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# Proximal point approach for solving convex minimization problems in positive curvature

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ABSTRACT. In this research, we offer a new proximal point technique for approximating a common member of the set of all solutions of the convex minimization problems in the setting of CAT(1) spaces. We further show that, in some modest circumstances, the resulting process is convergent. Our results extend and enhance other related results in the literature.

## 1. INTRODUCTION

The proximal point algorithm(PPA), was first developed by Martinet [16] in 1970. It is an efficient technique for tackling this problem. Later, in 1976, Rockafellar [18] showed that the PPA converges to the solution of the convex problems in Hilbert spaces. Let f be a proper, convex, and lower semi-continuous function on a Hilbert space $\Theta$. The PPA is generated by $\varphi_1 \in \Theta$ and

$$\varphi_{v+1} = \arg\min_{a\in\Theta} \left[\zeta(a) + \frac{1}{2\mu_v}\|a-\varphi_v\|^2\right], \tag{1.1}$$

where $\mu_v > 0$ for all $v \in \mathbb{N}$. It was proved that the sequence $\{\varphi_v\}$ converges weakly to a minimizer of $\zeta$ provided $\sum_{v=1}^{\infty} \mu_v = \infty$.

Bruck and Reich [7] introduced the nonexpansive projections and resolvents of accretive operators in Banach spaces in 1970. Next, Bačák and Reich [3] studied the asymptotic behavior of a class of nonlinear semigroups in Hadamard spaces. Later, Bargetz et al. [4] considered on a large geodesic metric space, including Banach space, hyperbolic space, and geodesic CAT($\kappa$) space, and investigate the space of nonexpansive mappings on either a convex or a star-shaped subset in these settings. They proved that the strict contractions form a negligible subset of this space, in the sense that they form a $\sigma$-porous subset.

The concept of $\Delta$-convergence in general metric spaces was first discussed by Lim [15] in 1976. Let $\kappa \in \mathbb{R}$. Then, a geodesic space that has a geodesic triangle that is sufficiently thinner than the comparable comparison triangle in a model space with curvature $\kappa$ is said to be a CAT($\kappa$) space.

Kirk [14] originally investigated the fixed point theory in CAT($\kappa$) spaces in 2003. Since each CAT($\kappa$) space is a CAT($\kappa'$) space for any $\kappa' \geq \kappa$, the results of a CAT(0) space can be applied to any CAT($\kappa$) space with $\kappa \leq 0$ (see in [6]). However, many researchers have studied CAT($\kappa$) spaces for $\kappa > 0$ (e.g., [1, 5, 8, 10, 17, 19, 20, 21, 22]).

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Kimura et al. [11] introduced the PPA in the following way in a CAT(1) space:

$$\begin{cases} \varphi_1 \in \Theta, \\ \varphi_{v+1} = \arg\min_{a\in\Theta}[\zeta(a) + \dfrac{1}{\mu_v}\tan(\omega(a,\varphi_v))\sin(\omega(a,\varphi_v))], \end{cases} \tag{1.2}$$

where $\mu_v > 0$ for all $v \in \mathbb{N}$. By the Fejér monotonicity, it was proved that, if $\zeta$ has a minimizer and $\sum_{v=1}^{\infty} \mu_v = \infty$, then $\{\varphi_v\}$ $\Delta$-converges to its minimizer (see in [3]).

We would like to solve the common solution of the set of solutions of the convex minimization problems in the setting of CAT(1) spaces by using a new proximal point technique.

## 2. Preliminaries

Let $\{\varphi_v\}$ be a bounded sequence in a complete CAT(1) space $\Theta$. For all $\varphi \in \Theta$, we denote:

$$r(\varphi, \{\varphi_v\}) = \limsup_{v\to\infty} \omega(\varphi, \varphi_v).$$

The asymptotic radius $r(\{\varphi_v\})$ is given by

$$r(\{\varphi_v\}) = \inf\{r(\varphi,\varphi_v) : \varphi \in \Theta\}$$

and the asymptotic center $A(\{\varphi_v\})$ of $\{\varphi_v\}$ is defined as:

$$A(\{\varphi_n\}) = \{\varphi \in \Theta : r(\varphi,\varphi_v) = r(\{\varphi_v\})\}.$$

An element $\varphi$ of $\Theta$ is said to be an asymptotic center of $\{\varphi_v\}$ if

$$r(\varphi, \{\varphi_v\}) = r(\{\varphi_v\}).$$

If the sequence $\{\varphi_v\}$ converges to $\varphi$ in $\Theta$, then $r(\Theta, \{\varphi_v\}) = 0$ and $A(\Theta, \{\varphi_n\}) = \{\varphi\}$ (see more detail in [9]).

Let $(\Theta, \omega)$ be a CAT(1) space. A sequence $\{\varphi_v\}$ in $\Theta$ is said to be $\Delta$-convergent to a point $\varphi \in \Theta$ if $\varphi$ is the unique asymptotic center of every subsequence $\{\varphi_{v_i}\}$ of $\{\varphi_v\}$. In this case, we write $\Delta\text{-}\lim_{v\to\infty} \varphi_v = \varphi$.

A mapping $\psi : \Theta \to \Theta$ is said to be demi-compact if, for any sequence $\{\varphi_v\}$ in $C$ such that $\lim_{v\to\infty} \omega(\varphi_v, \psi\varphi_v) = 0$, $\{\varphi_v\}$ has a convergent subsequence.

If the set $E = \{\varphi \in \Theta : \zeta(\varphi) \le \lambda\}$ is closed in $\Theta$ for all $\lambda \in \mathbb{R}$ then $\zeta$ is said to be lower semi-continuous.

The resolvent of a proper lower semi-continuous function $\zeta$ in admissible CAT(1) spaces for $\mu > 0$ as follows:

$$R_\mu(\varphi) = \arg\min_{b\in\Theta}[\zeta(b) + \frac{1}{\mu}\tan\omega(\varphi,b)\sin\omega(\varphi,b)], \text{ for all } \varphi \in \Theta. \tag{2.3}$$

The mapping$R_\mu$ is well defined and the set of fixed points of the resolvent associated with $\zeta$ coincides with the set of minimizers of $\zeta$ [12].

**Lemma 2.1.** [11] *Let $(\Theta, \omega)$ be an admissible complete CAT(1) space and $\zeta : \Theta \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. If $\mu > 0, \varphi \in \Theta$ and $\lambda^* \in \arg\min \zeta$, then the following inequalities hold:*

(2.4)

$$\frac{\pi}{2}\left(\frac{1}{\cos^2\omega(R_\mu\varphi,\varphi)} + 1\left(\cos\omega(R_\mu\varphi,\varphi)\cos\omega(\lambda^*,R_\mu\varphi) - \cos\omega(\lambda^*,\varphi)\right) \ge \mu\left(\zeta(R_\mu\varphi) - \zeta(\lambda^*)\right)\right.$$

*and*

$$\cos\omega(R_\mu\varphi,\varphi)\cos\omega(\lambda^*,R_\mu\varphi) \ge \cos\omega(\lambda^*,\varphi). \tag{2.5}$$

**Lemma 2.2.** [12] *Let $(\Theta, \omega)$ be a admissible complete CAT(1) space and $\zeta : \Theta \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. Then, $\zeta$ is $\Delta$-lower semi-continuous.*

Denote that $W_\Delta(\{\varphi_v\}) = \{\varphi \in \Theta :$ there exists $\{\sigma_v\} \subset \{\varphi_v\}$ such that $\{\sigma_v\}$ $\Delta-$converges to $\varphi\} \neq \emptyset$.

**Lemma 2.3.** [13] *Let $(\Theta, \omega)$ be a admissible complete CAT(1) space and $\{\varphi_v\}$ be a spherical bounded sequence in $\Theta$. If $\{\omega(\varphi_v, \lambda^*)\}$ is convergent for all $\lambda^* \in W_\Delta(\{\varphi_v\})$, then the sequence $\{\varphi_v\}$ is $\Delta$-convergent.*

**Lemma 2.4.** [17] *Let $\psi : \Theta \to \Theta$ be a nonexpansive mapping defined on a nonempty closed convex subset of a complete CAT(1) space $(\Theta, \omega)$. If $\{\varphi_v\}$ is a bounded sequence with $\lim_{v\to\infty} \omega(\varphi_v, \psi\varphi_v) = 0$ and $\Delta\text{-}\lim_{v\to\infty} \varphi_v = \varphi$, then $\varphi \in \Theta$ and $\psi\varphi = \varphi$.*

## 3. Main results

**Lemma 3.5.** *Let $(\Theta, \omega)$ be an admissible complete CAT(1) space and $\zeta, \xi : \Theta \to (-\infty, \infty]$ be proper lower semi-continuous convex functions. Let $\psi : \Theta \to \Theta$ be a nonexpansive mapping such that $\Lambda = F(\psi) \cap \arg\min_{b\in\Theta} \zeta(b) \cap \arg\min_{c\in\Theta} \xi(c) \neq \emptyset$. Let $\{\rho_v\}, \{\sigma_v\}$ and $\{\alpha_n\}$ be sequences in $[\gamma, \beta]$ for some $\gamma, \beta \in (0, 1)$ and for all $v \geq 1$ and let $\{\mu_v\}$ and $\{\tau_v\}$ be sequences in $(0, \infty)$ such that $0 < \mu \leq \mu_v < 1$ and $0 < \tau \leq \tau_v < 1$ for all $v \geq 1$. For $\varphi_1 \in \Theta$, let $\{\varphi_v\}$ be a sequence in $\Theta$ defined in the following way:*

$$\begin{cases} \delta_v = \arg\min_{b\in\Theta}[\zeta(b) + \dfrac{1}{\mu_v} \tan(\omega(b, \varphi_v)) \sin(\omega(b, \varphi_v))], \\ \eta_v = \arg\min_{c\in\Theta}[\xi(c) + \dfrac{1}{\tau_v} \tan(\omega(c, \delta_v)) \sin(\omega(c, \delta_v))], \\ \chi_v = \psi[(1-\rho_v)\delta_v \oplus \rho_v \psi\eta_v], \\ \phi_v = \psi[(1-\sigma_v)\eta_v \oplus \sigma_v \psi\chi_v], \\ \varphi_{v+1} = \psi[(1-\alpha_v)\psi\chi_v \oplus \alpha_v \psi\phi_v], \end{cases} \tag{3.6}$$

*for each $v \geq 1$. Then, we have the following:*

(1) $\lim_{v\to\infty} \omega(\varphi_v, \lambda^*)$ *exists for each* $\lambda^* \in \Lambda$;
(2) $\lim_{v\to\infty} \omega(\varphi_v, \psi\varphi_v) = 0$.

*Proof.* (1) We show that $\lim_{v\to\infty} \omega(\varphi_v, \lambda^*)$ exists for each $\lambda^* \in \Lambda$. Let $\lambda^* \in \Lambda$. Note that $\delta_v = R_{\mu_v}\varphi_v$ and $\eta_v = R_{\tau_v}\delta_v$ for all $v \geq 1$. Then, from Lemma 2.1, we have

$$\min(\cos\omega(\lambda^*, \delta_v), \cos\omega(\delta_v, \varphi_v)) \geq \cos\omega(\lambda^*, \delta_v)\cos\omega(\delta_v, \varphi_v) \geq \cos\omega(\lambda^*, \varphi_v) \tag{3.7}$$

and

$$\min(\cos\omega(\lambda^*, \eta_v), \cos\omega(\eta_v, \delta_v)) \geq \cos\omega(\lambda^*, \eta_v)\cos\omega(\eta_v, \delta_v) \geq \cos\omega(\lambda^*, \delta_v). \tag{3.8}$$

Thus,

$$\max\{\omega(\lambda^*, \delta_v), \omega(\delta_v, \varphi_v)\} \leq \omega(\lambda^*, \varphi_v), \tag{3.9}$$

and

$$\max\{\omega(\lambda^*, \eta_v), \omega(\eta_v, \delta_v, )\} \leq \omega(\lambda^*, \delta_v), \tag{3.10}$$

which implies that

$$\omega(\lambda^*, \delta_v) \leq \omega(\lambda^*, \varphi_v). \tag{3.11}$$

Using the definition of nonexpansive mapping, $\Theta$ is admissible, and (3.6), we have

$$\begin{aligned}\cos\omega(\lambda^*,\chi_v) &= \cos\omega(\lambda^*,\psi[(1-\rho_v)\delta_v\oplus\rho_v\psi\eta_v])\\ &\geq \cos\omega(\lambda^*,(1-\rho_v)\delta_v\oplus\rho_v\psi\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\psi\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\delta_v)\\ &= \cos\omega(\lambda^*,\delta_v)\\ &\geq \cos\omega(\lambda^*,\varphi_v),\end{aligned}\tag{3.12}$$

$$\begin{aligned}\cos\omega(\lambda^*,\phi_v) &= \cos\omega(\lambda^*,\psi[(1-\sigma_v)\eta_v\oplus\sigma_v\psi\chi_v])\\ &\geq \cos\omega(\lambda^*,(1-\sigma_v)\eta_v\oplus\sigma_v\psi\chi_v)\\ &\geq (1-\sigma_v)\cos\omega(\lambda^*,\eta_v)+\sigma_v\cos\omega(\lambda^*,\psi\chi_v)\\ &\geq (1-\sigma_v)\cos\omega(\lambda^*,\eta_v)+\sigma_v\cos\omega(\lambda^*,\chi_v)\\ &\geq (1-\sigma_v)\cos\omega(\lambda^*,\varphi_v)+\sigma_v\cos\omega(\lambda^*,\varphi_v)\\ &= \cos\omega(\lambda^*,\varphi_v)\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}\cos\omega(\lambda^*,\varphi_{v+1}) &= \cos\omega(\lambda^*,\psi[(1-\alpha_v)\psi\chi_v\oplus\alpha_v\psi\phi_v])\\ &\geq \cos\omega(\lambda^*,(1-\alpha_v)\psi\chi_v\oplus\alpha_v\psi\phi_v)\\ &\geq (1-\alpha_v)\cos\omega(\lambda^*,\psi\chi_v)+\alpha_v\omega(\lambda^*,\psi\phi_v)\\ &\geq (1-\alpha_v)\cos\omega(\lambda^*,\chi_v)+\alpha_v\omega(\lambda^*,\phi_v)\\ &\geq (1-\alpha_v)\cos\omega(\lambda^*,\varphi_v)+\alpha_v\omega(\lambda^*,\varphi_v)\\ &= \cos\omega(\lambda^*,\varphi_v).\end{aligned}\tag{3.14}$$

This has the effect of

$$\omega(\lambda^*,\varphi_{v+1})\leq\omega(\lambda^*,\varphi_v)\leq\omega(\lambda^*,\varphi_1)<\frac{\pi}{2}.\tag{3.15}$$

As a result of (3.11) and (3.15),

$$\limsup_{v\to\infty}\omega(\lambda^*,\delta_v)\leq\limsup_{v\to\infty}\omega(\lambda^*,\varphi_v)<\frac{\pi}{2}.\tag{3.16}$$

Hence, the sequences $\{\delta_v\}$ and $\{\varphi_v\}$ are spherically bounded. So, $\sup_{v\geq1}\omega(\varphi_v,\delta_v)<\frac{\pi}{2}$ and
$\lim_{v\to\infty}\omega(\lambda^*,\varphi_v)<\frac{\pi}{2}$ exists for all $\lambda^*\in\Lambda$.

(2) We show that $\lim_{v\to\infty}\omega(\varphi_v,\psi\varphi_v)=0$. Let

$$\lim_{v\to\infty}\omega(\lambda^*,\varphi_v)=d\geq0.\tag{3.17}$$

As a result of

$$\begin{aligned}\cos\omega(\lambda^*,\varphi_{v+1}) &= \cos\omega(\lambda^*,\psi[(1-\alpha_v)\psi\chi_v\oplus\alpha_v\psi\phi_v])\\ &\geq \cos\omega(\lambda^*,(1-\alpha_v)\psi\chi_v\oplus\alpha_v\psi\phi_v)\\ &\geq (1-\alpha_v)\cos\omega(\lambda^*,\psi\chi_v)+\alpha_v\omega(\lambda^*,\psi\phi_v)\\ &\geq (1-\alpha_v)\cos\omega(\lambda^*,\chi_v)+\alpha_v\omega(\lambda^*,\phi_v)\\ &\geq (1-\alpha_v)\cos\omega(\lambda^*,\varphi_v)+\alpha_v\omega(\lambda^*,\phi_v)\\ &= \cos\omega(\lambda^*,\varphi_v)-\alpha_v\cos\omega(\lambda^*,\varphi_v)+\alpha_v\omega(\lambda^*,\phi_v),\end{aligned}\tag{3.18}$$

which implies that

$$\alpha_v \cos\omega(\lambda^*, \varphi_v) \geq \cos\omega(\lambda^*, \varphi_v) - \cos\omega(\lambda^*, \varphi_{v+1}) + \alpha_v \omega(\lambda^*, \phi_v) \tag{3.19}$$

this is,

$$\cos\omega(\lambda^*, \varphi_v) \geq \frac{1}{\alpha_v}(\cos\omega(\lambda^*, \varphi_v) - \cos\omega(\lambda^*, \varphi_{v+1})) + \omega(\lambda^*, \phi_v). \tag{3.20}$$

Since $\alpha_v \geq \gamma > 0$ for each $v \geq 1$, we obtain

$$\cos\omega(\lambda^*, \varphi_v) \geq \frac{1}{\gamma}(\cos\omega(\lambda^*, \varphi_v) - \cos\omega(\lambda^*, \varphi_{v+1})) + \omega(\lambda^*, \phi_v). \tag{3.21}$$

Using (3.17), it also results in

$$d = \liminf_{v\to\infty} \cos\omega(\lambda^*, \varphi_v) \geq \liminf_{v\to\infty} \omega(\lambda^*, \phi_v). \tag{3.22}$$

From (3.13), we obtain

$$\limsup_{v\to\infty} \omega(\lambda^*, \phi_v) \geq \limsup_{v\to\infty} \cos\omega(\lambda^*, \varphi_v) = d. \tag{3.23}$$

As a result of (3.22) and (3.23), we obtain

$$\lim_{v\to\infty} \cos\omega(\lambda^*, \phi_v) = d. \tag{3.24}$$

Next, we will look at

$$\begin{aligned}
\cos\omega(\lambda^*, \phi_v) &= \cos\omega(\lambda^*, \psi[(1-\sigma_v)\eta_v \oplus \sigma_v \psi\chi_v]) \\
&\geq \cos\omega(\lambda^*, (1-\sigma_v)\eta_v \oplus \sigma_v \psi\chi_v) \\
&\geq (1-\sigma_v)\cos\omega(\lambda^*, \eta_v) + \sigma_v \cos\omega(\lambda^*, \psi\chi_v) \\
&\geq (1-\sigma_v)\cos\omega(\lambda^*, \varphi_v) + \sigma_v \cos\omega(\lambda^*, \psi\chi_v) \\
&\geq (1-\sigma_v)\cos\omega(\lambda^*, \varphi_v) + \sigma_v \cos\omega(\lambda^*, \chi_v) \\
&= \cos\omega(\lambda^*, \varphi_v) - \sigma_v \cos\omega(\lambda^*, \varphi_v) + \sigma_v \cos\omega(\lambda^*, \chi_v),
\end{aligned} \tag{3.25}$$

which implies that

$$\sigma_v \cos\omega(\lambda^*, \varphi_v) \geq \cos\omega(\lambda^*, \varphi_v) - \cos\omega(\lambda^*, \phi_v) + \sigma_v \cos\omega(\lambda^*, \chi_v), \tag{3.26}$$

this is,

$$\cos\omega(\lambda^*, \varphi_v) \geq \frac{1}{\sigma_v}(\cos\omega(\lambda^*, \varphi_v) - \cos\omega(\lambda^*, \phi_v)) + \cos\omega(\lambda^*, \chi_v). \tag{3.27}$$

Since $\sigma_v \geq \gamma > 0$ for each $v \geq 1$, we obtain

$$\cos\omega(\lambda^*, \varphi_v) \geq \frac{1}{\gamma}(\cos\omega(\lambda^*, \varphi_v) - \cos\omega(\lambda^*, \phi_v)) + \cos\omega(\lambda^*, \chi_v). \tag{3.28}$$

Using (3.17) and (3.24), it also results in

$$d = \liminf_{v\to\infty} \cos\omega(\lambda^*, \varphi_v) \geq \liminf_{v\to\infty} \omega(\lambda^*, \chi_v). \tag{3.29}$$

From (3.12), we obtain

$$\limsup_{v\to\infty} \omega(\lambda^*, \chi_v) \geq \limsup_{v\to\infty} \cos\omega(\lambda^*, \varphi_v) = d. \tag{3.30}$$

As a result of (3.29) and (3.30), we obtain

$$\lim_{v\to\infty} \cos\omega(\lambda^*, \chi_v) = d. \tag{3.31}$$

From (3.9)-(3.12), we obtain

$$\begin{aligned}\cos\omega(\lambda^*,\chi_v) &= \cos\omega(\lambda^*,\psi[(1-\rho_v)\delta_v\oplus\rho_v\psi\eta_v])\\ &\geq \cos\omega(\lambda^*,(1-\rho_v)\delta_v\oplus\rho_v\psi\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\psi\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\varphi_v)+\rho_v\cos\omega(\lambda^*,\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\varphi_v)+\rho_v\frac{\cos\omega(\lambda^*,\varphi_v)}{\cos\omega(\delta_v,\varphi_v)}\\ &= \cos\omega(\lambda^*,\varphi_v)+\rho_v\cos\omega(\lambda^*,\varphi_v)\left(\frac{1}{\cos\omega(\delta_v,\varphi_v)}-1\right),\end{aligned}\tag{3.32}$$

which implies that

$$\cos\omega(\lambda^*,\chi_v)-\cos\omega(\lambda^*,\varphi_v)\geq\rho_v\cos\omega(\lambda^*,\varphi_v)\left(\frac{1}{\cos\omega(\delta_v,\varphi_v)}-1\right),\tag{3.33}$$

this is,

$$\frac{\cos\omega(\lambda^*,\chi_v)}{\cos\omega(\lambda^*,\varphi_v)}-1=\rho_v\left(\frac{1}{\cos\omega(\delta_v,\varphi_v)}-1\right).\tag{3.34}$$

Since $\rho_v\geq\gamma>0$ for each $v\geq 1$. As a result of (3.17) and (3.31), we obtain

$$\lim_{v\to\infty}\omega(\delta_v,\varphi_v)=0,\tag{3.35}$$

this implies that

$$\lim_{v\to\infty}\omega(R_{\mu_v}\varphi_v,\varphi_v)=0.\tag{3.36}$$

So, as $\mu_v\geq\mu>0$ for each $v\geq 1$, we obtain

$$\lim_{v\to\infty}\omega(R_{\mu}\varphi_v,\varphi_v)=0.\tag{3.37}$$

Similar to that

$$\begin{aligned}\cos\omega(\lambda^*,\chi_v) &= \cos\omega(\lambda^*,\psi[(1-\rho_v)\delta_v\oplus\rho_v\psi\eta_v])\\ &\geq \cos\omega(\lambda^*,(1-\rho_v)\delta_v\oplus\rho_v\psi\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\psi\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\eta_v)\\ &\geq (1-\rho_v)\cos\omega(\lambda^*,\delta_v)+\rho_v\frac{\cos\omega(\lambda^*,\delta_v)}{\cos\omega(\eta_v,\delta_v)}\\ &= \cos\omega(\lambda^*,\delta_v)+\rho_v\cos\omega(\lambda^*,\delta_v)\left(\frac{1}{\cos\omega(\eta_v,\delta_v)}-1\right),\end{aligned}\tag{3.38}$$

which implies that

$$\cos\omega(\lambda^*,\chi_v)-\cos\omega(\lambda^*,\delta_v)\geq\rho_v\cos\omega(\lambda^*,\delta_v)\left(\frac{1}{\cos\omega(\eta_v,\delta_v)}-1\right),\tag{3.39}$$

this is,

$$\frac{\cos\omega(\lambda^*,\chi_v)}{\cos\omega(\lambda^*,\delta_v)}-1=\rho_v\left(\frac{1}{\cos\omega(\eta_v,\delta_v)}-1\right).\tag{3.40}$$

Since $\rho_v\geq\gamma>0$ for each $v\geq 1$. As a result of (3.17) and (3.31), we obtain

$$\lim_{v\to\infty}\omega(\eta_v,\delta_v)=0,\tag{3.41}$$

which implies that

$$\lim_{v\to\infty} \omega(R_{\tau_v}\delta_v, R_{\mu_v}\varphi_v) = 0. \tag{3.42}$$

So, as $\mu_v \geq \mu > 0$ and $\tau_v \geq \tau > 0$ for each $v \geq 1$, we obtain

$$\lim_{v\to\infty} \omega(R_{\tau}\delta_v, R_{\mu}\varphi_v) = 0. \tag{3.43}$$

From (2.2), we obtain

$$\begin{aligned}
\omega^2(\lambda^*, \chi_v) &= \omega^2(\lambda^*, \psi[(1-\rho_v)\delta_v \oplus \rho_v \psi\eta_v]) \\
&\leq \omega^2(\lambda^*, (1-\rho_v)\delta_v \oplus \rho_v \psi\eta_v) \\
&\leq (1-\rho_v)\omega^2(\lambda^*, \delta_v) + \rho_v\omega^2(\lambda^*, \psi\eta_v) - \frac{R}{2}(1-\rho_v)\rho_v\omega^2(\delta_v, \psi\eta_v) \\
&\leq (1-\rho_v)\omega^2(\lambda^*, \varphi_v) + \rho_v\omega^2(\lambda^*, \eta_v) - \frac{R}{2}\alpha\beta\omega^2(\delta_v, \psi\eta_v) \\
&\leq (1-\rho_v)\omega^2(\lambda^*, \varphi_v) + \rho_v\omega^2(\lambda^*, \varphi_v) - \frac{R}{2}\alpha\beta\omega^2(\delta_v, \psi\eta_v) \\
&= \omega^2(\lambda^*, \varphi_v) - \frac{R}{2}\alpha\beta\omega^2(\delta_v, \psi\eta_v),
\end{aligned}$$

this implies that

$$\omega^2(\delta_v, \psi\eta_v) \leq \frac{2}{R\alpha\beta}(\omega^2(\lambda^*, \varphi_v) - \omega^2(\lambda^*, \chi_v)).$$

Hence,

$$\lim_{v\to\infty} \omega(\delta_v, \psi\eta_v) = 0. \tag{3.44}$$

Also,

$$\begin{aligned}
\omega^2(\lambda^*, \phi_v) &= \omega^2(\lambda^*, \psi[(1-\sigma_v)\eta_v \oplus \sigma_v \psi\chi_v]) \\
&\leq \omega^2(\lambda^*, (1-\sigma_v)\eta_v \oplus \sigma_v \psi\chi_v) \\
&\leq (1-\sigma_v)\omega^2(\lambda^*, \eta_v) + \sigma_v\omega^2(\lambda^*, \psi\chi_v) - \frac{R}{2}(1-\sigma_v)\sigma_v\omega^2(\eta_v, \psi\chi_v) \\
&\leq (1-\sigma_v)\omega^2(\lambda^*, \varphi_v) + \sigma_v\omega^2(\lambda^*, \chi_v) - \frac{R}{2}\alpha\beta\omega^2(\delta_v, \psi\chi_v) \\
&\leq (1-\sigma_v)\omega^2(\lambda^*, \varphi_v) + \sigma_v\omega^2(\lambda^*, \varphi_v) - \frac{R}{2}\alpha\beta\omega^2(\delta_v, \psi\chi_v) \\
&= \omega^2(\lambda^*, \varphi_v) - \frac{R}{2}\alpha\beta\omega^2(\delta_v, \psi\chi_v),
\end{aligned}$$

this implies that

$$\omega^2(\delta_v, \psi\chi_v) \leq \frac{2}{R\alpha\beta}(\omega^2(\lambda^*, \varphi_v) - \omega^2(\lambda^*, \phi_v)).$$

Hence,

$$\lim_{v\to\infty} \omega(\delta_v, \psi\chi_v) = 0. \tag{3.45}$$

Using the triangle inequality along with (3.35), (3.41) and (3.44), we obtain

$$\begin{aligned}
\omega(\varphi_v, \psi\varphi_v) &\leq \omega(\varphi_v, \delta_v) + \omega(\delta_v, \psi\eta_v) + \omega(\psi\eta_v, \psi\delta_v) + \omega(\psi\delta_v, \psi\varphi_v) \\
&\leq 2\omega(\varphi_v, \delta_v) + \omega(\delta_v, \psi\eta_v) + \omega(\eta_v, \delta_v) \\
&\to 0 \quad \text{as} \quad v \to \infty.
\end{aligned} \tag{3.46}$$

□

**Theorem 3.1.** *Let $(\Theta, \omega)$ be an admissible complete CAT(1) space, $\zeta, \xi : \Theta \to (-\infty, \infty]$ be proper lower semi-continuous convex functions and $\psi : \Theta \to \Theta$ be a nonexpansive mapping. Then, the sequence $\{\varphi_v\}$ generated by* (3.6) *$\Delta$-converges to an element of $\Lambda$.*

*Proof.* Let $\lambda^* \in \Lambda$. Then $\zeta(\lambda^*) \leq \zeta(\delta_v)$ for each $v \geq 1$. Now, from Lemma 2.1, we have

(3.47)
$$\mu_v(\zeta(\delta_v) - \zeta(\lambda^*)) \leq \frac{\pi}{2}\left(\frac{1}{\cos^2 \omega(\delta_v, \varphi_v)} + 1\right)(\cos\omega(\delta_v, \varphi_v)\cos\omega(\lambda^*, \delta_v) - \cos\omega(\lambda^*, \varphi_v)),$$

which yields

(3.48)
$$0 \leq \mu_v(\zeta(\delta_v) - \zeta(\lambda^*)) \leq \frac{\pi}{2}\left(\frac{1}{\cos^2 \omega(\delta_v, \varphi_v)} + 1\right)(\cos\omega(\delta_v, \varphi_v)\cos\omega(\lambda^*, \delta_v) - \cos\omega(\lambda^*, \varphi_v)).$$

Since $0 < \mu \leq \mu_v$ for each $v \geq 1$, from Lemma 3.5, we have

$$\lim_{v\to\infty} \omega(\delta_v, \varphi_v) = 0, \quad \lim_{v\to\infty} \omega(\lambda^*, \varphi_v) \quad \text{and} \quad \lim_{v\to\infty} \omega(\lambda^*, \delta_v) \text{ exist.} \tag{3.49}$$

From (3.48) and (3.49), we have

$$\lim_{v\to\infty} \zeta(\delta_v) = \inf \zeta(\Theta). \tag{3.50}$$

Continuing in the same way as above, we have

$$\lim_{v\to\infty} \xi(\eta_v) = \inf \xi(\Theta). \tag{3.51}$$

Next we show that $W_\Delta(\delta_v) = \bigcup_{\{\sigma_v\}\subset\{\delta_v\}} A(\{\delta_v\}) \subset \Lambda$. Let $\bar{\lambda} \in W_\Delta(\delta_v)$. Then there exists a subsequence $\{\sigma_v\}$ of $\{\delta_v\}$ such that $A(\delta_v) = \{\delta\}$. Thus, there exists a subsequence $\{\gamma_v\}$ of $\{\delta_v\}$ such that $\Delta\text{-}\lim_{v\to\infty} \gamma_v = \gamma$ for some $\gamma \in \Lambda$. In view of Lemma 3.5, we have $\lim_{v\to\infty} \omega(\gamma_v, \psi\gamma_v) = 0$ and $\lim_{v\to\infty} \omega(\delta_v, \gamma_v) = 0$ which $\lim_{v\to\infty} \omega(\gamma, \gamma_v)$ and $\lim_{v\to\infty} \omega(\gamma, \delta_v)$ exist and by Lemma 2.4, we have $\delta = \gamma$. This shows that $W_\Delta(\varphi_v) \subset \Lambda$. By using Lemma 2.3, we obtain that $\{\varphi_v\}$ $\Delta$-converges to an element of $\Lambda$. □

**Theorem 3.2.** *Let $(\Theta, \omega)$ be an admissible complete CAT(1) space and $\zeta, \xi : \Theta \to (-\infty, \infty]$ be proper lower semi-continuous convex functions. If one of the mappings $R_\mu$ or $R_\tau$ or $\psi$ is demi-compact, then the sequence $\{\varphi_v\}$ generated by* (3.6) *converges strongly to an element of $\Lambda$.*

*Proof.* From Lemma 3.5, we have

$$\lim_{v\to\infty} \omega(\varphi_v, R_\mu\varphi_v) = \lim_{v\to\infty} \omega(\varphi_v, R_\tau\varphi_v) = \lim_{v\to\infty} \omega(\varphi_v, \psi\varphi_v) = 0. \tag{3.52}$$

We can assume, without losing generality, that $R_\mu$ or $R_\tau$ or $\psi$ is demi-compact, then there exists a subsequence $\{g_v\}$ of $\{\varphi_v\}$ such that $\{g_v\}$ converges strongly to $\bar{\lambda} \in \Theta$. By using (3.52) and the nonexpansiveness of the mappings $R_\mu, R_\tau, \psi$, then we obtain

$$\omega(\bar{\lambda}, R_\mu\bar{\lambda}) = \omega(\bar{\lambda}, R_\tau\bar{\lambda}) = \omega(\bar{\lambda}, \psi\bar{\lambda}) = 0,$$

which yields $\bar{\lambda} \in \Lambda$. Further, we can prove the strong convergence of $\{\varphi_v\}$ to an element of $\Lambda$. This completes the proof. □

## 4. Applications

We obtain various applications to the convex minimization issue and the common fixed point problem in CAT($\kappa$) space, where $\kappa$ is a bounded positive real number.

Throughout this section, we assume that the following assertions hold:

(a) $\Theta$ is a complete CAT($\kappa$) space with $\omega(\varphi_1, \varphi_2) < D_\kappa$ for all $\varphi_1, \varphi_2 \in \Theta$.
(b) $\kappa$ is a positive real number and $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$.
(c) $\zeta, \xi : \Theta \to (-\infty, \infty]$ be proper lower semi-continuous convex functions.

(d) $\widetilde{R}_\mu, \widetilde{R}_\tau$ be resolvent mappings on $\Theta$ defined as

$$\widetilde{R}_\mu = \arg\min_{b\in\Theta}[\zeta(b) + \frac{1}{\mu}\tan(\sqrt{\kappa}\omega(b,\varphi))\sin(\sqrt{\kappa}\omega(b,\varphi))]$$

and

$$\widetilde{R}_\tau = \arg\min_{c\in\Theta}[\xi(c) + \frac{1}{\tau}\tan(\sqrt{\kappa}\omega(c,\delta))\sin(\sqrt{\kappa}\omega(c,\delta))]$$

for all $\mu, \tau > 0$ and $\varphi, \delta \in \Theta$.

**Corollary 4.1.** *Let $(\Theta, \sqrt{\kappa}\omega)$ be an admissible complete CAT(1) space and $\zeta, \xi : \Theta \to (-\infty, \infty]$ be proper lower semi-continuous convex functions. Let $\psi : \Theta \to \Theta$ be a nonexpansive mapping such that $\Lambda = F(\psi) \cap \arg\min_{b\in\Theta}\zeta(b) \cap \arg\min_{c\in\Theta}\xi(c) \neq \emptyset$. Let $\{\rho_v\}, \{\sigma_v\}$ and $\{\alpha_n\}$ be sequences in $[\gamma, \beta]$ for some $\gamma, \beta \in (0,1)$ and for all $v \geq 1$ and let $\{\mu_v\}$ and $\{\tau_v\}$ be sequences in $(0,\infty)$ such that $0 < \mu \leq \mu_v < 1$ and $0 < \tau \leq \tau_v < 1$ for all $v \geq 1$. For $\varphi_1 \in \Theta$, let $\{\varphi_v\}$ be a sequence in $\Theta$ defined in the following way:*

$$(4.53)\qquad \begin{cases} \delta_v = \arg\min_{b\in\Theta}[\zeta(b) + \frac{1}{\mu_v}\tan(\sqrt{\kappa}\omega(b,\varphi_v))\sin(\sqrt{\kappa}\omega(b,\varphi_v))], \\ \eta_v = \arg\min_{c\in\Theta}[\xi(c) + \frac{1}{\tau_v}\tan(\sqrt{\kappa}\omega(c,\delta_v))\sin(\sqrt{\kappa}\omega(c,\delta_v))], \\ \chi_v = \psi[(1-\rho_v)\delta_v \oplus \rho_v\psi\eta_v], \\ \phi_v = \psi[(1-\sigma_v)\eta_v \oplus \sigma_v\psi\chi_v], \\ \varphi_{v+1} = \psi[(1-\alpha_v)\psi\chi_v \oplus \alpha_v\psi\phi_v], \end{cases}$$

*for each $v \geq 1$. If the assumptions (a)-(d) hold then, sequence $\{\varphi_v\}$ $\Delta$-converges to an element of $\Lambda$.*

*Proof.* The proof follows by using assumptions (a)-(d) and the proof of Lemma 3.5 and Theorem 3.1. □

**Corollary 4.2.** *Let $(\Theta, \sqrt{\kappa}\omega)$ be an admissible complete CAT(1) space, $\zeta, \xi : \Theta \to (-\infty, \infty]$ be proper lower semi-continuous convex functions and $\psi : \Theta \to \Theta$ be a nonexpansive mapping. If one of the mappings $\widetilde{R}_\mu$ or $\widetilde{R}_\tau$ or $\psi$ is demi-compact and assumptions (a)-(d) are true, then, the sequence $\{\varphi_v\}$ generated by* (4.53) *$\Delta$-converges to an element of $\Lambda$.*

*Proof.* The proof follows by using assumptions (a)-(d) and the proof of Lemma 3.5 and Theorem 3.2. □

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