

# Existence and approximation of fixed points of enriched contractions in quasi-Banach spaces

VASILE BERINDE

**ABSTRACT.** We obtain results on the existence and approximation of fixed points of enriched contractions in quasi-Banach spaces and thus extend the previous results for enriched contractions defined on Banach spaces [Berinde, V.; Păcurar, M. Approximating fixed points of enriched contractions in Banach spaces. *J. Fixed Point Theory Appl.* **22** (2020), no. 2, Paper No. 38, 10 pp.]. The theoretical results are illustrated by means of an appropriate example of enriched contraction on a quasi-Banach space which is not a Banach space and thus show that our new results are effective generalizations of the previous ones in literature.

## 1. INTRODUCTION

In the recent years, the study of quasi-Banach spaces has developed as an extremely flourishing area of research, see Kalton [31], as it was very natural to investigate which known properties and results in the well established theory of Banach spaces could be transposed or have a non trivial correspondent in the more general context of quasi-Banach spaces.

From the rich literature devoted to such investigations, we mention the following very recent contributions that cover most of the areas where quasi-Banach spaces form an important setting:

- stability of solutions to ordinary differential equations and partial differential equations with impulses: Zada and Mashal [62];
- harmonic analysis on Euclidean spaces (harmonic analysis in several variables; maximal functions, Littlewood-Paley theory etc.): Torres et al. [58];
- Bayesian inference and inverse problems for PDEs: Sullivan [56];
- geometry and structure of normed linear spaces (generalized modulus of convexity, modulus of smoothness, and modulus of Zou-Cui etc.): Kwun et al. [34]; Maligranda [36], Wei [61];
- interpolation of bilinear operators: Grafakos and Mastlyo [27];
- functional analysis in infinite-dimensional not locally convex spaces: Albiac and Ansorena [3], Kalton [30], Sánchez [51];
- ...

If we refer particularly to the study of partial differential equations, there are many function spaces of basic importance in this field that are not Banach spaces but merely quasi-Banach spaces. Amongst these, there are significant portions of the following familiar scales of spaces: Lebesgue spaces, weak Lebesgue spaces, Lorentz spaces, Hardy spaces, weak Hardy spaces, Lorentz-based Hardy spaces, Besov spaces, Triebel-Lizorkin spaces, and weighted versions of these spaces, see Mitrea et al. [39].

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On the other hand, some important results on the existence and approximation of the solutions of the fixed point problem

$$x = Tx,$$

as well as their applications to various nonlinear problems, have been recently established for several classes of enriched contractive mappings in the setting of a Banach space (or Hilbert space). We collect a few of them in order to illustrate the high interest for the study of these classes of mappings:

- enriched strictly pseudocontractive operators: Álvarez [6]; Berinde [10];
- enriched nonexpansive mappings: Deshmukh et al. [24]; Shukla and Pant [54]; Shukla and Panicker [52], [53]; Suantai et al. [55];
- enriched asymptotically nonexpansive mappings in  $CAT(0)$  spaces: Abbas et al. [2]; Salisu et al. [50];
- enriched Banach contractions: Berinde and Păcurar [12]; Mondal et al. [40];
- enriched Kannan mappings (with applications to split feasibility and variational inequality problems): Berinde and Păcurar [13];
- enriched Chatterjea contractions: Berinde and Păcurar [15];
- enriched Ćirić-Reich-Rus contractions: Berinde and Păcurar [14], [16];
- enriched  $\varphi$ -contractions in Banach spaces: Berinde et al. [11];
- enriched almost contractions: Berinde and Păcurar [18];
- enriched multivalued mappings: Abbas et al. [1];
- enriched nonexpansive mappings in geodesic spaces: Ali and Jubair [5];
- best proximity points for cyclic enriched contractions: Chandok [21];
- enriched contractions in partially ordered Banach spaces: Faraji and Radenović [25];
- enriched nonexpansive semigroups: Kesahorm and Sintunavarat [33];
- enriched Prešić operators: Păcurar [41];
- enriched ordered contractions in noncommutative Banach spaces: Rawat et al. [46];
- enriched Kannan mappings in  $CAT(0)$  spaces: Inuwa et al. [29];
- enriched contraction mappings in convex metric spaces: Panicker and Shukla [42];
- enriched contractions in  $CAT(0)$  spaces: Panwar et al. [43];
- $b$ -enriched nonexpansive mapping for solving split variational inclusion problem and fixed point problem in Hilbert spaces: Phairatchatniyom et al. [44];
- $(b, \theta)$ -enriched contractions by means of the degree of nondensifiability: García [26];
- modified Kannan enriched contraction pair: Anjum and Abbas [7];
- ...

Starting from the fact that Schauder's theorem, which is a topological fixed point theorem, holds for any compact convex set in a quasi-Banach space (in fact, in an  $F$ -space, see Kalton [30]), the following question naturally arises:

**Open question:** is it possible to transpose metric fixed point theorems for enriched type mappings from Banach spaces to quasi-Banach spaces ?

The aim of this paper is to answer the above question in the affirmative. More specifically, we are interested to extend some of the fixed point results established in the setting of Banach spaces for various classes of enriched contractions mentioned before to quasi-Banach spaces.

We start by considering enriched Banach contractions in the present paper.

The motivation of our approach is based on the fact that the standard basic results of Banach space theory which depend only on completeness, such as the Uniform Boundedness Principle, Open Mapping Theorem and Closed Graph Theorem, apply to quasi-Banach spaces, while applications of convexity such as the Hahn-Banach theorem are not applicable, see Maligranda [36], Mitrea et al. [39].

In this paper, we investigate the way in which we could extend the results regarding the existence and approximation of fixed points of enriched contraction mappings, established in Berinde and Păcurar [12] for a Banach space, to the more general setting of a quasi-Banach space.

One important tool in obtaining our fixed point results in the setting of a quasi-Banach space is Lemma 2.1 which is adapted to quasi-Banach spaces from its original version established in Miculescu and Mihail [37], see also Suzuki [57], in the case of  $b$ -metric (quasi-metric) spaces.

To our best knowledge, the present approach is brand new and brings relevant novelty to the fixed point theory, as it allows obtaining important fixed point results that complement many similar ones established in  $b$ -metric (quasi-metric) spaces, see Berinde and Păcurar [17] and references therein.

## 2. PRELIMINARIES

In this section, we present some notations, definitions and auxiliary lemmas which are associated to our main results.

All vector spaces will be real, although most arguments may be modified to the complex case.

**Definition 2.1.** A *quasi-norm* on a real vector space  $X$  is a map  $\|\cdot\| : X \rightarrow [0, \infty)$  satisfying the following conditions:

( $QN_0$ )  $\|x\| = 0$  if and only if  $x = 0$ ;

( $QN_1$ )  $\|\lambda x\| = |\lambda| \cdot \|x\|$ , for all  $x \in X$  and  $\lambda \in \mathbb{R}$ .

( $QN_2$ )  $\|x + y\| \leq C (\|x\| + \|y\|)$ , for all  $x, y \in X$ , where  $C \geq 1$  does not depend on  $x, y$ ;

According to Pietsch [45], the concept of quasi-norm has been introduced by Hyers [28] in 1938, under the name "pseudo-norm" (with ( $QN_2$ ) formulated differently but in an equivalent way) and, in the above form, by Bourgin [20] in 1943, who actually proposed the label "quasi-norm".

Obviously, in the particular case  $C = 1$ , a quasi-norm  $d$  reduces to the usual notion of *norm* but, in general, a norm is not a quasi-norm, see examples in this paper.

A related notion to a quasi-norm is that of  $b$ -metric or *quasi-metric*, which appears to have been first introduced by Vulpe et al. [60], but was first known mainly due to the papers by Bakhtin [8], Czerwik [22], [23] and Berinde [9], see Berinde and Păcurar [17] for a recent and well documented survey. It is given by the next definition.

**Definition 2.2.** A function  $d : X \times X \rightarrow \mathbb{R}$  is called a *quasimetric* on  $X$  if it satisfies

(1) (*positivity*)  $d(x, y) \geq 0$  and  $d(x, y) = 0$  if and only if  $x = y$ ;

(2) (*symmetry*)  $d(x, y) = d(y, x)$ , for all  $x, y \in X$ ;

(3) (*quasi-triangle inequality*) there exists a constant  $K \geq 1$  such that

$$(2.1) \quad d(x, z) \leq K [d(x, y) + d(y, z)], \text{ for all } x, y, z \in X.$$

If  $d$  is a quasimetric on  $X$  then the pair  $(X, d)$  is called a *quasimetric space*.

We note that, as in the case of the classical pair norm - metric (distance), the notions of quasi-norm and that of quasimetric are closely interrelated, in the sense that any quasi-norm  $\| \cdot \|$  on a vector space  $X$  induces a quasimetric  $d$  by the formula

$$d(x, y) = \|x - y\|, x, y \in X.$$

According to Pietsch [45], the first example of a quasi-Banach space has been given by Tychonoff [59] in 1935, e.g., the space  $l_{\frac{1}{2}}$ ,

$$l_{\frac{1}{2}} = \left\{ x = (x_1, x_2, \dots, x_n, \dots) : \sum_{i=1}^{\infty} \sqrt{|x_i|} < \infty \right\}$$

with the quasi-norm  $\| \cdot \|$  defined by

$$(2.2) \quad \|x\| = \left( \sum_{i=1}^{\infty} \sqrt{|x_i|} \right)^2,$$

where  $x = (x_1, x_2, \dots, x_n, \dots)$ .

It is easily to prove that the above quasinorm  $\| \cdot \|$  on  $l_{\frac{1}{2}}$  satisfies the quasi-triangle inequality  $(QN_2)$  with  $C = 2$ . A more general example can be given if we consider  $p \in (0, 1)$  instead of  $p = \frac{1}{2}$  in the considerations above, see for example [19].

The smallest possible constant  $C = C_X \geq 1$  in the quasi-triangle inequality  $(QN_2)$  is called the *quasi-triangle constant* of  $(X, \| \cdot \|)$ .

A quasi-norm induces a locally bounded topology on  $X$  and conversely, any locally bounded topology is given by a quasi-norm, see for example Maligranda [36].

Let  $\| \cdot \|$  be a quasi-norm and  $p \in (0, 1)$ . We call  $\| \cdot \|$  a *p-norm* if we also have

$$(2.3) \quad \|x + y\|^p \leq \|x\|^p + \|y\|^p, \forall x, y \in X.$$

A quasi-normed space  $X$  is said to be *p-normable*,  $0 < p \leq 1$ , if there exists an equivalent *p-norm*  $\| \cdot \|_*$  on  $X$  and a constant  $C_1$  such that

$$\|x_1 + \dots + x_n\|_* \leq C_1 (\|x_1\|_*^p + \dots + \|x_n\|_*^p)^{\frac{1}{p}},$$

for all  $x_1, \dots, x_n \in X$ .

An 1-normable space is simply called normable.

Obviously, any *p-normable* space is a quasi-normed space. In fact, we can always assume that the quasi-norm is a *p-norm* for some  $p > 0$ . Indeed, by a theorem of Aoki and Rolewicz (see Kalton [32]), we know that any quasi-norm is equivalent to a *p-norm*, where  $p$  satisfies  $C = 2^{\frac{1}{p}-1}$ . More exactly, the formula

$$\|x\|_1 = \inf \left( \sum_k \|x_k\|^p \right)^{\frac{1}{p}},$$

where the infimum is taken over all finite sequences  $\{x_n\} \subset X$  satisfying  $\sum_k x_k = x$ , defines a *p-norm* such that for any  $x \in X$ , one has

$$\|x\|_1 \leq \|x\| \leq 4^{\frac{1}{p}} \|x\|_1.$$

If a *p-norm*  $\| \cdot \|$  is a quasi-norm on  $X$  defining a complete metrizable topology, then  $X$  is called a *quasi-Banach space*.

Regarding the completeness of a quasi-normed space, we state the following important result (Theorem 1.1, Maligranda [36]) which emphasises the difference between the Cauchy sequences in a Banach space and Cauchy sequences in a quasi-Banach space.

**Theorem 2.1.** *A quasi-normed space  $(X, \|\cdot\|)$  with a quasi-triangle constant  $C \geq 1$  is complete (quasi-Banach space) if and only if for every series such that  $\sum_{n=1}^{\infty} \|x_n\| < \infty$  we have  $\sum_{n=1}^{\infty} x_n \in X$  and*

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} C^{n+1} \|x_n\|.$$

*Proof.* Using the quasi-triangle inequality, for any  $n, p \in \mathbb{N}$ , we have

$$\begin{aligned} \left\| \sum_{k=n}^{n+p} x_k \right\| &\leq C \left( \|x_n\| + C \left\| \sum_{k=n+1}^{n+p} x_k \right\| \right) \\ &\leq C \left[ \|x_n\| + C \left( \|x_{n+1}\| + \left\| \sum_{k=n+2}^{n+p} x_k \right\| \right) \right] \\ &\leq \dots \leq \sum_{k=n}^{n+p} C^{k-n+1} \|x_k\|. \end{aligned}$$

Assume now that  $X$  is a quasi-Banach space and let

$$\sum_{n=1}^{\infty} C^n \|x_n\| = S < \infty.$$

For each  $n \in \mathbb{N}$ , let  $\{S_n\}$  be the sequence of partial sums of the series  $\sum_{n=1}^{\infty} x_n$ , i.e.,

$$S_n = \sum_{k=1}^n x_k.$$

Then

$$\begin{aligned} \|S_{n+p} - S_n\| &= \left\| \sum_{k=n+1}^{n+p} x_k \right\| \leq \sum_{k=n+1}^{n+p} C^{k-n} \|x_k\| \leq \sum_{k=n+1}^{n+p} C^k \|x_k\| \\ &= \sum_{k=1}^{n+p} C^k \|x_k\| - \sum_{k=1}^n C^k \|x_k\| \rightarrow S - S = 0 \text{ as } n, p \rightarrow \infty, \end{aligned}$$

which proves that  $\{S_n\}$  is Cauchy and hence it converges in  $X$ . This means that

$$\sum_{n=1}^{\infty} x_n = x \in X.$$

We also have

$$\left\| \sum_{k=1}^n x_k \right\| \leq \sum_{k=1}^n C^k \|x_k\| \leq \sum_{k=1}^{\infty} C^k \|x_k\|,$$

and therefore

$$\limsup_{n \rightarrow \infty} \|S_n\| \leq \sum_{k=1}^{\infty} C^k \|x_k\|$$

which yields

$$\|x\| = \left\| \sum_{k=1}^{\infty} x_k \right\| \leq C \limsup_{n \rightarrow \infty} (\|x - S_n\| + \|S_n\|) \leq \sum_{n=1}^{\infty} C^{n+1} \|x_n\|.$$

and the first part of the proof is complete.

For the reverse implication, we refer to the complete proof in Maligranda [36]. □

The best known examples of quasi-Banach spaces are the spaces  $l_p$  and  $L_p(0, 1)$ , when  $0 < p < 1$ , which are  $p$ -normable. It is well known, see Kalton [31], that the dual of  $l_p$  is  $l_\infty$ , while the dual of  $L_p(0, 1)$  is  $\{0\}$ , i.e.,  $l_p$  has a separating dual, while  $L_p(0, 1)$  has a trivial dual.

For some examples of quasi-Banach spaces which are important for the study of Ulam's stability of  $n^{\text{th}}$  order nonlinear impulsive differential equations, we refer to Zada and Mashal [62].

We close this section by stating the following useful lemma which shall be useful in proving our main results.

**Lemma 2.1.** *If  $\{x_n\}$  is a sequence in a quasi-normed space  $(X, \|\cdot\|)$  which has the property that there exists  $\gamma \in [0, 1)$  such that*

$$(2.4) \quad \|x_{n+1} - x_n\| \leq \gamma \|x_n - x_{n-1}\|, \forall n \in \mathbb{N},$$

*then  $\{x_n\}$  is a Cauchy sequence.*

*Proof.* We simply take

$$d(x, y) = \|x - y\|, x, y \in X,$$

and apply Lemma 2.2 in Miculescu and Mihail [37] (or Lemma 6 in Suzuki [57]).  $\square$

### 3. ENRICHED CONTRACTIONS IN QUASI-BANACH SPACES

**Definition 3.3.** Let  $(X, \|\cdot\|)$  be a linear quasi-normed space. A mapping  $T : X \rightarrow X$  is said to be an *enriched contraction* if there exist  $b \in [0, +\infty)$  and  $\theta \in [0, b + 1)$  such that

$$(3.5) \quad \|b(x - y) + Tx - Ty\| \leq \theta \|x - y\|, \forall x, y \in X.$$

To indicate the constants involved in (3.5) we shall also call  $T$  a  $(b, \theta)$ -enriched contraction.

**Remark 3.1.** 1) As any Banach space is a quasi-Banach space (with  $C = 1$ ), the enriched contractions  $T$  in a Banach space introduced in Berinde and Păcurar [12] are enriched contractions in the sense of Definition 3.3.

2) It is worth mentioning that, like in the case of Banach spaces, any  $(b, \theta)$ -enriched contraction is continuous.

Considering a self-mapping  $T$  on a convex subset  $E$  of a linear space  $X$ , then for any  $\lambda \in (0, 1]$ , the so-called *averaged mapping*  $T_\lambda$  given by

$$(3.6) \quad T_\lambda x = (1 - \lambda)x + \lambda Tx, \forall x \in E,$$

has the property that  $\text{Fix}(T_\lambda) = \text{Fix}(T)$ , where  $\text{Fix}(T)$  denotes as usually the set of fixed points of  $T$ , i.e.,

$$\text{Fix}(T) = \{x \in E : Tx = x\}.$$

**Theorem 3.2.** *Let  $(X, \|\cdot\|)$  be a quasi-Banach space and  $T : X \rightarrow X$  a  $(b, \theta)$ -enriched contraction. Then*

(i)  $\text{Fix}(T) = \{p\}$ ;

(ii) *There exists  $\lambda \in (0, 1]$  such that the iterative method  $\{x_n\}_{n=0}^\infty$ , given by*

$$(3.7) \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, n \geq 0,$$

*converges to  $p$ , for any  $x_0 \in X$ ;*

*Proof.* We split the proof into two different cases.

**Case 1.**  $b > 0$ . In this case, let us denote  $\lambda = \frac{1}{b+1}$ . Obviously,  $0 < \lambda < 1$  and the enriched contractive condition (3.5) becomes

$$\left\| \left( \frac{1}{\lambda} - 1 \right) (x - y) + Tx - Ty \right\| \leq \theta \|x - y\|, \forall x, y \in X,$$

which can be written in an equivalent form as

$$(3.8) \quad \|T_\lambda x - T_\lambda y\| \leq c \cdot \|x - y\|, \forall x, y \in X,$$

where we denoted  $c = \lambda\theta$  and  $T_\lambda$  is the averaged mapping defined in (3.6).

Since  $\theta \in (0, b + 1)$ , it follows that  $c \in (0, 1)$  and therefore by (3.8)  $T_\lambda$  is a usual Banach contraction with the contraction coefficient  $c$  in the quasi-Banach space  $X$ .

In view of (3.6), the Krasnoselskij iterative process  $\{x_n\}_{n=0}^\infty$  defined by (3.7) is exactly the Picard iteration associated to  $T_\lambda$ , that is,

$$(3.9) \quad x_{n+1} = T_\lambda x_n, n \geq 0.$$

Take  $x = x_n$  and  $y = x_{n-1}$  in (3.8) to get

$$(3.10) \quad \|x_{n+1} - x_n\| \leq c \cdot \|x_n - x_{n-1}\|, n \geq 1.$$

Now, by Lemma 2.1 it follows that  $\{x_n\}_{n=0}^\infty$  is a Cauchy sequence and hence it is convergent in the quasi-Banach space  $(X, \|\cdot\|)$ . Let us denote

$$(3.11) \quad p = \lim_{n \rightarrow \infty} x_n.$$

By letting  $n \rightarrow \infty$  in (3.9) and using the continuity of  $T_\lambda$  (which follows by the continuity of  $T$ ), we immediately obtain

$$p = T_\lambda p,$$

that is,  $p \in \text{Fix}(T_\lambda)$ .

Next, we prove that  $p$  is the unique fixed point of  $T_\lambda$ . Assume that  $q \neq p$  is another fixed point of  $T_\lambda$ . Then, by (3.8)

$$0 < \|p - q\| \leq c \cdot \|p - q\| < \|p - q\|,$$

a contradiction. Hence  $\text{Fix}(T_\lambda) = \{p\}$  and since, by (3.9),  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ , claim (i) is proven.

Conclusion (ii) follows by (3.11).

**Case 2.**  $b = 0$ . In this case,  $\lambda = 1, c = \theta$  and we proceed like in Case 1 but with  $T(= T_1)$  instead of  $T_\lambda$ , when the Krasnoselskij iteration (3.7) reduces in fact to the simple Picard iteration associated to  $T$ ,

$$x_{n+1} = Tx_n, n \geq 0.$$

□

**Example 3.1.** Let  $X = \mathbb{R}^2$  and, for  $x = (x_1, x_2) \in \mathbb{R}^2$ , consider the functionals  $\|\cdot\|_{a,p}$

$$\|x\|_{a,p} = \begin{cases} \|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}, & \text{if } x_2 \neq 0 \\ a|x_1|, & \text{if } x_2 = 0, \end{cases}$$

where  $1 \leq p \leq \infty$  and  $a \neq 1$ . It is well known, see Maligranda [36], that  $\|\cdot\|_{a,p}$  is a quasi-norm on  $X$ , with the quasi-triangle inequality constant

$$C = \max \left\{ a, \frac{1}{a} \right\}.$$

Let  $E = [\frac{1}{2}, 2]^2 \subset X$  and  $T : E \rightarrow E$  be given by  $T(x, y) = (1 - x, 1 - y)$ , for all  $(x, y) \in E$ .

First, we observe that  $T$  is not a Banach contraction on  $(X, \|\cdot\|_{a,p})$ , since it is merely nonexpansive (in fact  $T$  is an isometry).

Next, we prove that  $T$  is an enriched contraction with respect to the quasi-norm  $\|\cdot\|_{a,p}$ . Indeed, for  $u = (x_1, y_1), v = (x_2, y_2) \in E$ , the left hand side of the enriched contraction condition (3.5) will be

$$\begin{aligned} \|Tu - Tv\| &= \|T(x_1, y_1) - T(x_2, y_2)\| \\ &= \|((b - 1)(x_1 - x_2), (b - 1)(y_1 - y_2))\| = |b - 1| \cdot \|u - v\|. \end{aligned}$$

Now, in order to have (3.5) satisfied, we take  $\theta = 1 - b$ , with  $0 < b < 1$ .

So, for any  $b \in (0, 1)$ ,  $T$  is a  $(b, 1 - b)$  enriched contraction on  $(X, \|\cdot\|_{a,p})$ .

It is easily seen that  $Fix(T) = \{(1/2, 1/2)\}$ .

**Remark 3.2.** In the particular case when the quasi-triangle constant  $C = 1$ , by Theorem 3.2 one obtains the first part of the main result in Berinde and Păcurar [12]. We also note that, by following the same technique like the one in proving Theorem 2.4 in Berinde and Păcurar [12], we can also obtain the error estimate that corresponds to (3.13) for the case of quasi-Banach spaces and thus cover the next result.

**Corollary 3.1** (Berinde and Păcurar [12], Theorem 2.4). *Let  $(X, \|\cdot\|)$  be a Banach space and  $T : X \rightarrow X$  a  $(b, \theta)$ -enriched contraction. Then*

(i)  $Fix(T) = \{p\}$ ;

(ii) There exists  $\lambda \in (0, 1]$  such that the iterative method  $\{x_n\}_{n=0}^\infty$ , given by

$$(3.12) \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n \geq 0,$$

converges to  $p$ , for any  $x_0 \in X$ ;

(iii) The following estimate holds

$$(3.13) \quad \|x_{n+i-1} - p\| \leq \frac{c^i}{1 - c} \cdot \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \dots; \quad i = 1, 2, \dots,$$

where  $c = \frac{\theta}{b + 1}$ .

#### 4. LOCAL AND ASYMPTOTIC VERSIONS OF ENRICHED CONTRACTION MAPPING PRINCIPLE

The following example shows that there exist mappings  $T$  which are not contractions but a certain iterate of them is a contraction.

**Example 4.2.** (Examples 1.3.1, [47]) Let  $X = \mathbb{R}$  and  $T : X \rightarrow X$  be given by  $Tx = 0$ , if  $x \in (-\infty, 2]$  and  $Tx = -\frac{1}{3}$ , if  $x \in (2, +\infty)$ . Then  $T$  is not a contraction (being discontinuous) but  $T^2$  is a contraction.

In such a case, we cannot apply the classical Picard-Banach contraction mapping principle and thus the following fixed point theorem is useful, see for example Theorem 1.3.2 in Rus [47].

**Theorem 4.3.** *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow X$  be a mapping. If there exists a positive integer  $N$  such that  $T^N$  is a contraction, then  $Fix(T) = \{x^*\}$ .*

Our first aim in this section is to obtain a similar result for the more general case of enriched contractions in the setting of a quasi-Banach space.

**Theorem 4.4.** *Let  $(X, \|\cdot\|)$  be a quasi-Banach space and let  $U : X \rightarrow X$  be a mapping with the property that there exists a positive integer  $N$  such that  $U^N$  is a  $(b, \theta)$ -enriched contraction. Then*

(i)  $Fix(U) = \{p\}$ ;



(ii) There exists  $\lambda \in (0, 1]$  such that the iterative method  $\{x_n\}_{n=0}^\infty$ , given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda U^N x_n, \quad n \geq 0,$$

converges to  $p$ , for any  $x_0 \in X$ .

*Proof.* We apply Theorem 3.2 (i) for the mapping  $T = U^N$  and obtain that  $Fix(U^N) = \{p\}$ . We also have

$$U^N(U(p)) = U^{N+1}(p) = U(U^N(p)) = U(p),$$

which shows that  $U(p)$  is a fixed point of  $U^N$ . But  $U^N$  has a unique fixed point,  $p$ , hence  $U(p) = p$  and so  $p \in Fix(U)$ .

The remaining part of the proof follows by Theorem 3.2.  $\square$

One of the most interesting generalizations of the contraction mapping principle is the so-called Maia fixed point theorem, see Maia [35], which was obtained by splitting the assumptions in the contraction mapping principle among two metrics defined on the same set.

Recall that a map  $T : X \rightarrow X$  satisfying

(p1)  $T$  has a unique fixed point  $p$  in  $X$ ;

(p2) The Picard iteration  $\{x_n\}_{n=0}^\infty$  converges to  $p$ , for any  $x_0 \in X$ ,

is said to be a *Picard operator*, see Rus [47], [48], [49] for examples and more details.

**Theorem 4.5.** (Rus [47], Theorem 1.3.10) Let  $X$  be a nonempty set,  $d$  and  $\rho$  two metrics on  $X$  and  $T : X \rightarrow X$  a mapping. Suppose that

(i)  $d(x, y) \leq \rho(x, y)$ , for each  $x, y \in X$ ;

(ii)  $(X, d)$  is a complete metric space;

(iii)  $T : X \rightarrow X$  is continuous with respect to the metric  $d$ ;

(iv)  $T$  is a contraction mapping with respect to the metric  $\rho$ .

Then  $T$  is a Picard operator.

The next theorem is an extension of Theorem 4.5 to the class of enriched contractions but it is established in the particular case of a quasi-Banach space.

**Theorem 4.6.** Let  $X$  be a linear space,  $\|\cdot\|_d$  and  $\|\cdot\|_\rho$  two norms on  $X$  and  $T : X \rightarrow X$  a mapping. Suppose that

(i)  $\|x - y\|_d \leq \|x - y\|_\rho$ , for each  $x, y \in X$ ;

(ii)  $(X, \|\cdot\|_d)$  is a quasi-Banach space;

(iii)  $T : X \rightarrow X$  is continuous with respect to the norm  $\|\cdot\|_d$ ;

(iv)  $T$  is a  $(b, \theta)$ -enriched contraction mapping with respect to the norm  $\|\cdot\|_\rho$ .

Then  $T$  is a Picard operator.

*Proof.* Let  $x_0 \in X$ . By (iv), we deduce similarly to the proof of Theorem 3.2 that  $\{T_\lambda^n x_0\}$  is a Cauchy sequence in  $(X, \|\cdot\|_\rho)$ , where as usually  $\lambda = \frac{1}{b+1}$ . By (i),  $\{T_\lambda^n x_0\}$  is a Cauchy sequence in  $(X, \|\cdot\|_d)$  and by (ii) it converges. Let

$$x^* = \lim_{n \rightarrow \infty} T_\lambda^n x_0.$$

By (iii) we obtain that  $x^* \in Fix(T_\lambda)$  and by (iv) that  $Fix(T_\lambda) = \{x^*\}$ . Since  $Fix(T_\lambda) = Fix(T)$ , the conclusion follows.  $\square$

## 5. CONCLUSIONS

1. We introduced the class of enriched contractions in quasi-Banach spaces and obtained an extension of the main result in Berinde and Păcurar [12] regarding the existence and approximation of fixed points of enriched contractions in Banach spaces;
2. We have shown that any enriched contraction in a quasi Banach space has a unique fixed point that can be approximated by means of some Krasnoselskij iteration. In particular, by our fixed point results established in this paper we obtain the classical Banach contraction principle in the setting of a Banach space.
3. It is worth mentioning that enriched contractions preserve a fundamental property of Picard-Banach contractions, namely any enriched contraction has a unique fixed point and is continuous (one can easily show this considering the definition).
4. We also obtained a fixed point theorem of Maia type which extends the Maia fixed point theorem (Theorem 4.5) from the class of enriched contractions in Banach spaces to the class of enriched contractions in quasi-Banach spaces (Theorem 4.6).
5. To our best knowledge, these are the first fixed point results obtained in the setting of a quasi-Banach space. They bring relevant novelty to the fixed point theory, as it allows obtaining important fixed point results that complement many similar in  $b$ -metric (quasi-metric) spaces which cannot be derived in such a setting, see Berinde and Păcurar [17] and references therein.
6. We do hope that our approach will open a new direction of study in the metrical fixed point theory, i.e., the one of extending known results from Banach spaces to relevant ones in quasi-Banach spaces.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE  
 NORTH UNIVERSITY CENTRE AT BAIA MARE  
 TECHNICAL UNIVERSITY OF CLUJ-NAPOCA  
 VICTORIEI 76, 430122 BAIA MARE ROMANIA  
 Email address: vasilie.berinde@mi.utcluj.ro

ACADEMY OF ROMANIAN SCIENTISTS, 3 ILFOV, 050044, BUCHAREST, ROMANIA  
 Email address: vasilie.berinde@gmail.com