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Coupled Best Proximity Points for Cyclic Kannan and Chatterjea Contractions in $\mathrm{CAT_p(0)}$ Spaces

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ABSTRACT. This paper considers the framework of $\mathrm{CAT_p(0)}$ spaces and analyzes coupled best proximity points for cyclic Kannan and cyclic Chatterjea contractions. The framework is substantial and versatile for employing metric notions and contractive conditions. For each mapping, we establish the existence and uniqueness of the coupled best proximity point. The work gives a step toward investigating the best proximity points in $\mathrm{CAT_p(0)}$ spaces.

1. INTRODUCTION

Fixed point theory plays a crucial role in understanding and solving various physical phenomena. This deals with the existence, uniqueness and approximation of points that are mapped onto themselves by a given mapping. The field gained prominence following the establishment of the Banach contraction mapping theorem, which guarantees the existence and uniqueness of a fixed point for a contraction self-mapping. Thereafter several authors analyzed fixed point properties of mappings based on distinct contractive conditions. The prominent contractive mappings whose fixed points are substantial include the *Kannan contraction*, introduced by Kannan [11] and the *Chatterjea contraction*, initiated by Chatterjea [5]. Several other contractive mappings are studied as detailed in Rhoades [21], and more recently, the notion of average mappings has been employed to enrich the classes of such mappings [10].

One significant concept that has emerged from fixed point theory is that of best proximity points, which deals with points closest to a given set. Scholars have extensively investigated best proximity points and their applications in various fields [16, 18, 19, 27]. In addition to best proximity points, coupled best proximity points have attracted attention due to their relevance in nonlinear and convex analysis [1, 8, 17, 26].

Additionally, mappings of cyclic contraction play a vital role in the study of dynamical systems, reflecting the behavior of complex systems. These mappings are defined based on conditions that ensure the distance between the images of any two points is smaller than the distance between the points themselves. Among the different types of contraction mappings, Kannan-type cyclic contraction and Chatterjea-type cyclic contraction mappings have recently captured significant attention due to their interesting properties and potential applications [6, 14]. Further types of contraction mappings that are of special interest to scholars more recently, can be found in [20, 23] and the references therein.

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On the other hand, geodesic metric spaces such as $\mathrm{CAT_p}(0)$ spaces (for $\mathrm{p} \geq 2$) play a crucial role in fixed point theory and optimization. The geometry of these spaces provides flexibility in employing metric notions and contractive conditions, allowing for the coverage of equivalent contractive conditions across more classes of mappings and problems. These frameworks have led to the development of powerful tools and techniques for studying geometry and topology, significantly advancing the study of fixed point theory. As a result, several authors have made immense contributions to this setting (for detailed discussions regarding this framework, refer to [24, 15, 22, 12] and the references therein). It is important to mention that these spaces include Hilbert spaces, uniformly convex metric spaces, Hadamard manifolds, ℓ_{p} spaces, and CAT(κ) spaces for $\kappa \leq 0$.

In this paper, we consider the framework of $\mathrm{CAT_p}(0)$ space and analyze coupled best proximity points of two mappings, namely cyclic Kannan contraction and cyclic Chatterjea contraction. For each mapping, we establish the existence and uniqueness of the coupled best proximity point. Our work is inspired by the research of [9] on best proximity of cyclic contraction in uniformly Banach spaces and the work of [2] in metric-like spaces. We provide examples in nonlinear $\mathrm{CAT_p}(0)$ and linear $\mathrm{CAT_p}(0)$ spaces to support our findings.

The paper is organized in such a way that we present basic concepts related to best proximity points and a $\mathrm{CAT_p}(0)$ space in Section 2, while Section 3 contains the main results with examples, followed by the conclusion in Section 4.

2. Preliminaries

In this section, we provide an overview of the fundamental notations and terminology utilized throughout this work.

2.1. **Best Proximity Points.** Let there be a metric space that is designated by (X, d), and let C and D be nonempty subsets of X. Define

$$\mathrm{Dist}(C, D) := \inf\{d(x, y) : x \in C, y \in D\}.$$

Definition 2.1. Let $S : C \cup D \to C \cup D$. A point $x \in C$ is said to be a *best proximity point* of S if $d(x, Sx) = \mathrm{Dist}(C, D)$.

Definition 2.2. A mapping $S : C \cup D \to C \cup D$ is said to be a *cyclic mapping* if $S(C) \subset D$ and $S(D) \subset C$.

In 2006, Eldred and Veeramani [7] presented the idea of cyclic contraction, which can be described as a generalized contraction mapping in the following manner:

Definition 2.3. [7] A mapping $S : C \cup D \to C \cup D$ is called a *cyclic contraction* if S is a cyclic mapping and there exists $\eta \in [0, 1)$ such that

$$d(Sx, Sy) \leq \eta d(x, y) + (1 - \eta)\mathrm{Dist}(C, D) \tag{2.1}$$

for all $x \in C$ and $y \in D$.

Let C and D be two nonempty subsets of a metric space X and let $S : (C \times D) \cup (D \times C) \to C \cup D$ be a mapping. Then $(u, v) \in C \times D$ is said to be a *coupled best proximity pair* of S if it satisfies

$$d(x, S(x, y)) = \mathrm{Dist}(C, D),$$
$$d(y, S(y, x)) = \mathrm{Dist}(C, D).$$

2.2. **$\mathrm{CAT}_{\mathrm{p}}$(0) Spaces.** Let $(\mathcal{M}, d)$ be a metric space. A mapping that is continuous from the interval $[0,1]$ to $\mathcal{M}$ is referred to as a *path*. If for each pair $a, b \in [0,1]$, $d(\gamma(a), \gamma(b)) = |a-b| d(\gamma(0), \gamma(1))$, then the path $\gamma : [0,1] \to \mathcal{M}$ is a *geodesic*. We refer to $(\mathcal{M}, d)$ as a geodesic space if every two points $x, y \in \mathcal{M}$ are connected by a geodesic, in other words, there exists a geodesic $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In this particular instance, we mention to such a geodesic as $[x, y]$. Note that, in general, such geodesic is not uniquely determined by its endpoints. We then use the notation $z = (1-t)x \oplus ty$ to describe a point $z \in [x, y]$. In this case, we assume that $x \neq y$. If there is a unique geodesic between any two points in $\mathcal{M}$, then the metric space $(\mathcal{M}, d)$ is said to be *uniquely geodesic*. In this case, we refer to the unique geodesic connecting x and y in $\mathcal{M}$ as $[x, y]$. A subset C of $\mathcal{M}$ is said to be a *convex* if all geodesic segments connecting any two points of C are in C.

In a metric space, if the space $\mathcal{M}$ is geodesically connected and every geodesic triangle in $\mathcal{M}$ is at least as "thin" as its comparison triangle in the Euclidean plane (that is, the distance between any two points on the geodesic triangle is no greater than the Euclidean distance between their corresponding comparison points), then $\mathcal{M}$ is considered to be a CAT(0) space. The term "CAT" was coined by M. Gromov and can be found on page 159 of [3].

The Gromov geometric definition of CAT(0) spaces was extended recently by Khamsi and Shukri in [13] to include the scenario in which the comparison triangles belong to a general Banach space, in particular, the situation where the Banach space is ℓ_{p}, $\mathrm{p} \geq 2$.

Recall that a *geodesic triangle*, denoted by the notation $\triangle(x, y, z)$ in a geodesic metric space $(\mathcal{M}, d)$, is formed by three points in x, y, z in $\mathcal{M}$, denoted as the *vertices* of $\triangle$, and a geodesic segment connecting each pair of vertices, denoted as the *edges* of $\triangle$. A *comparison triangle* of a geodesic triangle $\triangle(x, y, z)$ is a triangle in the Banach space ℓ_{p}, for $\mathrm{p} \geq 2$ denoted by $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ satisfying

$$d(x, y) = \|\bar{x} - \bar{y}\|, \quad d(y, z) = \|\bar{y} - \bar{z}\|, \quad \text{and } d(x, z) = \|\bar{x} - \bar{z}\|.$$

A point $\bar{w} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $w \in [x, y]$ if $d(x, w) = \|\bar{x} - \bar{w}\|$.

Definition 2.4. [13] Let $(\mathcal{M}, d)$ be a geodesic metric space. $\mathcal{M}$ is said to be a $\mathrm{CAT}_{\mathrm{p}}(0)$ space, for $\mathrm{p} \geq 2$, if for any geodesic triangle $\triangle$ in $\mathcal{M}$, there exists a comparison triangle $\bar{\triangle}$ in ℓ_{p} such that the comparison axiom is satisfied, i.e., for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$, we have

$$d(x, y) \leq \|\bar{x} - \bar{y}\|.$$

Any normed vector space ℓ_{p}, for $\mathrm{p} \geq 2$ is incontrovertibly a complete $\mathrm{CAT}_{\mathrm{p}}(0)$ space. See [13, 4], for additional information regarding $\mathrm{CAT}_{\mathrm{p}}(0)$ spaces.

Let $x, y, z \in \mathcal{M}$, and $\dfrac{y \oplus z}{2}$ is the midpoint of the geodesic $[y, z]$, then the comparison axiom implies

$$d^{\mathrm{p}}\left(x, \frac{y \oplus z}{2}\right) \leq \frac{1}{2} d^{\mathrm{p}}(x, y) + \frac{1}{2} d^{\mathrm{p}}(x, z) - \frac{1}{2^{\mathrm{p}}} d^{\mathrm{p}}(y, z). \tag{2.2}$$

Khamsi and Shukri [13] developed this inequality, which they called the $(\mathrm{CN}_{\mathrm{p}})$ inequality.

The following lemmas that are going to be presented below are extension results of Eldred and Veeramani's work [7] to the context of $\mathrm{CAT}_{\mathrm{p}}(0)$ metric spaces.

Lemma 2.1. [25] *Let $(\mathcal{M}, d)$ be a complete* $\mathrm{CAT}_{\mathrm{p}}(0)$ *metric space, with $\mathrm{p} \geq 2$. Let C be a nonempty closed and convex subset and D a nonempty closed subset of $\mathcal{M}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C and $\{y_n\}$ a sequence in D satisfying*

(i) $\lim_{n\to\infty} d(z_n, y_n) = \mathrm{Dist}(C, D)$.

(ii) *For every* $\epsilon > 0$, *there exists* N_0, *such that for all* $m > n \geq N_0$,

$$d(x_m, y_n) \leq \mathrm{Dist}(C, D) + \epsilon.$$

Then, for every $\epsilon > 0$, *there exists* N_1, *such that for all* $m > n \geq N_1$, $d(x_m, z_n) \leq \epsilon$.

Lemma 2.2. [25] *Let* $(\mathcal{M}, d)$ *be a complete* $\mathrm{CAT_p(0)}$ *metric space, with* $\mathrm{p} \geq 2$. *Let* C *be a nonempty closed and convex subset and* D *a nonempty closed subset of* $\mathcal{M}$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be sequences in* C *and* $\{y_n\}$ *a sequence in* D *satisfying*

(i) $\lim_{n\to\infty} d(x_n, y_n) = \mathrm{Dist}(C, D)$,
(ii) $\lim_{n\to\infty} d(z_n, y_n) = \mathrm{Dist}(C, D)$.

Then, $\lim_{n\to\infty} d(x_n, z_n) = 0$.

3. Main results

In this section, we present the idea of $p-$cyclic Kannan and Chatterjea contractions, and prove the existence of a unique coupled best proximity point in complete $\mathrm{CAT_p(0)}$ metric spaces, with $\mathrm{p} \geq 2$.

3.1. $p-$cyclic Kannan Contraction.

Let us first introduce the $p-$cyclic Kannan contraction mapping in $\mathrm{CAT_p(0)}$ spaces.

Definition 3.5. Let $(\mathcal{M}, d)$ be a $\mathrm{CAT_p(0)}$ metric space, with $\mathrm{p} \geq 2$. Let C and D be nonempty subsets of $\mathcal{M}$. A mapping $S : (C \times D) \cup (D \times C) \to C \cup D$ is said to be a *$p-$cyclic Kannan contraction* if it satisfies the following conditions:

(i) $S(C, D) \subset D$ and $S(D, C) \subset C$,
(ii) For $(x_1, y_1), (x_2, y_2) \in (C \times D) \cup (D \times C)$ there exists $\eta \in (0, 1/2)$ such that

$$d(S(x_1, y_1), S(x_2, y_2)) \leq \eta(d(x_1, S(x_1, y_1)) + d(x_2, S(x_2, y_2)) + (1 - 2\eta)\mathrm{Dist}(C, D). \tag{3.3}$$

Remark 3.1. The use of p in the term "p-cyclic" is meant to denote "pair cyclic" thereby indicating the necessity of pair sets and distinguishing it from the existing concept of cyclic contraction.

The subsequent results depend on the preceding ones to be proven.

Lemma 3.3. *Let* $(\mathcal{M}, d)$ *be a* $\mathrm{CAT_p(0)}$ *metric space, with* $\mathrm{p} \geq 2$. *Let* C *and* D *be nonempty closed and convex subsets of* $\mathcal{M}$ *and* $S : (C \times D) \cup (D \times C) \to C \cup D$ *a p-cyclic Kannan contraction mapping. Then, for any* $(x_0, y_0) \in C \times D$, *define* $x_n = S(y_{n-1}, x_{n-1})$ *and* $y_n = S(x_{n-1}, y_{n-1})$, *for each* $n \in \mathbb{N}$, *we have*

$$\lim_{n\to\infty} d(x_n, S(x_n, y_n)) = \mathrm{Dist}(C, D),$$
$$\lim_{n\to\infty} d(y_n, S(y_n, x_n)) = \mathrm{Dist}(C, D),$$

and

$$\lim_{n\to\infty} d(x_n, y_n) = \mathrm{Dist}(C, D).$$

Proof. Let $(x_0, y_0) \in C \times D$, define $x_n = S(y_{n-1}, x_{n-1})$ and $y_n = S(x_{n-1}, y_{n-1})$, for each $n \in \mathbb{N}$. In view of (3.3), we obtain

$$\begin{aligned} d(x_1, y_2) &= d(S(y_0, x_0), S(x_1, y_1)) \\ &\leq \eta(d(y_0, S(y_0, x_0)) + d(x_1, S(x_1, y_1))) + (1 - 2\eta)\mathrm{Dist}(C, D) \\ &= \eta(d(y_0, x_1) + d(x_1, y_2)) + (1 - 2\eta)\mathrm{Dist}(C, D). \end{aligned}$$

So, we get

$$d(x_1, y_2) \leq \frac{\eta}{1-\eta} d(y_0, x_1) + \frac{1 - 2\eta}{1 - \eta}\mathrm{Dist}(C, D). \tag{3.4}$$

From the inequality (3.3), we also get

$$\begin{aligned} d(y_1, x_2) &= d(S(x_0, y_0), S(y_1, x_1)) \\ &\leq \eta(d(x_0, S(x_0, y_0)) + d(y_1, S(y_1, x_1))) + (1-2\eta)\mathrm{Dist}(C, D) \\ &= \eta(d(x_0, y_1) + d(y_1, x_2)) + (1-2\eta)\mathrm{Dist}(C, D), \end{aligned}$$

which gives rise to the conclusion that

$$d(y_1, x_2) \leq \frac{\eta}{1-\eta} d(x_0, y_1) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D). \tag{3.5}$$

In a similar way, we have

$$d(x_2, y_3) \leq \frac{\eta}{1-\eta} d(y_1, x_2) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D). \tag{3.6}$$

Substituting (3.5) into (3.6), then we have

$$d(x_2, y_3) \leq \left(\frac{\eta}{1-\eta}\right)^2 d(x_0, y_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2}\right)(1-2\eta)\mathrm{Dist}(C, D). \tag{3.7}$$

We further have

$$d(y_2, x_3) \leq \frac{\eta}{1-\eta} d(x_1, y_2) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D). \tag{3.8}$$

Combining (3.4) and (3.8), we obtain

$$d(y_2, x_3) \leq \left(\frac{\eta}{1-\eta}\right)^2 d(y_0, x_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2}\right)(1-2\eta)\mathrm{Dist}(C, D). \tag{3.9}$$

In the same way, we can deduce that

$$d(x_3, y_4) \leq \frac{\eta}{1-\eta} d(y_2, x_3) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D), \tag{3.10}$$

and

$$d(y_3, x_4) \leq \frac{\eta}{1-\eta} d(x_2, y_3) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D). \tag{3.11}$$

By combining (3.10) and (3.11) with (3.9) and (3.7), respectively, we get

$$d(x_3, y_4) \leq \left(\frac{\eta}{1-\eta}\right)^3 d(y_0, x_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3}\right)(1-2\eta)\mathrm{Dist}(C, D)$$

and

$$d(y_3, x_4) \leq \left(\frac{\eta}{1-\eta}\right)^3 d(x_0, y_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3}\right)(1-2\eta)\mathrm{Dist}(C, D).$$

We are able to demonstrate, through the use of induction, that for every odd integer n, we possess

$$\begin{aligned} d(x_n, y_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(y_0, x_1) + \bigg(&\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3} + \ldots \\ &+ \frac{\eta^{n-1}}{(1-\eta)^n}\bigg)(1-2\eta)\mathrm{Dist}(C, D) \end{aligned} \tag{3.12}$$

and

$$d(y_n, x_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(x_0, y_1) + \bigg(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3} + \ldots$$

$$+\left.\frac{\eta^{n-1}}{(1-\eta)^n}\right)(1-2\eta)\mathrm{Dist}(C,D), \tag{3.13}$$

and for all even integer n, we have

$$\begin{aligned}d(x_n,y_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(x_0,y_1) + \left(\frac{1}{1-\eta}+\frac{\eta}{(1-\eta)^2}+\frac{\eta^2}{(1-\eta)^3}+\dots\right.\\ \left. +\frac{\eta^{n-1}}{(1-\eta)^n}\right)(1-2\eta)\mathrm{Dist}(C,D)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}d(y_n,x_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(y_0,x_1) + \left(\frac{1}{1-\eta}+\frac{\eta}{(1-\eta)^2}+\frac{\eta^2}{(1-\eta)^3}+\dots\right.\\ \left. +\frac{\eta^{n-1}}{(1-\eta)^n}\right)(1-2\eta)\mathrm{Dist}(C,D).\end{aligned} \tag{3.15}$$

According to equations (3.12), (3.13), (3.14), and (3.15), the following holds for every $n \in \mathbb{N}$:

$$\begin{aligned}d(x_n,S(x_n,y_n)) &= d(x_n,y_{n+1})\\ &\leq \left(\frac{\eta}{1-\eta}\right)^n \max\{d(x_0,y_1),d(y_0,x_1)\}\\ &\quad+\left(\frac{1}{1-\eta}+\frac{\eta}{(1-\eta)^2}+\frac{\eta^2}{(1-\eta)^3}+\dots+\frac{\eta^{n-1}}{(1-\eta)^n}\right)(1-2\eta)\mathrm{Dist}(C,D).\end{aligned}$$

From the fact that $\eta = (0,1/2)$ and taking $n\to\infty$, we obtain

$$\lim_{n\to\infty} d(x_n,S(x_n,y_n)) = \mathrm{Dist}(C,D). \tag{3.16}$$

Similarly,

$$\lim_{n\to\infty} d(y_n,S(y_n,x_n)) = \mathrm{Dist}(C,D). \tag{3.17}$$

Next, we prove that $\lim_{n\to\infty} d(x_{n+1},y_{n+1}) = \mathrm{Dist}(C,D)$. In view of (3.3), we have

$$\begin{aligned}\mathrm{Dist}(C,D) &\leq d(x_n,y_n)\\ &= d(S(y_{n-1},x_{n-1}),S(x_{n-1},y_{n-1}))\\ &\leq \eta(d(y_{n-1},S(y_{n-1},x_{n-1}))+d(x_{n-1},S(x_{n-1},y_{n-1})))+(1-2\eta)\mathrm{Dist}(C,D).\end{aligned}$$

Letting $n\to\infty$ together with (3.16) and (3.17), we get

$$d(x_n,y_n) = \mathrm{Dist}(C,D).$$

Therefore, the proof is completed. □

Now, we are ready to prove the existence of coupled best proximity points of $p-$cyclic Kannan contraction mapping.

Theorem 3.1. *Let $(\mathcal{M},d)$ be a complete* $\mathrm{CAT_p(0)}$ *metric space, with* $\mathrm{p}\geq 2$. *Let C and D be nonempty closed and convex subsets of $\mathcal{M}$ and $S:(C\times D)\cup(D\times C)\to C\cup D$ a p-cyclic Kannan contraction mapping. Then S has a unique coupled best proximity point.*

Proof. Let $(x_0,y_0)\in C\times D$ and define $x_n = S(y_{n-1},x_{n-1})$ and $y_n = S(x_{n-1},y_{n-1})$ for each $n\in\mathbb{N}$. We can establish the following by using Lemma 3.3:

$$\lim_{n\to\infty} d(x_n,y_{n+1}) = \lim_{n\to\infty} d(x_n,S(x_n,y_n)) = \mathrm{Dist}(C,D) \tag{3.18}$$

and

$$\lim_{n\to\infty} d(x_{n+1}, y_{n+1}) = \mathrm{Dist}(C, D). \tag{3.19}$$

In light of (3.18) and (3.19), and utilizing Lemma 2.2, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

The same reasoning process guarantees that

$$\lim_{n\to\infty} d(y_{n+1}, y_{n+2}) = 0.$$

Let us show that for any $\epsilon > 0$, there is a number $N_0 \in \mathbb{N}$ such that

$$d(x_m, y_{n+1}) \leq \mathrm{Dist}(C, D) + \epsilon \quad \text{for all} \quad m > n \geq N_0. \tag{3.20}$$

Assume the opposite is true. Then, for all $k \geq 1$, there is an integer $\epsilon_0 > 0$ such that there is a positive integer $m_k > n_k \geq k$ such that

$$d(x_{m_k}, y_{n_k+1}) > \mathrm{Dist}(C, D) + \epsilon_0 \text{ and } d(x_{m_k-1}, y_{n_k+1}) \leq \mathrm{Dist}(C, D) + \epsilon_0.$$

Consider,

$$\begin{aligned}
\mathrm{Dist}(C, D) + \epsilon_0 &< d(x_{m_k}, y_{n_k+1}) \\
&\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, y_{n_k+1}) \\
&\leq d(x_{m_k}, x_{m_k-1}) + \mathrm{Dist}(C, D) + \epsilon_0.
\end{aligned}$$

This implies that $\lim_{k\to\infty} d(x_{m_k}, y_{n_k+1}) = \mathrm{Dist}(C, D) + \epsilon_0$. From S is a $p-$cyclic Kannan contraction mapping and the triangle inequality, we can derive

$$\begin{aligned}
d(x_{m_k}, y_{n_k+1}) &= d(S(y_{m_k-1}, x_{m_k-1}), S(x_{n_k}, y_{n_k})) \\
&\leq \eta\,(d(y_{m_k-1}, S(y_{m_k-1}, x_{m_k-1})) + d(x_{n_k}, S(x_{n_k}, y_{n_k}))) + (1 - 2\eta)\mathrm{Dist}(C, D).
\end{aligned}$$

Letting $k \to \infty$ and using Lemma 3.3, we get

$$\mathrm{Dist}(C, D) + \epsilon_0 \leq 2\eta\mathrm{Dist}(C, D) + (1 - 2\eta)\mathrm{Dist}(C, D) = \mathrm{Dist}(C, D),$$

which is a contradiction. It follows from (3.18), (3.20) and Lemma 2.1 that there exists a positive integer $N_1 \in \mathbb{N}$ such that $d(x_m, x_n) \leq \epsilon$ for all $m > n \geq N_1$. This means that $\{x_n\}$ is Cauchy sequence in C. In a similar vein, $\{y_n\}$ is Cauchy sequence in D. Therefore, $\{(x_n, y_n)\}$ has a convergent subsequence in $C \times D$.

Let $\{(x_{n_k}, y_{n_k})\}$ be a of subsequence $\{(x_n, y_n)\}$ such that

$$\lim_{k\to\infty} (x_{n_k}, y_{n_k}) = (u^*, v^*)$$

for some $(u^*, v^*) \in C \times D$. This implies that $x_{n_k} \to u^*$ and $y_{n_k} \to v^*$ as $k \to \infty$. Consider

$$\begin{aligned}
d(u^*, S(u^*, v^*)) &\leq d(u^*, x_{n_k}) + d(x_{n_k}, S(u^*, v^*)) \\
&= d(u^*, x_{n_k}) + d(S(y_{n_k-1}, x_{n_k-1}), S(u^*, v^*)) \\
&\leq d(u^*, x_{n_k}) + \eta(d(y_{n_k-1}, S(y_{n_k-1}, x_{n_k-1})) + d(u^*, S(u^*, v^*))) \\
&\quad + (1 - 2\eta)\mathrm{Dist}(C, D) \\
&= d(u^*, x_{n_k}) + \eta d(y_{n_k-1}, S(y_{n_k-1}, x_{n_k-1})) + \eta d(u^*, S(u^*, v^*)) \\
&\quad + (1 - 2\eta)\mathrm{Dist}(C, D).
\end{aligned}$$

Thus, it can be deduced that

$$(1 - \eta)d(u^*, S(u^*, v^*)) \leq d(u^*, x_{n_k}) + \eta d(y_{n_k-1}, S(y_{n_k-1}, x_{n_k-1})) + (1 - 2\eta)\mathrm{Dist}(C, D)$$

and hence

$$d(u^*, S(u^*, v^*)) \leq \frac{1}{1-\eta} d(u^*, x_{n_k}) + \frac{\eta}{1-\eta} d(y_{n_k-1}, S(y_{n_k-1}, x_{n_k-1})) + \frac{1-2\eta}{1-\eta} \mathrm{Dist}(C, D).$$

In view of the last inequality and the definition of $\mathrm{Dist}(C, D)$, we get

$$\begin{aligned} \mathrm{Dist}(C, D) &\leq d(u^*, S(u^*, v^*)) \\ &\leq \frac{1}{1-\eta} d(u^*, x_{n_k}) + \frac{\eta}{1-\eta} d(y_{n_k-1}, S(y_{n_k-1}, x_{n_k-1})) + \frac{1-2\eta}{1-\eta} \mathrm{Dist}(C, D). \end{aligned}$$

By letting $k \to \infty$, we obtain from Lemma 3.3 that

$$\begin{aligned} \mathrm{Dist}(C, D) &\leq d(u^*, S(u^*, v^*)) \\ &\leq \frac{\eta}{1-\eta} \mathrm{Dist}(C, D) + \frac{1-2\eta}{1-\eta} \mathrm{Dist}(C, D) \\ &= \mathrm{Dist}(C, D). \end{aligned}$$

Therefore

$$d(u^*, S(u^*, v^*)) = \mathrm{Dist}(C, D).$$

Similarly, we can obtain

$$d(v^*, S(v^*, u^*)) = \mathrm{Dist}(C, D).$$

That is (u^*, v^*) is a coupled best proximity of S.

Let $(\bar{u}, \bar{v})$ be another coupled best proximity of S such that $u^* \neq \bar{u}$ and $v^* \neq \bar{v}$, and we show that the coupled best proximity of S is unique. So,

$$d(\bar{u}, S(\bar{u}, \bar{v})) = \mathrm{Dist}(C, D) \tag{3.21}$$

and

$$d(\bar{v}, S(\bar{v}, \bar{u})) = \mathrm{Dist}(C, D). \tag{3.22}$$

Since S is $p-$cyclic Kannan contraction, we have

$$\begin{aligned} d(u^*, S(\bar{u}, \bar{v})) &\leq d(u^*, x_{n_k+1}) + d(x_{n_k+1}, S(\bar{u}, \bar{v})) \\ &= d(u^*, x_{n_k+1}) + d(S(y_{n_k}, x_{n_k}), S(\bar{u}, \bar{v})) \\ &\leq d(u^*, x_{n_k+1}) + \eta d(y_{n_k}, S(y_{n_k}, x_{n_k})) + \eta d(\bar{u}, S(\bar{u}, \bar{v})) + (1-2\eta)\mathrm{Dist}(C, D) \\ &= d(u^*, x_{n_k+1}) + \eta d(y_{n_k}, S(y_{n_k}, x_{n_k})) + \eta \mathrm{Dist}(C, D) + (1-2\eta)\mathrm{Dist}(C, D) \\ &= d(u^*, x_{n_k+1}) + \eta d(y_{n_k}, S(y_{n_k}, x_{n_k})) + (1-\eta)\mathrm{Dist}(C, D). \end{aligned}$$

Letting $k \to \infty$, we deduce that

$$\begin{aligned} d(u^*, S(\bar{u}, \bar{v})) &\leq \eta d(v^*, G(v^*, u^*)) + (1-\eta)\mathrm{Dist}(C, D) \\ &= \eta \mathrm{Dist}(C, D) + (1-\eta)\mathrm{Dist}(C, D) \\ &= \mathrm{Dist}(C, D). \end{aligned}$$

Therefore, by the definition of $\mathrm{Dist}(C, D)$, we have

$$d(u^*, S(\bar{u}, \bar{v})) = \mathrm{Dist}(C, D). \tag{3.23}$$

Similarly, we also have

$$d(v^*, S(\bar{v}, \bar{u})) = \mathrm{Dist}(C, D). \tag{3.24}$$

Let $\frac{\bar{u} \oplus u^*}{2}$ is the midpoint of the geodesic $[\bar{u}, u^*]$. Then by the ($\mathrm{CN_p}$) inequality, we have

$$d^p\left(\frac{\bar{u} \oplus u^*}{2}, S(\bar{u}, \bar{v})\right) \leq \frac{1}{2} d^p(S(\bar{u}, \bar{v}), \bar{u}) + \frac{1}{2} d^p(S(\bar{u}, \bar{v}), u^*) - \frac{1}{2^p} d^p(\bar{u}, u^*).$$

We are able to rewrite it as

$$d^p(\bar{u}, u^*) \leq 2^{p-1} d^p(S(\bar{u}, \bar{v}), \bar{u}) + 2^{p-1} d^p(S(\bar{u}, \bar{v}), u^*) - 2^p d^p \left(\frac{\bar{u} \oplus u^*}{2}, S(\bar{u}, \bar{v}) \right).$$

In view of (3.21), (3.23) and the definition of $\mathrm{Dist}(C, D)$, we obtain

$$\begin{aligned} d^p(\bar{u}, u^*) &\leq 2^{p-1}\mathrm{Dist}(C, D)^p, +2^{p-1}\mathrm{Dist}(C, D)^p - 2^p\mathrm{Dist}(C, D)^p \\ &= 2(2^{p-1})\mathrm{Dist}(C, D)^p - 2^p\mathrm{Dist}(C, D)^p \\ &= 0. \end{aligned}$$

Hence, $d(\bar{u}, u^*) = 0$, this implies that $u^* = \bar{u}$. Based on equations (3.22) and (3.24), we can demonstrate in a similar manner that $v^* = \bar{v}$. As a consequence, S has a unique coupled best proximity. □

3.2. $p-$**cyclic Chatterjea Contraction.** Now, we introduce the $p-$cyclic Chatterjea contraction mapping in the setting of a complete $\mathrm{CAT_p}(0)$ metric space, with $\mathrm{p} \geq 2$.

Definition 3.6. Let $(\mathcal{M}, d)$ be a $\mathrm{CAT_p}(0)$ metric space, with $\mathrm{p} \geq 2$. Let C and D be nonempty subsets of $\mathcal{M}$. A mapping $S : (C \times D) \cup (D \times C) \to C \cup D$ is said to be a $p-$*cyclic Chatterjea contraction* if it satisfies the following conditions:

(i) $S(C, D) \subset D$ and $S(D, C) \subset C$,

(ii) For $(x_1, y_1), (x_2, y_2) \in (C \times D) \cup (D \times C)$ there exists $\eta \in (0, 1/4)$ such that

$$d(S(x_1, y_1), S(x_2, y_2)) \leq \eta(d(x_2, S(x_1, y_1)) + d(x_1, S(x_2, y_2))) + (1 - 4\eta)\mathrm{Dist}(C, D). \tag{3.25}$$

The following lemma is going to be very important in proving our main results.

Lemma 3.4. *Let $(\mathcal{M}, d)$ be a $\mathrm{CAT_p}(0)$ metric space, with $\mathrm{p} \geq 2$. Let C and D be nonempty closed and convex subsets of $\mathcal{M}$ and $S : (C \times D) \cup (D \times C) \to C \cup D$ a p-cyclic Chatterjea contraction mapping. Then, for any $(x_0, y_0) \in C \times D$, define $x_n = S(y_{n-1}, x_{n-1})$ and $y_n = S(x_{n-1}, y_{n-1})$, for each $n \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} d(x_n, S(x_n, y_n)) = \mathrm{Dist}(C, D),$$
$$\lim_{n\to\infty} d(y_n, S(y_n, x_n)) = \mathrm{Dist}(C, D),$$

and

$$\lim_{n\to\infty} d(x_n, y_n) = \mathrm{Dist}(C, D).$$

Proof. Let $(x_0, y_0) \in C \times D$, define $x_n = S(y_{n-1}, x_{n-1})$ and $y_n = S(x_{n-1}, y_{n-1})$, for each $n \in \mathbb{N}$. From S is $p-$cyclic Chatterjea contraction mapping, we have

$$\begin{aligned} d(x_1, y_2) &= d(S(y_0, x_0), S(x_1, y_1)) \\ &\leq \eta(d(x_1, S(y_0, x_0)) + d(y_0, S(x_1, y_1))) + (1 - 4\eta)\mathrm{Dist}(C, D) \\ &< \eta(d(x_1, S(y_0, x_0)) + d(y_0, S(x_1, y_1))) + (1 - 2\eta)\mathrm{Dist}(C, D) \\ &= \eta(d(x_1, x_1) + d(y_0, y_2)) + (1 - 2\eta)\mathrm{Dist}(C, D) \\ &\leq \eta(d(y_0, x_1) + d(x_1, y_2)) + (1 - 2\eta)\mathrm{Dist}(C, D). \end{aligned}$$

Consequently, we arrive at

$$d(x_1, y_2) \leq \frac{\eta}{1 - \eta} d(y_0, x_1) + \frac{1 - 2\eta}{1 - \eta}\mathrm{Dist}(C, D). \tag{3.26}$$

In view of (3.25), we get

$$\begin{aligned} d(y_1, x_2) &= d(S(x_0, y_0), S(y_1, x_1)) \\ &\leq \eta(d(y_1, S(x_0, y_0)) + d(x_0, S(y_1, x_1))) + (1 - 4\eta)\mathrm{Dist}(C, D) \end{aligned}$$

$$< \eta(d(y_1, S(x_0, y_0)) + d(x_0, S(y_1, x_1))) + (1 - 2\eta)\mathrm{Dist}(C, D)$$
$$= \eta(d(y_1, y_1) + d(x_0, x_2)) + (1 - 2\eta)\mathrm{Dist}(C, D)$$
$$\leq \eta(d(x_0, y_1) + d(y_1, x_2)) + (1 - 2\eta)\mathrm{Dist}(C, D).$$

This implies that

$$d(y_1, x_2) \leq \frac{\eta}{1-\eta} d(x_0, y_1) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D). \tag{3.27}$$

In a similar way, we have

$$d(x_2, y_3) \leq \frac{\eta}{1-\eta} d(y_1, x_2) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D).$$

Substituting (3.27) into the last inequality, then

$$d(x_2, y_3) \leq \left(\frac{\eta}{1-\eta}\right)^2 d(x_0, y_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2}\right)(1 - 2\eta)\mathrm{Dist}(C, D). \tag{3.28}$$

We further have

$$d(y_2, x_3) \leq \frac{\eta}{1-\eta} d(x_1, y_2) + \frac{1-2\eta}{1-\eta}\mathrm{Dist}(C, D).$$

Combining (3.26) and the above inequality, we obtain

$$d(y_2, x_3) \leq \left(\frac{\eta}{1-\eta}\right)^2 d(y_0, x_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2}\right)(1 - 2\eta)\mathrm{Dist}(C, D). \tag{3.29}$$

We are able to show, through the process of induction, that for all even integers n that are odd, we are in possession of

$$d(x_n, y_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(y_0, x_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3} + \dots + \frac{\eta^{n-1}}{(1-\eta)^n}\right)(1 - 2\eta)\mathrm{Dist}(C, D)$$

and

$$d(y_n, x_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(x_0, y_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3} + \dots + \frac{\eta^{n-1}}{(1-\eta)^n}\right)(1 - 2\eta)\mathrm{Dist}(C, D),$$

and for all even integers n, we have

$$d(x_n, y_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(x_0, y_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3} + \dots + \frac{\eta^{n-1}}{(1-\eta)^n}\right)(1 - 2\eta)\mathrm{Dist}(C, D)$$

and

$$d(y_n, x_{n+1}) \leq \left(\frac{\eta}{1-\eta}\right)^n d(y_0, x_1) + \left(\frac{1}{1-\eta} + \frac{\eta}{(1-\eta)^2} + \frac{\eta^2}{(1-\eta)^3} + \dots + \frac{\eta^{n-1}}{(1-\eta)^n}\right)(1 - 2\eta)\mathrm{Dist}(C, D).$$

By analogy with the proof of Lemma 3.3, we can show that

$$\lim_{n\to\infty} d(x_n, S(x_n, y_n)) = \mathrm{Dist}(C, D) \tag{3.30}$$

and

$$\lim_{n\to\infty} d(y_n, S(y_n, x_n)) = \text{Dist}(C, D). \tag{3.31}$$

Next, we show that $\lim_{n\to\infty} d(x_n, y_n) = \text{Dist}(C, D)$. Based on S is $p-$cyclic Chatterjea contraction, we have

$$\begin{aligned} d(x_n, y_n) &= d(S(y_{n-1}, x_{n-1}), S(x_{n-1}, y_{n-1})) \\ &\le \eta(d(x_{n-1}, S(y_{n-1}, x_{n-1})) + d(y_{n-1}, S(x_{n-1}, y_{n-1}))) + (1-4\eta)\text{Dist}(C, D) \\ &= \eta(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) + (1-4\eta)\text{Dist}(C, D) \\ &\le \eta(d(x_{n-1}, y_n) + d(y_n, x_n) + d(y_{n-1}, x_n) + d(x_n, y_n)) + (1-4\eta)\text{Dist}(C, D) \\ &= 2\eta d(x_n, y_n) + \eta(d(x_{n-1}, S(x_{n-1}, y_{n-1})) + d(y_{n-1}, S(y_{n-1}, x_{n-1}))) \\ &\quad + (1-4\eta)\text{Dist}(C, D), \end{aligned}$$

and we further have

$$d(x_n, y_n) \le \frac{\eta}{1-2\eta}(d(x_{n-1}, S(x_{n-1}, y_{n-1})) + d(y_{n-1}, S(y_{n-1}, x_{n-1}))) + \frac{1-4\eta}{1-2\eta}\text{Dist}(C, D).$$

Letting $n \to \infty$ together with the definition of $\text{Dist}(C, D)$ and (3.30), (3.31), we get

$$\begin{aligned} \text{Dist}(C, D) &\le d(x_n, y_n) \\ &\le \frac{\eta}{1-2\eta}(\text{Dist}(C, D) + \text{Dist}(C, D)) + \frac{1-4\eta}{1-2\eta}\text{Dist}(C, D) \\ &= \frac{2\eta}{1-2\eta}\text{Dist}(C, D) + \frac{1-4\eta}{1-2\eta}\text{Dist}(C, D) \\ &= \text{Dist}(C, D), \end{aligned}$$

and hence

$$\lim_{n\to\infty} d(x_n, y_n) = \text{Dist}(C, D).$$

Therefore, the proof is completed. □

Next, we prove the existence of a unique coupled best proximity point of the $p-$cyclic Chatterjea contraction mapping.

Theorem 3.2. *Let $(\mathcal{M}, d)$ be a complete $\text{CAT}_\text{p}(0)$ metric space, with $\text{p} \ge 2$. Let C and D be nonempty closed and convex subsets of $\mathcal{M}$ and $S : (C \times D) \cup (D \times C) \to C \cup D$ be a $p-$cyclic Chatterjea contraction mapping. Then S has a unique coupled best proximity point.*

Proof. Let $(x_0, y_0) \in C \times D$ and define $x_n = S(y_{n-1}, x_{n-1})$ and $y_n = S(x_{n-1}, y_{n-1})$ for each $n \in \mathbb{N}$. By making use of Lemma 3.4, we are able to establish the following:

$$\lim_{n\to\infty} d(x_n, y_{n+1}) = \lim_{n\to\infty} d(x_n, S(x_n, y_n)) = \text{Dist}(C, D) \tag{3.32}$$

and

$$\lim_{n\to\infty} d(x_{n+1}, y_{n+1}) = \text{Dist}(C, D). \tag{3.33}$$

In light of (3.32) and (3.33), and utilizing Lemma 2.2, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Following the same method, we are able to prove

$$\lim_{n\to\infty} d(y_{n+1}, y_{n+2}) = 0.$$

We next prove that for any $\epsilon > 0$, there is a number $N_0 \in \mathbb{N}$ such that

$$d(x_m, y_{n+1}) \le \text{Dist}(C, D) + \epsilon, \text{ and for all } \quad m > n \ge N_0. \tag{3.34}$$

Suppose the contrary. Then, for all $k \geq 1$, there is an integer $\epsilon_0 > 0$ such that there is a positive integer $m_k > n_k \geq k$ such that

$$d(x_{m_k}, y_{n_k+1}) > \text{Dist}(C, D) + \epsilon_0, \quad d(x_{m_k-1}, y_{n_k+1}) \leq \text{Dist}(C, D) + \epsilon_0.$$

Consider,

$$\begin{aligned}\text{Dist}(C, D) + \epsilon_0 &< d(x_{m_k}, y_{n_k+1}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, y_{n_k+1}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \text{Dist}(C, D) + \epsilon_0.\end{aligned}$$

This implies that $\lim_{k\to\infty} d(x_{m_k}, y_{n_k+1}) = \text{Dist}(C, D) + \epsilon_0$. Since S is a $p-$cyclic Chatterjea contraction mapping and by the triangle inequality, we get

$$\begin{aligned}d(x_{m_k}, y_{n_k+1}) &= d(S(y_{m_k-1}, x_{m_k-1}), S(x_{n_k}, y_{n_k})) \\ &\leq \eta\left(d(x_{n_k}, S(y_{m_k-1}, x_{m_k-1})) + d(y_{m_k-1}, S(x_{n_k}, y_{n_k}))\right) + (1-4\eta)\text{Dist}(C, D) \\ &\leq \eta\left(d(x_{n_k}, y_{n_k+1}) + d(y_{n_k+1}, x_{m_k}) + d(y_{m_k-1}, x_{m_k}) + d(x_{m_k}, y_{n_k+1})\right) \\ &\quad + (1-4\eta)\text{Dist}(C, D) \\ &\leq 2\eta d(x_{m_k}, y_{n_k+1}) + \eta(d(x_{n_k}, S(x_{n_k}, y_{n_k})) + d(y_{m_k-1}, S(y_{m_k-1}, x_{m_k-1}))) \\ &\quad + (1-4\eta)\text{Dist}(C, D).\end{aligned}$$

By rearrangement the last inequality, we have

$$d(x_{m_k}, y_{n_k+1}) \leq \frac{\eta}{1-2\eta}(d(x_{n_k}, S(x_{n_k}, y_{n_k})) + d(y_{m_k-1}, S(y_{m_k-1}, x_{m_k-1}))) + \frac{1-4\eta}{1-2\eta}\text{Dist}(C, D).$$

Letting $k \to \infty$ together with Lemma 3.4, we get

$$\begin{aligned}\text{Dist}(C, D) + \epsilon_0 &\leq d(x_{m_k}, y_{n_k+1}) \\ &\leq \frac{\eta}{1-2\eta}(\text{Dist}(C, D) + \text{Dist}(C, D)) + \frac{1-4\eta}{1-2\eta}\text{Dist}(C, D) \\ &= \frac{2\eta}{1-2\eta}(\text{Dist}(C, D)) + \frac{1-4\eta}{1-2\eta}\text{Dist}(C, D) \\ &= \text{Dist}(C, D),\end{aligned}$$

which is a contradiction. It follows from (3.32), (3.34) and Lemma 2.4 that for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $d(x_m, x_n) \leq \epsilon$. Thus, $\{x_n\}$ is Cauchy sequence in C. Similarly, $\{y_n\}$ is also Cauchy sequence in D. Therefore, $\{(x_n, y_n)\}$ is a convergent subsequence in $C \times D$.

Let $\{(x_{n_k}, y_{n_k})\}$ be a subsequence of $\{(x_n, y_n)\}$ such that

$$\lim_{k\to\infty}(x_{n_k}, y_{n_k}) = (u^*, v^*) = \text{Dist}(C, D)$$

for some $(u^*, v^*) \in C \times D$. This suggests that $x_{n_k} \to u^*$ and $y_{n_k} \to v^*$ as $k \to \infty$. For the sake of (3.25), we get

$$\begin{aligned}d(u^*, S(u^*, v^*)) &\leq d(u^*, x_{n_k}) + d(x_{n_k}, S(u^*, v^*)) \\ &= d(u^*, x_{n_k}) + d(S(y_{n_k-1}, x_{n_k-1}), S(u^*, v^*)) \\ &\leq d(u^*, x_{n_k}) + \eta(d(u^*, S(y_{n_k-1}, x_{n_k-1})) + d(y_{n_k-1}, S(u^*, v^*))) \\ &\quad + (1-4\eta)\text{Dist}(C, D) \\ &\leq d(u^*, x_{n_k}) + \eta(d(u^*, y_{n_k-1}) + d(y_{n_k-1}, S(y_{n_k-1}, x_{n_k-1})) + d(y_{n_k-1}, u^*) \\ &\quad + d(u^*, S(u^*, v^*))) + (1-4\eta)\text{Dist}(C, D).\end{aligned}$$

Accordingly, one can draw the conclusion that

$$d(u^*,S(u^*,v^*)) \leq \frac{1}{1-\eta}d(u^*,x_{n_k}) + \frac{\eta}{1-\eta}(d(u^*,y_{n_k-1}) + d(y_{n_k-1},S(y_{n_k-1},x_{n_k-1})) + d(y_{n_k-1},u^*)) + \frac{1-4\eta}{1-\eta}\mathrm{Dist}(C,D).$$

After considering Lemma 3.4 as well as the definition of $\mathrm{Dist}(C,D)$, we have

$$\begin{aligned}\mathrm{Dist}(C,D) &\leq d(u^*,S(u^*,v^*))\\ &\leq \frac{1}{1-\eta}d(u^*,x_{n_k}) + \frac{\eta}{1-\eta}(d(u^*,y_{n_k-1}) + d(y_{n_k-1},S(y_{n_k-1},x_{n_k-1})) + d(y_{n_k-1},u^*))\\ &\quad + \frac{1-4\eta}{1-\eta}\mathrm{Dist}(C,D).\end{aligned}$$

By taking $k \to \infty$, we obtain

$$\mathrm{Dist}(C,D) \leq d(u^*,S(u^*,v^*)) \leq \frac{3\eta}{1-\eta}\mathrm{Dist}(C,D) + \frac{1-4\eta}{1-\eta}\mathrm{Dist}(C,D) = \mathrm{Dist}(C,D).$$

Therefore,

$$d(u^*,S(u^*,v^*)) = \mathrm{Dist}(C,D).$$

Similarly, we can obtain

$$d(v^*,S(v^*,u^*)) = \mathrm{Dist}(C,D).$$

That is (u^*,v^*) is a coupled best proximity of S.

Next, we show that the coupled best proximity of S is unique. Let $(\bar{u},\bar{v})$ be another coupled best proximity of S such that $u^* \neq \bar{u}$ and $v^* \neq \bar{v}$. Then, we have

$$d(\bar{u},S(\bar{u},\bar{v})) = \mathrm{Dist}(C,D) \tag{3.35}$$

and

$$d(\bar{v},S(\bar{v},\bar{u})) = \mathrm{Dist}(C,D). \tag{3.36}$$

From S is $p-$cyclic Chatterjea contraction, we get

$$\begin{aligned}d(S(\bar{u},\bar{v}),S(v^*,u^*)) &\leq \eta(d(v^*,S(\bar{u},\bar{v})) + d(\bar{u},S(v^*,u^*))) + (1-4\eta)\mathrm{Dist}(C,D)\\ &\leq \eta(d(v^*,S(v^*,u^*)) + d(S(v^*,u^*),S(\bar{u},\bar{v})) + d(\bar{u},S(\bar{u},\bar{v}))\\ &\quad + d(S(\bar{u},\bar{v}),S(v^*,u^*))) + (1-4\eta)\mathrm{Dist}(C,D),\end{aligned}$$

and we further get

$$\begin{aligned}d(S(\bar{u},\bar{v}),S(v^*,u^*)) &\leq \frac{2\eta}{1-2\eta}\mathrm{Dist}(C,D) + \frac{1-4\eta}{1-2\eta}\mathrm{Dist}(C,D)\\ &= \mathrm{Dist}(C,D).\end{aligned}$$

Consider,

$$\begin{aligned}d(u^*,S(\bar{u},\bar{v})) &\leq d(u^*,x_{n_k+1}) + d(x_{n_k+1},S(\bar{u},\bar{v}))\\ &= d(u^*,x_{n_k+1}) + d(S(y_{n_k},x_{n_k}),S(\bar{u},\bar{v}))\\ &\leq d(u^*,x_{n_k+1}) + \eta(d(\bar{u},S(y_{n_k},x_{n_k})) + d(y_{n_k},S(\bar{u},\bar{v}))) + (1-4\eta)\mathrm{Dist}(C,D)\\ &\leq d(u^*,x_{n_k+1}) + \eta(d(\bar{u},S(\bar{u},\bar{v})) + d(S(\bar{u},\bar{v}),S(y_{n_k},x_{n_k})) + d(y_{n_k},u^*)\\ &\quad + d(u^*,S(\bar{u},\bar{v}))) + (1-4\eta)\mathrm{Dist}(C,D).\end{aligned}$$

By rearrangement the last inequality, we obtain

$$\begin{aligned}d(u^*, S(\bar{u}, \bar{v})) \leq \frac{1}{1-\eta} d(u^*, x_{n_k+1}) + \frac{\eta}{1-\eta}(d(\bar{u}, S(\bar{u}, \bar{v})) + d(S(\bar{u}, \bar{v}), S(y_{n_k}, x_{n_k})) + d(y_{n_k}, u^*)) \\ + \frac{1-4\eta}{1-\eta}\mathrm{Dist}(C, D).\end{aligned}$$

Letting $k \to \infty$, we deduce that

$$\begin{aligned}d(u^*, S(\bar{u}, \bar{v})) &\leq \frac{\eta}{1-\eta}(\mathrm{Dist}(C, D) + d(S(\bar{u}, \bar{v}), S(v^*, u^*)) + d(y_{n_k}, u^*)) + \frac{1-4\eta}{1-\eta}\mathrm{Dist}(C, D) \\ &= \frac{3\eta}{1-\eta}\mathrm{Dist}(C, D) + \frac{1-4\eta}{1-\eta}\mathrm{Dist}(C, D) \\ &= \mathrm{Dist}(C, D).\end{aligned}$$

Therefore, by the definition of $\mathrm{Dist}(C, D)$, we have

$$d(u^*, S(\bar{u}, \bar{v})) = \mathrm{Dist}(C, D). \tag{3.37}$$

Similarly, we also have

$$d(v^*, S(\bar{v}, \bar{u})) = \mathrm{Dist}(C, D). \tag{3.38}$$

By applying equations (3.35), (3.36), (3.37) and (3.38) and repeating the same process used in Theorem 3.1, it can be deduced that $u^* = \bar{u}$ and $v^* = \bar{v}$. As a result, S has a unique coupled best proximity. ☐

3.3. **Illustrative Examples.** Now, we give examples to support the findings.

Example 3.1. Let $\mathcal{M} = \mathbb{R}^2$ endowed with the metric d defined by

$$d(x, y) = \sqrt{(x_1 + y_2^2 - y_1 - x_2^2)^2 + (x_2 - y_2)^2},$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. It is not difficult to see that $(\mathbb{R}^2, d)$ is a metric space. Moreover, for $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, consider $\gamma_x^y : [0, 1] \to \mathbb{R}^2$ such that

$$\gamma_x^y(t) = \big(x_1 + t\,(y_1 - x_1) - t(1-t)(y_2 - x_2)^2,\ (1-t)x_2 + ty_2\big).$$

Then simple calculations yield that γ is a geodesic connecting x and y. It follows that,

$$(1-t)w \oplus tz = \gamma_w^z(t) = \Big(\big((1-t)w_2 + tz_2\big)^2 - (1-t)(w_2^2 - w_1) - t(z_2^2 - z_1),\ (1-t)w_2 + tz_2\Big).$$

Moreover, it is easy to see that the (2.2) is satisfied with $\mathrm{p} = 2$. Indeed, for any $u = (x_2,\ x_1), v = (y_2,\ y_1), u = (u_1,\ u_2) \in \mathcal{H}$, we have

$$\frac{1}{2}x \oplus \frac{1}{2}y = \left(\left(\frac{x_2 + y_2}{2}\right)^2 - \frac{x_2^2 - x_1}{2} - \frac{y_2^2 - y_1}{2},\ \frac{x_2 + y_2}{2}\right).$$

Consequently, we get

$$\begin{aligned}d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y,\ u\right) &= \left[\left(\frac{x_2 + y_2}{2}\right)^2 - \left(\left(\frac{x_2 + y_2}{2}\right)^2 - \frac{x_2^2 - x_1}{2} - \frac{y_2^2 - y_1}{2}\right) - u_2^2 + u_1\right]^2 \\ &\quad + \left[\frac{x_2 + y_2}{2} - u_2\right]^2 \\ &\leq \left[\frac{x_2 - u_2}{2} + \frac{y_2 - u_2}{2}\right]^2 + \left[\frac{x_2^2 - x_1 - u_2^2 + u_1}{2} + \frac{y_2^2 - y_1 - u_2^2 + u_1}{2}\right]^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{(x_2-u_2)^2}{2} - \frac{(u_2-y_2)^2}{2} - \frac{(x_2-y_2)^2}{4} + \frac{(x_2^2-x_1-u_2^2+u_1)^2}{2} \\
&\quad + \frac{(u_2^2-u_1-y_2^2+y_1)^2}{2} - \frac{(x_2^2-x_1-y_2^2+y_1)^2}{4} \\
&= \frac{1}{2}\left[(x_2-u_2)^2 + \left(x_2^2-x_1-u_2^2+u_1\right)^2\right] + \frac{1}{2}\left[(u_2-y_2)^2\right. \\
&\quad \left. + \left(u_2^2-u_1-y_2^2+y_1\right)^2\right] - \frac{1}{4}\left[(x_2-y_2)^2 + \left(x_2^2-x_1-y_2^2+y_1\right)^2\right] \\
&\leq \frac{1}{2}d^2(x,\ u) + \frac{1}{2}d^2(y,\ u) - \frac{1}{4}d^2(x,\ y).
\end{aligned}
$$

Thus $(\mathcal{M}, d)$ is a non-linear $\mathrm{CAT_p}(0)$ space for $\mathrm{p} = 2$.

Consider $C = \mathbb{R} \times \left[-\frac{3}{2}, -1\right]$ and $D = \mathbb{R} \times [1, 2]$. Then C and D are nonempty closed subsets of $\mathcal{M}$. Moreover, it is not difficult to see that $\mathrm{Dist}(C, D) = 2$. Let $S : (C \times D) \cup (D \times C) \to C \cup D$ be defined by

$$
S(\bar{x}, \bar{y}) = \begin{cases} \left(\dfrac{(1-x_2)^2}{4},\ \dfrac{1-x_2}{2}\right), & \text{if } \bar{x} = (x_1, x_2) \in C \text{ and } \bar{y} = (y_1, y_2) \in D, \\ \left(\dfrac{(1-y_2)^2}{4},\ \dfrac{y_2-1}{2}\right), & \text{otherwise.} \end{cases}
$$

Next, we show that S is a $p-$cyclic Kannan contraction mapping. For that, we consider the subsequent possibilities.

Case 1: Let $\bar{x}^1 = \left(\bar{x}_1^1, \bar{x}_2^1\right),\ \bar{x}^2 = \left(\bar{x}_1^2, \bar{x}_2^2\right) \in C$ and $\bar{y}^1 = \left(\bar{y}_1^1, \bar{y}_2^1\right),\ \bar{y}^2 = \left(\bar{y}_1^2, \bar{y}_2^2\right) \in D$, then

$$
S(\bar{x}^1, \bar{y}^1) = \left(\frac{(1-\bar{x}_2^1)^2}{4},\ \frac{1-\bar{x}_2^1}{2}\right) \quad \text{and} \quad S(\bar{x}^2, \bar{y}^2) = \left(\frac{(1-\bar{x}_2^2)^2}{4},\ \frac{1-\bar{x}_2^2}{2}\right).
$$

Thus,

$$
\begin{aligned}
d\left(S(\bar{x}^1, \bar{y}^1), S(\bar{x}^2, \bar{y}^2)\right) &= d\left(\left(\frac{(1-\bar{x}_2^1)^2}{4},\ \frac{1-\bar{x}_2^1}{2}\right), \left(\frac{(1-\bar{x}_2^2)^2}{4},\ \frac{1-\bar{x}_2^2}{2}\right)\right) \\
&= \left|\frac{1-\bar{x}_2^1}{2} - \frac{1-\bar{x}_2^2}{2}\right| = \frac{1}{2}\left|\bar{x}_2^1 - \bar{x}_2^2\right| \\
&\leq 1 = \left(1 - \frac{2}{4}\right)\mathrm{Dist}(C, D) \\
&\leq \frac{1}{4}\left[d\left(\bar{x}^1,\ S(\bar{x}^1, \bar{y}^1)\right) + d\left(\bar{x}^2,\ S(\bar{x}^2, \bar{y}^2)\right)\right] + \left(1 - \frac{2}{4}\right)\mathrm{Dist}(C, D).
\end{aligned}
$$

Case 2: Let $\bar{x}^1 = \left(\bar{x}_1^1, \bar{x}_2^1\right),\ \bar{x}^2 = \left(\bar{x}_1^2, \bar{x}_2^2\right) \in D$ and $\bar{y}^1 = \left(\bar{y}_1^1, \bar{y}_2^1\right),\ \bar{y}^2 = \left(\bar{y}_1^2, \bar{y}_2^2\right) \in C$, then

$$
S(\bar{x}^1, \bar{y}^1) = \left(\frac{(1-\bar{y}_2^1)^2}{4},\ \frac{\bar{y}_2^1-1}{2}\right) \quad \text{and} \quad S(\bar{x}^2, \bar{y}^2) = \left(\frac{(1-\bar{y}_2^2)^2}{4},\ \frac{\bar{y}_2^2-1}{2}\right).
$$

Thus,

$$
\begin{aligned}
d\left(S(\bar{x}^1, \bar{y}^1), S(\bar{x}^2, \bar{y}^2)\right) &= d\left(\left(\frac{(1-\bar{y}_2^1)^2}{4},\ \frac{\bar{y}_2^1-1}{2}\right), \left(\frac{(1-\bar{y}_2^2)^2}{4},\ \frac{\bar{y}_2^2-1}{2}\right)\right) \\
&= \left|\frac{1-\bar{y}_2^1}{2} - \frac{1-\bar{y}_2^2}{2}\right| = \frac{1}{2}\left|\bar{y}_2^1 - \bar{y}_2^2\right| \\
&\leq 1 = \left(1 - \frac{2}{4}\right)\mathrm{Dist}(C, D) \\
&\leq \frac{1}{4}\left[d\left(\bar{x}^1,\ S(\bar{x}^1, \bar{y}^1)\right) + d\left(\bar{x}^2,\ S(\bar{x}^2, \bar{y}^2)\right)\right] + \left(1 - \frac{2}{4}\right)\mathrm{Dist}(C, D).
\end{aligned}
$$

Case 3: Let $\bar{x}^1 = \left(\bar{x}^1_1, \bar{x}^1_2\right),\ \bar{y}^2 = \left(\bar{y}^2_1, \bar{y}^2_2\right) \in C$ and $\bar{y}^1 = \left(\bar{y}^1_1, \bar{y}^1_2\right),\ \bar{x}^2 = \left(\bar{x}^2_1, \bar{x}^2_2\right) \in D$, then

$$S(\bar{x}^1, \bar{y}^1) = \left(\frac{(1-\bar{x}^1_2)^2}{4},\ \frac{1-\bar{x}^1_2}{2}\right) \quad \text{and} \quad S(\bar{x}^2, \bar{y}^2) = \left(\frac{(1-\bar{y}^2_2)^2}{4},\ \frac{\bar{y}^2_2-1}{2}\right).$$

Thus,

$$\begin{aligned}
d\left(S(\bar{x}^1, \bar{y}^1), S(\bar{x}^2, \bar{y}^2)\right) &= d\left(\left(\frac{(1-\bar{x}^1_2)^2}{4},\ \frac{1-\bar{x}^1_2}{2}\right), \left(\frac{(1-\bar{y}^2_2)^2}{4},\ \frac{\bar{y}^2_2-1}{2}\right)\right) \\
&= \left|\frac{1-\bar{x}^1_2}{2} - \frac{\bar{y}^2_2-1}{2}\right| = \frac{1}{2}\left|-\bar{x}^1_2 - \bar{y}^2_2\right| \\
&\leq \frac{3}{2} = \frac{1}{2} + \left(1 - \frac{2}{4}\right)\mathrm{Dist}(C, D) \\
&\leq \frac{1}{4}\left[d\left(\bar{x}^1,\ S(\bar{x}^1, \bar{y}^1)\right) + d\left(\bar{x}^2,\ S(\bar{x}^2, \bar{y}^2)\right)\right] + \left(1 - \frac{2}{4}\right)\mathrm{Dist}(C, D).
\end{aligned}$$

Case 4: This case, that is, $\bar{y}^1 = \left(\bar{y}^1_1, \bar{y}^1_2\right),\ \bar{x}^2 = \left(\bar{x}^2_1, \bar{x}^2_2\right) \in C$ and $\bar{x}^1 = \left(\bar{x}^1_1, \bar{x}^1_2\right),\ \bar{y}^2 = \left(\bar{y}^2_1, \bar{y}^2_2\right) \in C$, follows similar fashion as in Case 3.

Observe that the point $(\bar{u}, \bar{v})$ given by $\bar{u} = (1, -1)$ and $\bar{v} = (1, 1)$ is the unique couple best proximity point of S. In fact, it is easy to see that

$$\begin{aligned}
&d\big(\bar{u}, S(\bar{u}, \bar{v})\big) = d\big((1,-1),(1,-1)\big) = 2 = \mathrm{Dist}(C, D) \quad \text{and} \\
&d\big(\bar{v}, S(\bar{v}, \bar{u})\big) = d\big((1,1),(1,-1)\big) = 2 = \mathrm{Dist}(C, D).
\end{aligned}$$

Example 3.2. Let $(\mathcal{M}, d)$ be a space defined as in Example 3.1. Consider $C = \mathbb{R} \times [-1, 0]$ and $D = \mathbb{R} \times [0, 1]$. Then C and D are nonempty, closed subsets of $\mathcal{M}$. Moreover, it is not difficult to see that $\mathrm{Dist}(C, D) = 0$. For $\sigma > 5$, let $S : (C \times D) \cup (D \times C) \to C \cup D$ be defined by

$$S(\bar{x}, \bar{y}) = \left(\frac{(x_2)^2}{\sigma^2},\ \frac{-x_2}{\sigma}\right), \qquad \text{for all} \quad \bar{x} = (x_1, x_2) \text{ and } \bar{y} = (y_1, y_2).$$

Then S is a $p-$cyclic Chatterjea contraction. Indeed, for $\bar{x}^1 = \left(\bar{x}^1_1, \bar{x}^1_2\right),\ \bar{x}^2 = \left(\bar{x}^2_1, \bar{x}^2_2\right)$, $\bar{y}^1 = \left(\bar{y}^1_1, \bar{y}^1_2\right),\ \bar{y}^2 = \left(\bar{y}^2_1, \bar{y}^2_2\right)$, we get

$$S(\bar{x}^1, \bar{y}^1) = \left(\frac{(\bar{x}^1_2)^2}{\sigma^2},\ \frac{-\bar{x}^1_2}{\sigma}\right) \quad \text{and} \quad S(\bar{x}^2, \bar{y}^2) = \left(\frac{(\bar{x}^2_2)^2}{\sigma^2},\ \frac{-\bar{x}^2_2}{\sigma}\right).$$

Thus,

$$\begin{aligned}
d\left(S(\bar{x}^1, \bar{y}^1), S(\bar{x}^2, \bar{y}^2)\right) &= d\left(\left(\frac{(\bar{x}^1_2)^2}{\sigma^2},\ \frac{-\bar{x}^1_2}{\sigma}\right), \left(\frac{(\bar{x}^2_2)^2}{\sigma^2},\ \frac{-\bar{x}^2_2}{\sigma}\right)\right) \\
&= \left|\frac{-\bar{x}^1_2}{\sigma} - \frac{-\bar{x}^2_2}{\sigma}\right| = \frac{1}{\sigma}\left|\bar{x}^2_2 - \bar{x}^1_2\right| \\
&= \frac{1}{\sigma}\left|\bar{x}^2_2 + \frac{\bar{x}^1_2}{\sigma} + \frac{\bar{x}^2_2}{\sigma} - \frac{\bar{x}^1_2}{\sigma} - \left(\bar{x}^1_2 + \frac{\bar{x}^2_2}{\sigma}\right)\right| \\
&\leq \frac{1}{\sigma}\left(\left|\bar{x}^2_2 + \frac{\bar{x}^1_2}{\sigma}\right| + \left|\frac{\bar{x}^2_2}{\sigma} - \frac{\bar{x}^1_2}{\sigma}\right| + \left|\bar{x}^1_2 + \frac{\bar{x}^2_2}{\sigma}\right|\right) \\
&= \frac{1}{\sigma}\left|\bar{x}^2_2 - \frac{-\bar{x}^1_2}{\sigma}\right| + \frac{1}{\sigma}d\left(S(\bar{x}^1, \bar{y}^1), S(\bar{x}^2, \bar{y}^2)\right) + \frac{1}{\sigma}\left|\bar{x}^1_2 - \frac{-\bar{x}^2_2}{\sigma}\right|.
\end{aligned}$$

This implies that

$$d\left(S(\bar{x}^1, \bar{y}^1), S(\bar{x}^2, \bar{y}^2)\right) \leq \frac{1}{\sigma - 1}\left(\left|\bar{x}^2_2 - \frac{-\bar{x}^1_2}{\sigma}\right| + \left|\bar{x}^1_2 - \frac{-\bar{x}^2_2}{\sigma}\right|\right)$$

$$\le \frac{1}{\sigma-1}\left[d\left(\bar{x}^2,\ S(\bar{x}^1,\bar{y}^1)\right)+d\left(\bar{x}^1,\ S(\bar{x}^2,\bar{y}^2)\right)\right]$$
$$=\frac{1}{\sigma-1}\left[d\left(\bar{x}^2,S(\bar{x}^1,\bar{y}^1)\right)+d\left(\bar{x}^1,S(\bar{x}^2,\bar{y}^2)\right)\right]+\left(1-\frac{4}{\sigma-1}\right)\mathrm{Dist}(C,D).$$

Observe that the point $(\bar{u},\bar{v})$ given by $\bar{u}=(0,0)$ and $\bar{v}=(0,0)$ is the unique couple best proximity point of S. It is easy to see that

$$d\big(\bar{u},S(\bar{u},\bar{v})\big)=d\big(\bar{v},S(\bar{v},\bar{u})\big)=d\big((0,0),(0,0)\big)=0=\mathrm{Dist}(C,D).$$

Example 3.3. For $\mathrm{p}\ge 2$, consider $\mathcal{M}=\ell_{\mathrm{p}}$, $C=\prod_{i\ge1}[-1,0]$ and $D=\prod_{i\ge1}[0,1]$ and d is the metric induced by $\|\cdot\|_{\mathrm{p}}$. Then C and D are nonempty, closed and bounded subsets of $\mathcal{M}$. Moreover, it is not difficult to see that $\mathrm{Dist}(C,D)=0$. Let $S:(C\times D)\cup(D\times C)\to C\cup D$ be defined by

$$S(\bar{x},\bar{y})=-\frac{1}{6}\bar{x},\qquad \text{for all}\quad \bar{x}=(x_1,x_2,x_3,\dots)\text{ and }\bar{y}=(y_1,y_2,y_3,\dots).$$

Then S is a $p-$cyclic Chatterjea contraction. Indeed, it is easy to see that

$$\begin{aligned}
d\left(S(\bar{x}^1,\bar{y}^1),S(\bar{x}^2,\bar{y}^2)\right)&=d\left(-\frac{1}{6}\bar{x}^1,\ -\frac{1}{6}\bar{x}^2\right)=\frac{1}{6}\left\|\bar{x}^2-\bar{x}^1\right\|_{\mathrm{p}}\\
&=\frac{1}{6}\left\|\bar{x}^2+\frac{1}{6}\bar{x}^1+\frac{1}{6}\bar{x}^2-\frac{1}{6}\bar{x}^1-\bar{x}^1-\frac{1}{6}\bar{x}^2\right\|_{\mathrm{p}}\\
&\le\frac{1}{6}\left(\left\|\bar{x}^2+\frac{1}{6}\bar{x}^1\right\|_p+\left\|\frac{1}{6}\bar{x}^2-\frac{1}{6}\bar{x}^1\right\|_p+\left\|\bar{x}^1+\frac{1}{6}\bar{x}^2\right\|_{\mathrm{p}}\right)\\
&=\frac{1}{6}d\left(\bar{x}^2,S(\bar{x}^1,\bar{y}^1)\right)+\frac{1}{6}d\left(S(\bar{x}^1,\bar{y}^1),S(\bar{x}^2,\bar{y}^2)\right)+\frac{1}{6}d\left(\bar{x}^1,S(\bar{x}^2,\bar{y}^2)\right).
\end{aligned}$$

This implies that

$$d\left(S(\bar{x}^1,\bar{y}^1),S(\bar{x}^2,\bar{y}^2)\right)\le\frac{1}{5}\left[d\left(\bar{x}^2,\ S(\bar{x}^1,\bar{y}^1)\right)+d\left(\bar{x}^1,\ S(\bar{x}^2,\bar{y}^2)\right)\right]+\left(1-\frac{4}{5}\right)\mathrm{Dist}(C,D).$$

Observe that the point $(\bar{u},\bar{v})$ given by $\bar{u}=0\in\ell_{\mathrm{p}}$ and $\bar{v}=0\in\ell_{\mathrm{p}}$ is the unique couple best proximity point of S. It is easy to see that

$$d\big(\bar{u},S(\bar{u},\bar{v})\big)=d\big(\bar{v},S(\bar{v},\bar{u})\big)=0=\mathrm{Dist}(C,D).$$

4. Conclusion

This article discusses coupled best proximity points in the framework of generalized CAT(0) spaces, specifically $\mathrm{CAT_p}(0)$ spaces. In particular, we present and investigate the $p-$cyclic Kannan and Chatterjea contraction mappings, focusing on an analysis of the existence and uniqueness of the coupled best proximity point associated with these intriguing mappings. To illuminate our findings, we provide a comprehensive example that illustrates the practical implications of our research. Furthermore, our study builds upon and enriches the existing body of research in this domain, contributing novel insights and advancements.

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References

[1] Ajeti, L.; Ilchev, A.; Zlatanov, B. On coupled best proximity points in reflexive banach spaces. *Mathematics* **10** (2022), 1–19.

[2] Aydi, H.; Felhi, A. Best proximity points for cyclic Kannan-Chatterjea-Ćirić type contractions on metric-like spaces. *J. Nonlinear Sci. Appl.* **9** (2016), 2458–2466.

[3] Bridson, M. R.; Haefliger, A. *Metric spaces of non-positive curvature*. Fundamental Principles of Mathematical Sciences. Springer-Verlag, Berlin, 1999.

[4] Calderón, K.; Khamsi, M. A.; Martínez-Moreno, J. Perturbed approximations of fixed points of nonexpansive mappings in $\mathrm{CAT_p}(0)$ spaces. *Carpathian J. Math.* **37** (2021), 65–79.

[5] Chatterjea, S. K. Fixed-point theorems. *C. R. Acad. Bulgare Sci.* **25** (1972), 727–730.

[6] Choudhury, B. S.; Das, K.; Bhandari, S. K. Cyclic contraction result in 2-Menger space. *Bull. Int. Math. Virtual Inst.* **2** (2012), 223–234.

[7] Eldred, A. A.; Veeramani, P. Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **323** (2006), 1001–1006.

[8] Gardašević-Filipović, M.; Kukić, K.; Gardašević, D.; Mitrović, Z. D. Some best proximity point results in the orthogonal 0-complete b-metric-like spaces. *Izv. Nats. Akad. Nauk Armenii Mat.* **58** (2023), 14–27.

[9] Gupta, A.; Rohilla, M. On coupled best proximity points and Ulam-Hyers stability. *J. Fixed Point Theory Appl.* **22** (2020), 1–21.

[10] Inuwa, A. Y.; Kumam, P.; Chaipunya, P.; Salisu, S. Fixed point theorems for enriched Kannan mappings in $\mathrm{CAT}(0)$ spaces. *Fixed Point Theory Algorithms Sci. Eng.* **2023** (2023), 1–21, 21.

[11] Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **60** (1968), 71–76.

[12] Khammahawong, K.; Salisu, S. Modified Halpern and viscosity methods for hierarchical variational inequalities on Hadamard manifolds. *Comput. Appl. Math.*, **43** (2024), 1–34.

[13] Khamsi, M. A.; Shukri, S. A. Generalized CAT(0) spaces. *Bull. Belg. Math. Soc. Simon Stevin*, **24** (2017), 417–426.

[14] Kirk, W. A.; Srinivasan, P. S.; Veeramani, P. Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory*, **4** (2003), 79–89.

[15] Kratuloek, K.; Kumam, P.; Amnuaykarn, K.; Nantadilok, J.; Salisu, S. Iterative approximation of common fixed points of two nonself asymptotically nonexpansive mappings in $\mathrm{CAT}(0)$ spaces with numerical examples. *Math. Methods Appl. Sci.* **46** (2023), 13521–13539.

[16] Kumam, P.; Roldán López de Hierro, A. F. On existence and uniqueness of g-best proximity points under $(\varphi, \theta, \alpha, g)$-contractivity conditions and consequences. *Abstr. Appl. Anal.* **2014** (2014), 1–14.

[17] Lateef, D. Best proximity points in $\mathcal{F}$-metric spaces with applications. *Demonstr. Math.* **56** (2023), 1–14.

[18] Mongkolkeha, C.; Kumam, P. Best proximity point theorems for generalized cyclic contractions in ordered metric spaces. *J. Optim. Theory Appl.* **155** (2012), 215–226.

[19] Mongkolkeha, C.; Kumam, P. Some common best proximity points for proximity commuting mappings. *Optim. Lett.*, **7** (2013), 1825–1836.

[20] Panwar, A.; Lamba, P.; Rakočević, V.; Gopal, D. New fixed point results of some enriched contractions in cat(0) spaces. *Miskolc Mathematical Notes* **24** (2023), 1477–1493.

[21] Rhoades, B. E. A comparison of various definitions of contractive mappings. *Trans. Amer. Math. Soc.* **226** (1977), 257–290.

[22] Salisu, S.; Kumam, P.; Sriwongsa, S.; Abubakar, J. On minimization and fixed point problems in Hadamard spaces. *Comput. Appl. Math.* **41** (2022), 1–22.

[23] Salisu, S.; Kumam, P.; Sriwongsa, S.; Gopal, D. Enriched asymptotically nonexpansive mappings with center zero. *Filomat* **38** (2024), 343–356.

[24] Salisu, S.; Minjibir, M. S.; Kumam, P.; Sriwongsa, S. Convergence theorems for fixed points in $\mathrm{CAT_p}(0)$ spaces. *J. Appl. Math. Comput.* **69** (2023), 631–650.

[25] Shukri, S. Existence and convergence of best proximity points in $\mathrm{CAT_p}(0)$ spaces. *J. Fixed Point Theory Appl.*, **22** (2020), 1–10.

[26] Sintunavarat, W.; Kumam, P. Coupled best proximity point theorem in metric spaces. *Fixed Point Theory Appl.* **2012** (2012), 1–16.

[27] Sintunavarat, W.; Kumam, P. The existence and convergence of best proximity points for generalized proximal contraction mappings. *Fixed Point Theory Appl.* **2014** (2014), 1–16.

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