x^*\rangle + \lambda_n\langle B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n)), x^*\rangle + \frac{1}{q}\Big[||J^p_{E_1}(v_n)||^q \\
&\quad - \lambda_n q\langle B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(Sbv_n)), v_n\rangle + C_q\lambda_n^q||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q\Big] \\
&= \frac{1}{p}||x^*||^p + \frac{1}{q}||v_n||^p - \langle J^p_{E_1}(v_n), x^*\rangle - \lambda_n\langle J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n), Bv_n - Bx^*\rangle \\
&\quad + \frac{C_q\lambda_n^q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q \\
&\leq \Delta_p(v_n, x^*) - \frac{\lambda_n}{\xi}\Delta_p(Bv_n, S(Bv_n)) + \frac{C_q\lambda_n^q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q \\
&\leq \Delta_p(v_n, x^*) - \lambda_n\left(\frac{\pi_p}{\xi}||Bv_n - SBv_n||^p - \frac{C_q\lambda_n^{q-1}}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q\right). \quad (1.45)
\end{aligned}$$

On the other hand from the step size λ_n in (1.34), we have

$$\lambda_n^{q-1} \leq \frac{\pi_p q||Bv_n - SBv_n||^p}{C_q\xi||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q} - \epsilon,$$

if and only if

(1.46)
$$\epsilon C_q||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q \leq \frac{\pi_p q}{\xi}||Bv_n - SBv_n||^p - \lambda_n^{q-1}C_q||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q.$$

Thus with left side of λ_n in (1.34) and together with (1.46), we get

$$\begin{aligned}
\frac{\epsilon^2 C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q &\leq \frac{\lambda_n\epsilon C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q \\
&\leq \lambda_n[\frac{\pi_p}{\xi}||Bv_n) - (SBv_n))||^p \\
&\quad - \frac{\lambda_n^{q-1}C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q]. \quad (1.47)
\end{aligned}$$

Combining (1.45) and (1.47), we obtain

$$\Delta_p(z_n, x^*) \leq \Delta_p(v_n, x^*) - \frac{\epsilon^2 C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q. \quad (1.48)$$

Hence, combining (1.41), (1.44) and (1.48), we obtain

$$\begin{aligned}
\Delta_p(z_n, x^*) &\leq \Big(1 + \frac{2^{p-1}\mu_n}{p\pi_p}\Big)\Delta_p(x_n, x^*) - \Big(1 - \frac{2^{p-1}\mu_n}{p\pi_p}\Big)\Delta_p(x_n, y_n) + \frac{\mu_n}{q} \\
&\quad - (1 - C_q q^{-1})||J^p_{E_1}y_n - J^p_{E_1}v_n||^q - \frac{\epsilon^2 C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q. \quad (1.49)
\end{aligned}$$

Since, from (C5) taking $\zeta \in \left(0, \frac{p\pi_p}{2^{p-1}}\right)$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$

$$\frac{\mu_n 2^{p-1}}{p\pi_p} < \alpha_n \zeta.$$

For some constant $M > 0$, it which follows from (1.49) that

$$\Delta_p(z_n, x^*) \leq (1 + \alpha_n\zeta)\Delta_p(x_n, x^*) + \alpha_n M. \tag{1.50}$$

Using the definition of (x_{n+1}), (1.50) and the fact that T is a Bregman strongly relative nonexpansive mapping, we obtain

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p(J^q_{E_1^8}[\alpha_n J^p_{E_1} u + (1-\alpha_n) J^p_{E_1} T z_n], x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(Tz_n, x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(z_n, x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)[(1+\alpha_n\zeta)\Delta_p(x_n, x^*) + \alpha_n M] \\
&\leq \alpha_n \Delta_p(u, x^*) + [1 - \alpha_n(1-\zeta)]\Delta_p(x_n, x^*) + \alpha_n M \\
&= [1 - \alpha_n(1-\zeta)]\Delta_p(x_n, x^*) + \alpha_n(1-\zeta)\frac{\Delta_p(u, x^*) + M}{1-\zeta} \\
&\leq \max\left\{\Delta_p(x_n, x^*), \frac{\Delta_p(u, x^*) + M}{1-\zeta}\right\} \\
&\ \ \vdots \\
&\leq \max\left\{\Delta_p(x_N, x^*), \frac{\Delta_p(u, x^*) + M}{1-\zeta}\right\}.
\end{aligned} \tag{1.51}$$

By induction, we get

$$\Delta_p(x_n, x^*) \leq \max\left\{\Delta_p(x_N, x^*), \frac{\Delta_p(u, x^*) + M}{1-\zeta}\right\}, \text{ for all } n \geq N.$$

This implies that $\{\Delta_p(x_n, x^*)\}$ is bounded. From (1.23), we know that $\pi_p||x_n - x^*||^p \leq \Delta_p(x_n, x^*)$, so $\{x_n\}$ is also bounded. Hence $\{y_n\}, \{v_n\}, \{w_n\}$ and $\{z_n\}$ are bounded. Furthermore, combining (1.49) and (1.51), we get

$$\begin{aligned}
\Delta_p(x_{n+1}, p) \leq\ & \alpha_n \Delta_p(u, x^*) + (1 = \alpha_n\zeta)\Delta_p(x_n, x^*) - (1-\alpha_n\zeta)\Delta_p(x_n, y_n) + \alpha_n M \\
& -(1 - C_q q^{-1}))||J^p_{E_1} y_n - J^p_{E_1} v_n||^q - \frac{\epsilon^2 C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q.
\end{aligned}$$

Thus

$$\Psi_n \leq \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*) + \alpha_n M_1, \tag{1.52}$$

where $\Psi_n := (1-\alpha_n\zeta)\Delta_p(x_n, y_n) + (1 - C_q q^{-1})||J^p_{E_1} y_n - J^p_{E_1} v_n||^q + \frac{\epsilon^2 C_q}{q}||B^*(J^p_{E_2}(Bv_n) - J^p_{E_2}(SBv_n))||^q$ and $M_1 := \sup_{n \geq N}\{\Delta_p(x_n, x^*), M\}$. □

Theorem 1.3. *The sequence $\{x_n\}$ generated by Algorithm 1.2 converges strongly to a point $x^* \in \Gamma$, where $x^* = \Pi_\Gamma u$.*

Proof. **Claim 1.** Let $\{y_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 1.2 under Assumption 1.1. If there exist subsequences $\{y_{n_k}\}$ and $\{w_{n_k}\}$ of $\{y_n\}$ and $\{w_n\}$, respectively such that $\{y_{n_k}\}$ converges weakly to a point say z in H_1 and $\lim_{k\to\infty}||w_{n_k} - y_{n_k}|| = 0$, then $z \in VI(C, A)$.

Claim 2.

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &\leq [1-\alpha_n(1-\zeta)]\Delta_p(x_n,x^*)\\ &+\alpha_n(1-\zeta)\Big[(1-\zeta)^{-1}\Big(\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-x^*\rangle+\frac{\mu_n}{\alpha_n}\Big)\Big].\end{aligned} \quad (1.53)$$

Indeed, using Lemma 1.5 (ii), (iii), (1.49), (1.50) and $\mu_n = \circ(\alpha_n)$ in (C5), we get

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &= \Delta_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(z_n)),x^*)\\ &= V_p(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(z_n),x^*)\\ &\leq V_p(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(z_n)-\alpha_n(J^p_{E_1}(u)-J^p_{E_1}(x^*)),x^*)\\ &\quad -\langle J^q_{E_1^*}(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(z_n))-x^*,-\alpha_n(J^p_{E_1}(u)-J^p_{E_1}(x^*))\rangle\\ &= V_p(\alpha_n J^p_{E_1}(z_0)+(1-\alpha_n)J^p_{E_1}(z_n),x^*)+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-z_0\rangle\\ &= \Delta_p(J^q_{E_1^*}(\alpha_n+(1-\alpha_n)J^p_{E_1}(z_n)),x^*)+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\leq \alpha_n\Delta_p(x^*,x^*)+(1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-x^*\rangle\\ &\leq [1-\alpha_n(1-\zeta)]\Delta_p(x_n,x^*)+\alpha_n(1-\zeta)\Big[(1-\zeta)^{-1}\Big(\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n+1}-x^*\rangle+\frac{\mu_n}{\alpha_n}\Big)\Big].\end{aligned}$$

Claim 2. $\{\Delta_p(x_n,x^*)\}$ converges to zero. That is by Lemma 1.9 and Claim 3, we require only to show that $\limsup_{k\to\infty}\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),x_{n_k+1}-x^*\rangle \leq 0$ for every subsequence $\{\Delta_p(x_{n_k},x^*)\}$ of $\{\Delta_p(x_n,x^*)\}$ satisfying

$$\liminf_{k\to\infty}(\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)) \geq 0.$$

Now, suppose that $\{\Delta_p(x_{n_k},x^*)\}$ is a subsequence of $\{\Delta_p(x_n,x^*)\}$ such that $\liminf_{k\to\infty}(\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)) \geq 0$. Then from (1.52), we get

$$\begin{aligned}\limsup_{k\to\infty}\Psi_{n_k} &\leq \liminf_{k\to\infty}[\Delta_p(x_{n_k},x^*)-\Delta_p(x_{n_k+1},x^*)+\alpha_{n_k}M_1]\\ &\leq \limsup_{k\to\infty}[\Delta_p(x_{n_k},x^*)-\Delta_p(x_{n_k+1},x^*)]+\limsup_{k\to\infty}\alpha_{n_k}M_1\\ &\leq -\liminf_{k\to\infty}[\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)]+\limsup_{k\to\infty}\alpha_{n_k}M_1\\ &\leq 0.\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\Psi_{n_k}=0,$$

where $\Psi_{n_k} := (1-\alpha_n\zeta)\Delta_p(x_{n_k},y_{n_k})+(1-C_q q^{-1})||J^p_{E_1}y_n-J^p_{E_1}v_n||^q+\frac{\epsilon^2 C_q}{q}||B^*(J^p_{E_2}(Bv_{n_k})-J^p_{E_2}(SBv_{n_k}))||^q$. Hence

$$\lim_{k\to\infty}\Delta_p(x_{n_k},y_{n_k})=\lim_{k\to\infty}||J^p_{E_1}y_n-J^p_{E_1}v_n||=\lim_{k\to\infty}||B^*(J^p_{E_2}(Bv_{n_k})-J^p_{E_2}(SBv_{n_k}))||=0. \quad (1.54)$$

Thus, we obtain from (1.54) that

$$\lim_{k\to\infty}||x_{n_k}-y_{n_k}||=\lim_{k\to\infty}||y_{n_k}-v_{n_k}||=0, \quad (1.55)$$

which implies that

$$\lim_{k\to\infty}||x_{n_k}-v_{n_k}||=0. \quad (1.56)$$

Since S is $\xi-$demigeneralized mapping, by (1.23) and for some $M_2 > 0$, we get

$$\begin{aligned}||B^*(J^p_{E_2}(Bv_{n_k}) - J^p_{E_2}(SBv_{n_k}))||M_2 &\geq ||B^*(J^p_{E_2}(Bv_{n_k}) - J^p_{E_2}(SBv_{n_k}))||||v_{n_k} - x^*|| \\ &\geq \langle B^*(J^p_{E_2}(Bv_{n_k}) - J^p_{E_2}(SBv_{n_k})), v_{n_k} - x^*\rangle \\ &= \langle J^p_{E_2}(Bv_{n_k}) - J^p_{E_2}(SBv_{n_k}), Bv_{n_k} - Bx^*\rangle \\ &\geq \frac{1}{\xi}\Delta_p(Bv_{n_k}, SBv_{n_k}) \\ &\geq \frac{\pi_p}{\xi}||Bv_{n_k} - SBv_{n_k}||^p,\end{aligned}$$

it follows from (1.54) that

(1.57) $$\lim_{k\to\infty} ||Bv_{n_k} - SBv_{n_k}|| = 0.$$

By the definition of (z_n) and (1.54), we get

$$\begin{aligned}0 &\leq ||J^p_{E_1} z_{n_k} - J^p_{E_1} v_{n_k}|| \\ &\leq \lambda_{n_k}||B^*(J^p_{E_2}(Bv_{n_k}) - J^p_{E_2}(SBv_{n_k}))|| \to 0\end{aligned}$$

as $k \to \infty$, relying on the fact that $J^q_{E_1}$ is norm-to-norm uniformly continuous on bounded subset of E_1^*, we obtain

(1.58) $$\lim_{k\to\infty} ||z_{n_k} - v_{n_k}|| = 0.$$

Combining (1.56) and (1.58), we get

(1.59) $$\lim_{k\to\infty} ||z_{n_k} - x_{n_k}|| = 0.$$

Using (1.50), we get

$$\begin{aligned}\Delta_p(z_n, x^*) - \Delta_p(Tz_n, x^*) &= \Delta_p(z_n, x^*) - \Delta_p(x_{n+1}, x^*) + \Delta_p(x_{n+1}, x^*) - \Delta_p(Tz_n, x^*) \\ &\leqslant \Delta_p(z_n, x^*) - \Delta_p(x_{n+1}, x^*) + \alpha_n\Delta_p(u, x^*) \\ &+ (1 - \alpha_n)\Delta_p(Tz_n, x^*) - \Delta_p(Tz_n, x^*) \\ &\leq (1 + \alpha_n\zeta)\Delta_p(x_n, x^*) + \alpha_n M - \Delta_p(x_{n+1}, x^*) - \alpha_n\Delta_p(Tz_n, x^*) \\ &= [\Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*) + \alpha_n[\zeta\Delta_p(x_n, x^*) + M - \Delta_p(Tz_n, x^*)]\end{aligned}$$

it follows that

$$\begin{aligned}\limsup_{k\to\infty}[\Delta_p(z_{n_k}, x^*) - \Delta_p(Tz_{n_k}, x^*)] &\leq \limsup_{k\to\infty}[\Delta_p(x_{n_k}, x^*) - \Delta_p(x_{n_k+1}, x^*)] \\ &\quad + \limsup_{k\to\infty} \alpha_{n_k}[\zeta\Delta_p(x_{n_k}, x^*) + M - \Delta_p(Tz_{n_k}, x^*)] \\ &\leq -\liminf_{k\to\infty}[\Delta_p(x_{n_k+1}, x^*) - \Delta_p(x_{n_k}, x^*)] \\ &\leq 0\end{aligned}$$

hence

(1.60) $$\lim_{k\to\infty}[\Delta_p(z_{n_k}, x^*) - \Delta_p(Tz_{n_k}, x^*)] = 0.$$

Thus, by the definition of T, we obtain

(1.61) $$\lim_{k\to\infty} \Delta_p(Tz_{n_k}, z_{n_k}) = 0.$$

Therefore

(1.62) $$\lim_{k\to\infty} ||Tz_{n_k} - z_{n_k}|| = 0.$$

Indeed, from (1.33) and (1.61), we get

$$\Delta_p(x_{n_k+1}, z_{n_k}) \leq \alpha_{n_k}\Delta_p(u, z_{n_k}) + (1-\alpha_{n_k})\Delta_p(Tz_{n_k}, z_{n_k}) \to 0,$$

as $k \to \infty$, hence

$$\lim_{k\to\infty} ||x_{n_k+1} - z_{n_k}|| = 0. \tag{1.63}$$

Combining (1.59) and (1.64), we get

$$\lim_{k\to\infty} ||x_{n_k+1} - x_{n_k}|| = 0. \tag{1.64}$$

Furthermore, from (1.31), we get

$$||J^p_{E_1}y_n - J^p_{E_1}w_n||^q = \frac{\rho_n^{q-1}||J^p_{E_1}d_n||^q}{K^*} = \frac{||\rho_n J^p_{E_1}d_n||^q}{K\rho_n} = \frac{||J^p_{E_1}v_n - J^p_{E_1}y_n||^q}{K^*\rho_n} \tag{1.65}$$

And by (1.35), we get

$$\frac{||J^p_{E_1}y_n - J^p_{E_1}w_n||^q}{||J^p_{E_1}d_n||^q} \geq \frac{1}{2\Big(1 + C_q\Big(\frac{\tau L}{K}\Big)^q\Big)}. \tag{1.66}$$

Therefore, we get

$$\rho_n^{q-1} = K^*\frac{||J^p_{E_1}y_n - J^p_{E_1}w_n||^q}{||J^p_{E_1}d_n||^q} \geq \frac{K^*}{2\Big(1 + C_q\Big(\frac{\tau L}{K}\Big)^q\Big)}$$

thus

$$\frac{1}{\rho_n} \leq \Big(\frac{2\Big(1 + C_q\Big(\frac{\tau L}{K}\Big)^q\Big)}{K^*}\Big)^{\frac{1}{q-1}} =: \kappa \tag{1.67}$$

Combining (1.65) and (1.67), we obtain

$$||J^p_{E_1}y_n - J^p_{E_1}w_n||^q \leq \frac{\kappa}{K^*}||J^p_{E_1}v_n - J^p_{E_1}y_n||^q \tag{1.68}$$

Hence, from (1.54) and (1.68), we get

$$||J^p_{E_1}y_{n_k} - J^p_{E_1}w_{n_k}||^q \leq \frac{\kappa}{K^*}||J^p_{E_1}v_{n_k} - J^p_{E_1}y_{n_k}||^q \to 0$$

as $k \to \infty$. That is,

$$\lim_{k\to\infty} ||J^p_{E_1}y_{n_k} - J^p_{E_1}w_{n_k}|| = 0$$

and by property of $J^p_{E_1}$, we get

$$\lim_{k\to\infty} ||y_{n_k} - w_{n_k}|| = 0. \tag{1.69}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence say $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to say $z \in E_1$ as $j \to \infty$. Then, from (1.55), we have that $\{y_{n_{k_j}}\}$ converges weakly to z as $j \to \infty$, so by (1.69) and Claim 1, we obtain that $z \in VI(C, A)$.
Next, we show that $Bz \in F(S)$. From (1.55), we get that $\{v_{n_{k_j}}\}$ converges weakly to z as $j \to \infty$ and since B is a bounded linear operator, then $\{Bv_{n_{k_j}}\}$ converges weakly to $Bz \in E_2$. Thus, combining (1.57) and demiclosedness of S, we obtain that $Bz \in F(S)$. Also, from (1.59), we get $z_{n_{k_j}}$ converges weakly to z, it follows from (1.62) that $z \in F(T)$,

since $F(T) = \hat{F}(T)$. Hence, from Claim 2, we get $z \in \Gamma := VI(C, A) \cap F(T) \cap B^{-1}(F(T))$. Since $x^* = \Pi_\Gamma u$, from (1.19), we get

$$\begin{aligned} \limsup_{k\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p x^*, x_{n_k} - x^*\rangle &= \lim_{j\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p x^*, x_{n_{k_j}} - x^*\rangle \\ &= \langle J_{E_1}^p(u) - J_{E_1}^p x^*, z - x^*\rangle \leq 0. \end{aligned} \tag{1.70}$$

Combining (1.64) and (1.70), we get

$$\begin{aligned} \limsup_{k\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p x^*, x_{n_k+1} - x^*\rangle &= \limsup_{k\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p x^*, x_{n_k} - x^*\rangle \\ &= \langle J_{E_1}^p(u) - J_{E_1}^p x^*, z - x^*\rangle \leq 0. \end{aligned} \tag{1.71}$$

Therefore, using Lemma 1.9 and Claim 3, we obtain that $\Delta_p(x_n, x^*) \to 0$ as $n \to \infty$, by (1.23), we know that $\pi_p||x_n - x^*||^p \leq \Delta_p(x_n, x^*) \to 0$. Hence, $x_n \to x^*$, where $x^* = \Pi_\Gamma u$. □

2. Numerical Illustration

In this section, we provide some numerical examples for implementing our algorithm.

Example 2.1. We consider this example in $(\mathbb{R}^3, \|\cdot\|_2)$ of the problem considered in Theorem 1.3. For this example, let $C := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle = b\}$, where $a = (2, -1, 5)$ and $b = 1$ then

$$\Pi_C(x) = P_C(x) = \frac{b - \langle a, x\rangle}{\|a\|_2^2}.$$

We choose the operators as follows: Let $T = P_C$, then T is a Bregman strongly nonexpansive mapping (see [31, 32]). We also define the ξ-Bregman demigeneralized type mapping $S : \mathbb{R}^3 \to \mathbb{R}^3$ by $S(x) = \frac{1}{2}x + 1$ for all $x \in \mathbb{R}^3$. Let the mapping $A : C \to E_1^*$ be given by $A(x) = \frac{x}{2}$. Also, we define the bounded linear operator B by

$$B = \begin{pmatrix} 3 & -3 & -5 \\ -4 & 2 & -4 \\ -5 & -2 & 3 \end{pmatrix},$$

then

$$B^* = \begin{pmatrix} 3 & -4 & -5 \\ -3 & 2 & -2 \\ -5 & -4 & 3 \end{pmatrix}.$$

For this example, choose $\alpha_n = \frac{1}{150n+1}$, $\mu_n = \frac{1}{n^{1.1}}$, $\theta = \frac{1}{2}$, $\tau = \frac{1}{5}$ and $\lambda = 0.001$. We make different choices of the initial points x_0 and x_1 with $u = 0.1$ and a stopping criterion $\|x_{n+1} - x_n\| < 10^{-3}$.

Case 1: $x_0 = (1.01, 1.23, 0.01)$ and $x_1 = (1.19, -0.96, 1.01)$;

Case 2: $x_0 = (1, 1, 1)$ and $x_1 = (3, 0, 4)$;

Case 3: $x_0 = (-1, 1, -1)$ and $x_1 = (3.1, 0.78, 1)$;

Case 4: $x_0 = (0, 0.75, 0.25)$ and $x_1 = (0, 2, 0.2)$.

The report of this example is displayed in Figure 1.

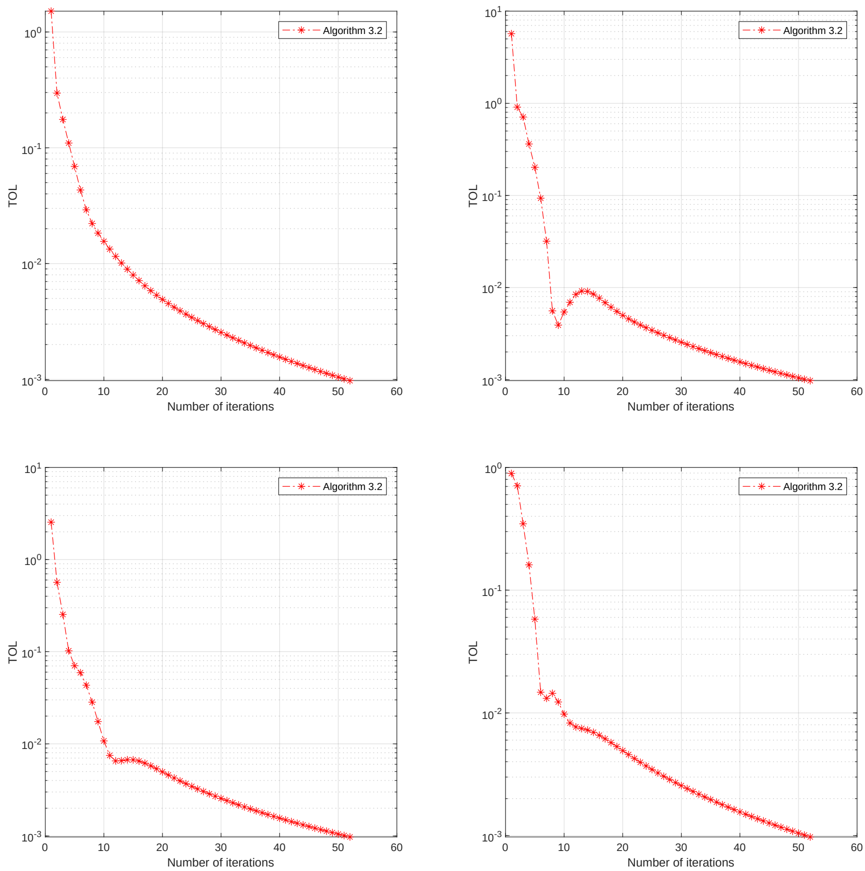

FIGURE 1. Example 2.1, Top left: Case 1; Top right: Case 2; Bottom left : Case 3; Bottom right: Case 4.

Example 2.2. Let $E_1 = E_2 = L_2([0,1])$ with the inner product and norm given by $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ and $\|x\| = \int_0^1 |x(t)|^2 dt$, respectively. Let $C := \{x \in L_2([0,1]) : \langle x, a\rangle = b\}$ where $a = 2t^2$ and $b = 1$. Then,

$$P_C(x) = \max\left\{0, \frac{b - \langle a, x\rangle}{\|a\|^2}\right\} a + x.$$

Let $A : C \to E_1^*$ be given by $A(x) = \max\{0, x(t)\}$ for all $x \in L_2([0,1])$ and $t \in [0,1]$. Define the bounded linear operator $B : E_1 \to E_2$ by $B(x) = \frac{x(t)}{2}$ and $T = P_C$. Also, let $S : E_2 \to E_2$ be defined by $S(y) = 2y(t)$ for all $y \in L_2([0,1])$ and $t \in [0,1]$.
For this example, choose $\alpha_n = \frac{1}{150n+1}$, $\mu_n = \frac{1}{3n+1.1}$, $\theta = \frac{1}{2}$, $\tau = \frac{1}{5}$ and $\lambda = 0.001$. We make different choices of the initial points x_0 and x_1 with $u = 0.1$ and a stopping criterion $\|x_{n+1} - x_n\| < 10^{-3}$.

Case I: $x_0 = t + 3$ and $x_1 = 2t$;

Case II: $x_0 = \sin(2t)$ and $x_1 = \cos(t)$;

Case III: $x_0 = e^{-t}$ and $x_1 = 2t^2$;

Case IV: $x_0 = \log(3t)$ and $x_1 = 11t + 1$.

The report of this example is displayed in Figure 2.

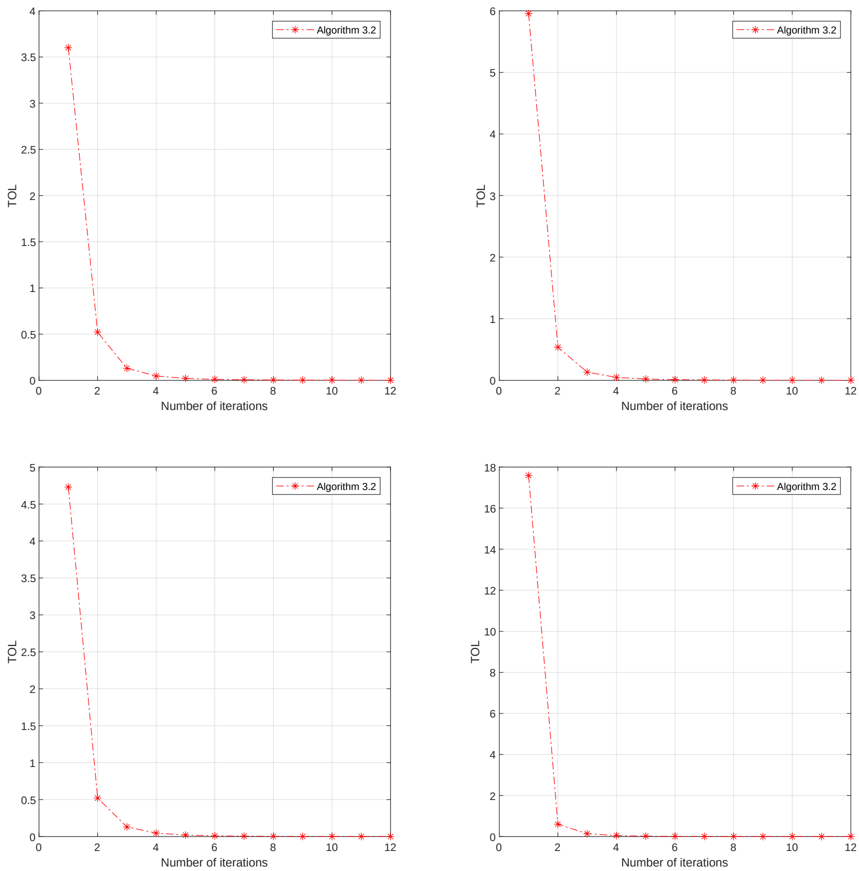

FIGURE 2. Example 2.2, Top left: Case I; Top right: Case II; Bottom left : Case III; Bottom right: Case IV.

3. CONCLUSIONS

We investigated a split variational inequality problem with a pseudomonotone operator and a fixed point issue in the context of real, uniformly convex, and uniformly real Banach spaces. To approximatingly solve the split variational inequality, a pseudomonotone problem, and the common fixed point of Bregman demigeneralized mapping and Bregman strongly nonexpansive mapping, we proposed a contraction and projection method, which is known to be one of the most effective methods for solving variational inequality. A method called inertial extrapolation was incorporated into our suggested algorithm to hasten the rate of convergence of our iterative approach which is new in Banach space. Strong convergence results were achieved by combining our algorithm with Halpern's method, and several numerical examples were used to demonstrate how well our method performed in comparison with some related ones.

Acknowledgments. The first author is grateful to Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University, Pretoria 0204, South Africa and Department of Mathematics and Statistics, Tshwane University of Technology, Staatsartillerie Rd, Pretoria West, Pretoria, 0183, South Africa for supporting this research work, and third author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Post-Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

References

[1] Abass, H. A.; Izuchukwu, C.; Mewomo, O. T.; Dong, Q. L. Strong convergence of an inertial forward-backward splitting method for accretive operators in real Banach spaces. *Fixed Point Theory* **21** (2020), no. 2, 397–412.

[2] Abass, H. A.; Ugwunnadi, G. C.; Narain, O. K.; V. Darvish, V. Inertial extrapolation method for solving variational inequality and fixed point problems of a Bregman demigeneralized mapping in a reflexive Banach space. *Optimization* **2022**, 1–28.

[3] Abass, H. A.; Ugwunnadi G. C.; Narain, O. K. A modified inertial Halpern method for solving split monotone variational inclusion problems in Banach spaces. *Rend. Circ. Mat. Palermo II ser. 2* **2022**, 1–24.

[4] Abass, H. A.; Aremu, K. O.; Jolaoso, L. O.; Mewomo, O. T. An inertial forward-backward splitting method for approximating solutions of certain optimization problem. *J. Nonlinear Funct. Anal.* 2020, **2020**, Article ID 6.

[5] Alansari, M.; Ali, R.; Farid, M. Strong convergence of an inertial iterative algorithm for variational inequality problem, generalized equilibrium problem and fixed point problem in a Banach space. *J. Inequal. Appl.* **2020**, 2020:42.

[6] Antipin, A. S. On a method for convex programs using a symmetrical modification of Lagrange function. *Ekonom I Mate Metod.* **12** (1976), 1164–1173.

[7] Ali, B.; Ugwunnadi, G. C.; Lawan, M. S.; Khan, A. R. Modified inertial subgradient extragradient method in reflexive Banach spaces, *Boletin de la sociedad Matematica Mexicana*. **27** (2021), no. 1, 1–26.

[8] Alvarez, F.; Attouch, H. An Inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.* **9** (2001), 3–11.

[9] Baiocchi, C.; A Capello, A. *Variational and Quasi-variational inequalities.* Wiley, New York 1984.

[10] Bensoussan, A.; Lions, J. L. *Applications of variational inequalities to stochastic control.* North-Holland, Amsterdam 1982.

[11] Bregman, L. M. The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming. *U.S.S.R Comput. Math. Phys.* **7** (1967), 200–217.

[12] Censor, Y.; Gibali, A.; Reich, S. Algorithms for the split variational inequality problem. *Numer. Algor.* **59** (2012), 301–323.

[13] Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **148** (2011), 318–335.

[14] Censor, Y.; Gibali, A.; Reich, S. Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. *Optim. Methods Softw.* **26** (2011), 827–845.

[15] Cheng, Q.; Su, Y.; Zhang, J. Duality fixed point and zero point theorem and application. *Abstr Appl Anal.* Article ID 391301, **2012**, 11 pp.

[16] Chidume, C. E. *Geometric properties of Banach spaces and nonlinear iterations*. Springer Verlag Series, Lecture Notes in Mathematics, ISBN 978-1-84882-189-7, 2009.

[17] Chidume, C. E.; Ikechukwu, S. I.; Adamu, A. Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps. *Fixed Point Theory Appl.* **2018**, Paper No. 9, 9 pp.

[18] Cholamjiak, P.; Thong, D. V.; Cho, Y. J. A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems. *Acta Applicandae Mathematicae* **169** (2020), no. 1, 217–245.

[19] Cholamjiak, P.; Sunthrayuth, P. A Halpern-type iteration for solving the split feasibility problem and fixed point problem of Bregman relatively nonexpansive semigroup in Banach spaces. *Filomat* **32** (2018), no. 9, 3211–3227.

[20] Cioranescu, I. *Geometry of Banach spaces, Duality Mappings and Nonlineqar Problems*. Kluwer Academic, Dordrecht, 1990.
[21] Dong, Q. L. Cho, Y. T. Rassias, T. M. The projection and contraction methods for finding common solutions to variational inequality problems. *Optim. Letter.* **12** (2018), 1871–1896.
[22] Dong, Q. L.; Jiang, D.; Gibali, A. A modified subgradient extragradient method for solving the variational inequality problem. *Numer. Algor.* **9 9** (2018), 927–940.
[23] Dong, Q. L.; Cho, Y. J.; Zhong, L. L.; Rassias, M. Th. Inertial projection and contraction algorithms for variational inequalities. *J. Glob. Optim.* **70** (2018), 687–704.
[24] Fichera, G. Problemi elasstostatici con vincoli unilaterali: II problema di signorini ambigue condizione al contorno. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Nat. Sez. Ia **7** (1963), no. 8, 91–140.
[25] He, B. S. A class of projection and contraction methods for monotone variational inequalities. *Appl. Math. Optim.* **35** (1997), 69–76.
[26] Jia, H.; Liu, S. ; Dang, Y. An inertial iterative algorithm with strong convergence for solving modified split feasibility problem in Banach spaces. *J. Math.* Article ID 9974351, **2021**, 12 pp.
[27] Jolaoso, L. O. An inertial projection and contraction method with a line search technique for variational inequality and fixed point problems. *Optimization* (2021), 1–30.
[28] Jolaoso, L. O.; Alphane, M. Strong convergence inertial projection and contraction method with self adaptive stepsize for pseudomonotone variational inequalities and fixed point problems. *J. Inequal. Appl.* **2020**, Paper No. 261, 22 pp.
[29] Korpelevich, G. M. The extragradient method for finding saddle points and other problems. *Ekonom Mate Metod.* **12** (1976), 747–756.
[30] Mashreghi, J.; Nasri, M. Forcing strong convergence of Korpelevich's method in Banach spaces with its applications in game theory. *Nonlinear Anal*. **72** (2010), no. 3-4, 2086–2099.
[31] Martín-Márquez, V.; Reich, S.; Sabach, S. Bregman strongly nonexpansive operators in reflexive Banach spaces. *J. Math. Anal. Appl.* **400** (2013), 597–614.
[32] Martín-Márquez, V.; Reich. S.; Sabach, S. Right Bregman nonexpansive operators in Banach spaces. *Nonlinear Anal.* **75** (2012), 5448–5465.
[33] Oyewole, O. K.; Mewomo, O. T. A subgradient extragradient algorithm for solving split equilibrium and fixed point problems in reflexive Banach spaces. *J. Nonlinear Funct. Anal.* (2020), DOI:10.23952/jnfa.2020.37.
[34] Pang, J. S. Asymmetric variational inequalities over product of sets" applications and iterative methods. *Math. Program* **31** (1985), 206–219.
[35] Polyak, B. T. Some methods of speeding up the convergence of iterates methods. *U.S.S.R Comput. Math. Phys.* 4(5) (1964), 1–17.
[36] Stampacchia, G. Formes bilinearieres coercitivities sur les ensembles convexes. *C. R. Acad. Sci. Paris*. **258** (1964), 4413–4416.
[37] Taiwo, A.; Jolaoso, L. O.; Mewomo, O. T. A modified Halpern algorithm for approximating a common solution of split equality convex minimization problem and fixed point problem in Uniformly convex Banach spaces. *Comput. Appl. Math.* **38** (2019), no. 2, Paper No. 77, 28 pp.
[38] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control. Optim.* **38** (2000), 431–446.
[39] Schopfer, F.; Schuster, T.; Louis, A. K.; An iterative regularization method for solving the split feasibility problem in Banach spaces. *Inverse Probl.* **24** (2008), no. 5, 055008.
[40] Shehu, Y.; Ogbuisi, F. U.; Iyiola, O. S. Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces. *Optimization* **65** (2016), 299–323.
[41] Saejung, S.; Yotkaew, P. Approximation of zeroes of inverse strongly monotone operators in Banach spaces. *Nonlinear Anal.* **75** (2012), 742–750.
[42] Ugwunnadi, G. C.; Izuchukwu, C.; Khan, A. R.; Dynamical Technique for Split common fixed point problem in Banach spaces. *Comput. Appl. Math.* **41** (2022), no. 4, Paper No. 162, 27 pp.

[1,2,3,4]Department of Mathematics and Applied Mathematics
Sefako Makgato Health Science University
P. O. Box 94, Pretoria 0204, South Africa
Email address: [1]rosemaluleka@rocketmail.com; malulekar@tut.ac.za
Email address: [3]maggie.aphane@smu.ac.za

[1]Department of Mathematics and Statistics
Tshwane University of Technology
Staatsartillerie Rd, Pretoria West, Pretoria, 0183, South Africa
Email address: malulekar@tut.ac.za

[2]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ESWATINI
PRIVATE BAG 4, KWALUSENI, ESWATINI
Email address: ugwunnadi4u@yahoo.com

[4] DSI-NRF CENTER OF EXCELLENCE IN MATHEMATICAL AND STATISTICAL SCIENCES (COE-MASS)
Email address: hammedabass548@gmail.com, AbassH@ukzn.ac.za

[5]DEPARTMENT OF MATHEMATICS AND STATISTICS
INSTITUTE OF SOUTHERN PUNJAB, MULTAN, PAKISTAN
Email address: abdulrahimkhan@isp.edu.pk