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Modified inertial Tseng type method for zeros of the sum of monotone operators in Hilbert spaces

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ABSTRACT. In this work, we propose a modified inertial Tseng type method for finding a solution to the monotone inclusion problem in Hilbert spaces. The strong convergence of the algorithm is guaranteed by sufficient conditions on the control sequences of related parameters. The forms of proposed algorithms for solving some important applications of the monotone inclusion problem models are provided. Also, the numerical experiments of the proposed algorithm are discussed.

1. INTRODUCTION

Let H be a real Hilbert space and let $B : H \to 2^H$ be a set-valued operator. The variational inclusion problem, which was introduced by Martinet [18], is the problem of finding a point $x^* \in H$ such that

$$0 \in Bx^*. \tag{1.1}$$

If B is a maximal monotone operator, the elements in the solution set of the problem (1.1) are called the zeros of a maximal monotone operator. The variational inclusion problems are being used as mathematical programming models to study a large number of optimization problems that arise in finance, economics, network, transportation, and engineering science. For solving the problem (1.1), many authors considered the following proximal point method: for a given $x_1 \in H$,

$$x_{n+1} = J^B_{\lambda_n} x_n, \qquad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ is the resolvent of the maximal monotone operator B corresponding to λ_n; see [12, 19, 33, 36] for more information.

A type of generalization of (1.1) is the following inclusion problem: finding a point $x^* \in H$ such that

$$0 \in Ax^* + Bx^*, \tag{1.3}$$

where $A : H \to H$ is a single-valued operator and $B : H \to 2^H$ is a set-valued operator. The elements in the solution set of the monotone inclusion problem (1.3) are called the zeros of the sum of monotone operators; see [6, 8, 20, 25, 32] and the references therein. Note that, when A is a continuous and monotone operator and B is a maximal monotone operator, we have $A + B$ is a maximal monotone operator; see [26]. A popular iterative method for solving the problem (1.3) is the so-called the forward-backward splitting method, which defines a sequence $\{x_n\}$ by the following algorithm: for any $x_1 \in H$,

$$x_{n+1} = J^B_{\lambda_n}(I - \lambda_n A)x_n, \qquad \forall n \in \mathbb{N}, \tag{1.4}$$

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where $\{\lambda_n\}$ is a sequence of positive real numbers, $A : H \to H$ and $B : H \to 2^H$ are maximal monotone operators, see Passty [22].

Furthermore, the study of the inertial technique was first presented by Polyak in 1964, to speed up the rate of convergence; see [24]. This technique is a two-step iterative method, in which each iteration involves the previous two iterates. Consequently, many authors considered the inertial method because of this faster convergence rate property of the algorithm; see [1, 4, 9, 10, 11, 14, 23, 27] for more information.

In 2001, Alvarez and Attouch [3] proposed the inertial proximal point method for finding the solution of the problem (1.1): for arbitrary x_0, $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J^B_{\lambda_n} y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where $\{\lambda_n\}$ is non-decreasing sequence and $\{\mu_n\} \subset [0,1)$ satisfy with $\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty$, and presented the weakly convergence results.

In 2015, Lorenz and Pock [16] studied the monotone inclusion problem (1.3) and proposed the following inertial forward-backward algorithm: for arbitrary x_0, $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J^B_{\lambda_n}(I - \lambda_n A) y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.6}$$

where $\{\lambda_n\}$ is a positive real sequence and $\{\mu_n\} \subset [0,1)$. By suitable conditions, they proved that the sequence $\{x_n\}$ converges weakly to a solution of the problem (1.3).

In 2018, Cholamjiak et al. [9] proposed the Halpern type inertial forward-backward method for solving the problem (1.3): for arbitrary u, x_0, $x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= \alpha_n u + \beta_n y_n + \gamma_n J^B_{\lambda_n}(I - \lambda_n A) y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.7}$$

where $\{\mu_n\} \subset [0, \mu)$ with $\mu \in [0,1)$, the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$ and some suitable conditions. They proved that the sequence $\{x_n\}$ converges strongly to a solution of the problem (1.3).

On the other hand, in 2000, Tseng [32] proposed a modified forward-backward splitting method for solving the problem (1.3), also known as Tseng's splitting algorithm: let $C \subseteq H$ be closed and convex set which intersects the solution set of (1.3), for any $x_1 \in C$,

$$\begin{aligned} y_n &= J^B_{\lambda_n}(I - \lambda_n A) x_n, \\ x_{n+1} &= P_C\big(y_n - \lambda_n(Ay_n - Ax_n)\big), \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.8}$$

where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \dots\}$ satisfying $\lambda\|Ay_n - Ax_n\| \le \mu\|y_n - x_n\|$, when $\delta > 0$ and $l, \mu \in (0,1)$. Tseng proved that the sequence $\{x_n\}$ converges weakly to the zeros of $A + B$. Subsequently, the study of the strong convergence methods with Tseng's splitting algorithm for the problem (1.3) are studied; see [2, 13, 14, 28] for more details.

In [14], Kaewyong and Sitthithakerngkiet studied the monotone inclusion problem (1.3). They introduced the following modified Tseng type algorithm: for arbitrary x_0, $x_1 \in H$,

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ w_n &= J^B_{\lambda_n}(I - \lambda_n A) z_n, \\ y_n &= w_n - \lambda_n(Aw_n - Az_n), \\ x_{n+1} &= \alpha_n \nabla h(x_n) + (1 - \alpha_n) y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\nabla h : H \to H$ is σ-Lipschitz continuous with $\sigma \in [0,1)$, $\{\mu_n\} \subset [0,1)$ and $\{\lambda_n\}$ is defined by

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\theta\|z_n - w_n\|}{\|Az_n - Aw_n\|}, \lambda_n\right\}, & \text{if } Az_n - Aw_n \neq 0; \\ \lambda_n, & \text{otherwise,} \end{cases}$$

when $\lambda_1 > 0$ and $\theta \in (0,1)$. They showed that if the sequence $\{\mu_n\} \in [0,1)$ such that $\lim_{n\to\infty}\dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty}\mu_n\|x_n - x_{n-1}\| = 0$, then the generated sequence $\{x_n\}$ converges strongly to a solution point of the problem (1.3).

In this paper, motivated and inspired by the above literature and the presented algorithm in [14], we are going to consider problem (1.3) by aiming to provide the modified inertial Tseng type algorithms for finding a solution of the problem and provide some suitable conditions to guarantee that the constructed sequence $\{x_n\}$ converges strongly to a solution point.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. For a sequence $\{x_n\}$ in H, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $T : H \to H$ be a mapping. Then T is said to be

(i) *Lipschitz* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in H.$$

The number L, associated with T, is called a Lipschitz constant. Moreover, if $L \in [0,1)$, we say that T is contraction. And, if $L = 1$, we say that T is nonexpansive.

(ii) *Firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \qquad \forall x, y \in H.$$

(iii) *β-inverse strongly monotone (β-ism)* if for a positive real number β,

$$\langle Tx - Ty, x - y\rangle \geq \beta\|Tx - Ty\|^2, \qquad \forall x, y \in H.$$

The set of fixed points of a self-mapping T will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We note that if T is nonexpansive, then $F(T)$ is closed and convex.

Now, we collect some important properties for our proof.

Lemma 2.1. [6, 34] *The following are true:*
(i) If $A : H \to H$ is β-ism, then A is $\frac{1}{\beta}$-Lipschitz continuous and monotone mapping.
(ii) If $A : H \to H$ is β-ism and $\lambda \in (0,\beta]$, then $T := I - \lambda A$ is firmly nonexpansive.

Let $B : H \to 2^H$ be a set-valued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. Recall that B is said to be monotone if

$$\langle x - y, u - v\rangle \geq 0, \qquad \forall x, y \in D(B), u \in Bx, v \in By.$$

A monotone mapping B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. To a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, its resolvent J_λ^B is defined by

$$J_\lambda^B := (I + \lambda B)^{-1} : H \to D(B).$$

Notice that the resolvent J_λ^B is a single-valued and firmly nonexpansive mapping, and $F(J_\lambda^B) = B^{-1}0 \equiv \{x \in H : 0 \in Bx\}, \quad \forall \lambda > 0$; see [29, 30].

Lemma 2.2. [5] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and $A : C \to H$ be an operator. If $B : H \to 2^H$ is a maximal monotone operator, then $F\big(J_\lambda^B(I - \lambda A)\big) = (A+B)^{-1}0$.*

The following known results are needed in our proof.

For each $x, y, z \in H$, the following facts are valid for inner product spaces,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{2.10}$$

and

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad -\alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2,\end{aligned} \tag{2.11}$$

for any $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$; see [21, 29].

Let C be a nonempty closed convex subset of H. For a point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

P_C is called a metric projection of H onto C; see [31]. The following property of P_C is well known and useful:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H, y \in C.$$

We also use the following lemma for proving the main theorems.

Lemma 2.3. [15, 33] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n + c_n, \qquad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying
(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \le 0$;
(iii) $c_n \ge 0$, *for each* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} c_n < \infty$.
Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

In this section, we start by introducing the following algorithm that combines the inertial method with Tseng type and viscosity type algorithm for solving the monotone inclusion problem, the problem (1.3).

Algorithm 3.1. *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ with $\alpha_n + \beta_n + \delta_n = 1$ and the initial $x_0, x_1 \in H$ be arbitraries, define*

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ w_n &= J_{\lambda_n}^B(I - \lambda_n A)z_n, \\ y_n &= w_n - \lambda_n(Aw_n - Az_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \delta_n y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{3.12}$$

where $\{\mu_n\} \subset [0,\mu)$ with $\mu \in [0,1)$ and $\{\lambda_n\}$ is defined by

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\theta\|z_n - w_n\|}{\|Az_n - Aw_n\|}, \lambda_n\right\}, & \text{if } Az_n - Aw_n \neq 0; \\ \lambda_n, & \text{otherwise,} \end{cases} \tag{3.13}$$

when $\lambda_1 > 0$ *and* $\theta \in (0,1)$.

We will consider the Algorithm 3.1 under the following assumptions.

(A1) $A : H \to H$ is a ν-Lipschitz continuous and monotone operator;
(A2) $B : H \to 2^H$ is a maximal monotone operator;
(A3) $f : H \to H$ is a contraction mapping with coefficient $\kappa \in [0,1)$.

We denote $\Omega := (A+B)^{-1}0$ for the solution set of the problem (1.3) and assume that Ω is nonempty.

Now, we will present Lemmas and the strong convergence theorem (Theorem 3.1), by using the above assumptions to Algorithm 3.1.

Lemma 3.4. *The sequence* $\{\lambda_n\}$ *generated by (3.13) is a non-increasing sequence and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\theta}{\nu}\right\}.$$

Proof. By the definition of $\{\lambda_n\}$ in (3.13), it follows immediately that the sequence $\{\lambda_n\}$ is non-increasing. Moreover, $Az_n - Aw_n \neq 0$ implies $\dfrac{\theta\|z_n - w_n\|}{\|Az_n - Aw_n\|} \geq \dfrac{\theta}{\nu}$. Therefore, it is obvious that $\{\lambda_n\}$ has a lower bound $\min\left\{\lambda_1, \dfrac{\theta}{\nu}\right\}$. This completes the proof. $\square$

Lemma 3.5. *Let* H *be a real Hilbert space and let* $\{y_n\}$ *be a sequence which is appeared in Algorithm 3.1. If assumptions (A1)-(A3) hold, then*

$$\|y_n - p\|^2 \leq \|z_n - p\|^2 - \left(1 - \theta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2, \tag{3.14}$$

for all $p \in \Omega$ *and*

$$\|y_n - w_n\| \leq \theta \frac{\lambda_n}{\lambda_{n+1}} \|z_n - w_n\|. \tag{3.15}$$

Proof. By the choice of $\{\lambda_n\}$ in (3.13), if $Az_n \neq Aw_n$, we have

$$\lambda_{n+1} = \min\left\{\frac{\theta\|z_n - w_n\|}{\|Az_n - Aw_n\|}, \lambda_n\right\} \leq \frac{\theta\|z_n - w_n\|}{\|Az_n - Aw_n\|},$$

for each $n \in \mathbb{N}$. This implies that

$$\|Az_n - Aw_n\| \leq \frac{\theta\|z_n - w_n\|}{\lambda_{n+1}}. \tag{3.16}$$

Note that, in fact (3.16) holds for all $n \in \mathbb{N}$.

Now, consider

$$\begin{aligned}
\|y_n - p\|^2 &= \|w_n - \lambda_n(Aw_n - Az_n) - p\|^2 \\
&= \|w_n - p\|^2 + \lambda_n^2\|Aw_n - Az_n\|^2 - 2\lambda_n\langle w_n - p, Aw_n - Az_n\rangle \\
&= \|z_n - p\|^2 + \|z_n - w_n\|^2 + 2\langle w_n - z_n, z_n - p\rangle \\
&\quad + \lambda_n^2\|Aw_n - Az_n\|^2 - 2\lambda_n\langle w_n - p, Aw_n - Az_n\rangle \\
&= \|z_n - p\|^2 - \|z_n - w_n\|^2 + 2\langle w_n - z_n, w_n - p\rangle \\
&\quad + \lambda_n^2\|Aw_n - Az_n\|^2 - 2\lambda_n\langle w_n - p, Aw_n - Az_n\rangle \\
&= \|z_n - p\|^2 - \|z_n - w_n\|^2 - 2\big\langle z_n - w_n - \lambda_n(Az_n - Aw_n), w_n - p\big\rangle \\
&\quad + \lambda_n^2\|Aw_n - Az_n\|^2,
\end{aligned} \tag{3.17}$$

for each $n \in \mathbb{N}$. By using (3.16), we obtain

$$\begin{aligned} \|y_n - p\|^2 &= \|z_n - p\|^2 - \left(1 - \theta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2 \\ &\quad -2\big\langle z_n - w_n - \lambda_n(Az_n - Aw_n), w_n - p\big\rangle. \end{aligned} \tag{3.18}$$

Next, we will show that $\big\langle z_n - w_n - \lambda_n(Az_n - Aw_n), w_n - p\big\rangle \geq 0$. Note that, $w_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)z_n$, implies $(I - \lambda_n A)z_n \in (I + \lambda_n B)w_n$. Since B is a maximal monotone operator, there exists $v_n \in Bw_n$ such that $(I - \lambda_n A)z_n = w_n + \lambda_n v_n$. This gives

$$v_n = \frac{1}{\lambda_n}(z_n - w_n - \lambda_n Az_n). \tag{3.19}$$

Furthermore, since $0 \in (A+B)p$ and $Aw_n + v_n \in (A+B)w_n$, together with the maximality of $A + B$, we get

$$\langle Aw_n + v_n, w_n - p\rangle \geq 0. \tag{3.20}$$

Substituting (3.19) into (3.20), we obtain

$$\frac{1}{\lambda_n}\langle z_n - w_n - \lambda_n Az_n + \lambda_n Aw_n, w_n - p\rangle \geq 0.$$

So,

$$\big\langle z_n - w_n - \lambda_n(Az_n - Aw_n), w_n - p\big\rangle \geq 0.$$

Thus, from (3.18), we have

$$\|y_n - p\|^2 \leq \|z_n - p\|^2 - \left(1 - \theta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2,$$

for each $n \in \mathbb{N}$.

In addition, by using (3.16), we obtain

$$\begin{aligned} \|y_n - w_n\| &= \|w_n - \lambda_n(Aw_n - Az_n) - w_n\| \\ &\leq \lambda_n\|Aw_n - Az_n\| \\ &\leq \theta\frac{\lambda_n}{\lambda_{n+1}}\|z_n - w_n\|, \end{aligned}$$

for each $n \in \mathbb{N}$. This completes the proof. □

Theorem 3.1. *Let H be a real Hilbert space and let $\{x_n\}$ be generated by Algorithm 3.1. Suppose that the assumptions (A1)-(A3) hold and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) There exist a positive real number a with $0 < a \leq \beta_n$ and $0 < a \leq \delta_n$, for each $n \in \mathbb{N}$;

(C3) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega f(p)$.

Proof. Firstly, we show that the sequence $\{x_n\}$ is bounded. Let $z \in \Omega$. Then, we have $z \in (A + B)^{-1}0$, and this implies that $J^B_{\lambda_n}(I - \lambda_n A)z = z$.

Since $\lim_{n\to\infty}\left(1 - \theta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \theta^2 > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \theta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \tag{3.21}$$

for each $n \geq n_0$. Thus, from (3.14), we have

$$\|y_n - z\| \leq \|z_n - z\|, \tag{3.22}$$

for each $n \geq n_0$.

Next, by the definition of z_n and the condition (C3), we obtain that for each $n \geq n_0$,

$$\begin{aligned} \|z_n - z\| &= \big\|x_n + \mu_n(x_n - x_{n-1}) - z\big\| \\ &\leq \|x_n - z\| + \mu_n\|x_n - x_{n-1}\| \\ &= \|x_n - z\| + \alpha_n \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - z\| + \alpha_n M_1, \end{aligned} \tag{3.23}$$

for some $M_1 > 0$. It follows by using (3.23) that

$$\|y_n - z\| \leq \|x_n - z\| + \alpha_n M_1. \tag{3.24}$$

By using (3.24) and the definition of x_{n+1}, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \big\|\alpha_n f(x_n) + \beta_n x_n + \delta_n y_n - z\big\| \\ &\leq \alpha_n\big\|f(x_n) - z\big\| + \beta_n\|x_n - z\| + \delta_n\|y_n - z\| \\ &\leq \alpha_n\big\|f(x_n) - f(z)\big\| + \alpha_n\big\|f(z) - z\big\| + \beta_n\|x_n - z\| + \delta_n\|y_n - z\| \\ &\leq \alpha_n\kappa\|x_n - z\| + \alpha_n\big\|f(z) - z\big\| + \beta_n\|x_n - z\| + \delta_n\big(\|x_n - z\| + \alpha_n M_1\big) \\ &\leq \big(\alpha_n\kappa + \beta_n + \delta_n\big)\|x_n - z\| + \alpha_n\Big(\big\|f(z) - z\big\| + M_1\Big) \\ &= \big(1 - \alpha_n(1-\kappa)\big)\|x_n - z\| + \alpha_n(1-\kappa)\left(\frac{\big\|f(z) - z\big\| + M_1}{1-\kappa}\right) \\ &\leq \max\left\{\|x_n - z\|, \frac{\big\|f(z) - z\big\| + M_1}{1-\kappa}\right\} \\ &\ \ \vdots \\ &\leq \max\left\{\|x_{n_0} - z\|, \frac{\big\|f(z) - z\big\| + M_1}{1-\kappa}\right\}, \end{aligned} \tag{3.25}$$

for each $n \geq n_0$. This means $\big\{\|x_n - z\|\big\}$ is a bounded sequence, and it follows that $\{x_n\}$ is bounded. Subsequently, the sequences $\{z_n\}$, $\{y_n\}$ and $\{f(x_n)\}$ are also bounded.

Next, we note that $P_\Omega f(\cdot)$ is a contraction mapping. Let p be a unique fixed point of $P_\Omega f(\cdot)$, that is $p = P_\Omega f(p)$. For each $n \geq n_0$, we see that

$$\begin{aligned} \|z_n - p\|^2 &= \big\|x_n + \mu_n(x_n - x_{n-1}) - p\big\|^2 \\ &= \big\|x_n - p\big\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 + 2\mu_n\langle x_n - p, x_n - x_{n-1}\rangle \\ &\leq \big\|x_n - p\big\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 + 2\mu_n\|x_n - p\|\|x_n - x_{n-1}\|. \end{aligned} \tag{3.26}$$

By using (3.26), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \big\langle \alpha_n f(x_n)+\beta_n x_n+\delta_n y_n-p, x_{n+1}-p\big\rangle \\
&= \alpha_n\big\langle f(x_n)-f(p), x_{n+1}-p\big\rangle+\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\
&\quad +\beta_n\big\langle x_n-p, x_{n+1}-p\big\rangle+\delta_n\big\langle y_n-p, x_{n+1}-p\big\rangle \\
&\leq \frac{\alpha_n}{2}\Big(\big\|f(x_n)-f(p)\big\|^2+\big\|x_{n+1}-p\big\|^2\Big) \\
&\quad +\frac{\beta_n}{2}\Big(\big\|x_n-p\big\|^2+\big\|x_{n+1}-p\big\|^2\Big) \\
&\quad +\frac{\delta_n}{2}\Big(\big\|y_n-p\big\|^2+\big\|x_{n+1}-p\big\|^2\Big) \\
&\quad +\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\
&\leq \Big(\frac{\alpha_n\kappa^2}{2}+\frac{\beta_n}{2}\Big)\|x_n-p\|^2+\frac{\delta_n}{2}\big\|z_n-p\big\|^2+\frac{1}{2}\|x_{n+1}-p\|^2 \\
&\quad +\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\
&\leq \Big(\frac{\alpha_n\kappa^2+\beta_n+\delta_n}{2}\Big)\|x_n-p\|^2+\frac{1}{2}\|x_{n+1}-p\|^2 \\
&\quad +\frac{\delta_n\mu_n^2}{2}\|x_n-x_{n-1}\|^2+\delta_n\mu_n\|x_n-p\|\|x_n-x_{n-1}\| \\
&\quad +\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle,
\end{aligned}
$$

for each $n \geq n_0$. Thus,

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-p\|^2+\mu_n^2\|x_n-x_{n-1}\|^2 \\
&\quad +2\mu_n\|x_n-p\|\|x_n-x_{n-1}\|+2\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\
&= \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-p\|^2 \\
&\quad +\mu_n\|x_n-x_{n-1}\|\big(\mu_n\|x_n-x_{n-1}\|+2\|x_n-p\|\big) \\
&\quad +2\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\
&\leq \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-p\|^2+M_2\mu_n\|x_n-x_{n-1}\| \\
&\quad +2\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\
&\leq \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-p\|^2 \\
&\quad +\alpha_n(1-\kappa^2)\Big(\frac{M_2}{1-\kappa^2}\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|+\frac{2}{1-\kappa^2}\big\langle f(p)-p, x_{n+1}-p\big\rangle\Big), \qquad (3.27)
\end{aligned}
$$

where $M_2 = 3\sup_n\big\{\mu\|x_n-x_{n-1}\|, \|x_n-p\|\big\} > 0$.

Now, we consider the following two cases.

Case 1: Suppose that $\big\{\|x_n-p\|\big\}$ is monotonically non-increasing for all $n \geq n_0$. Since $\big\{\|x_n-p\|\big\}$ is bounded, in this situation we can confirm that it is a convergent sequence.

Now, from (3.23) we have

$$
\begin{aligned}
\|z_n-p\|^2 &= \big(\|x_n-p\|+\alpha_n M_1\big)^2 \\
&= \|x_n-p\|^2+2\alpha_n M_1\|x_n-p\|+\alpha_n^2 M_1^2 \\
&= \|x_n-p\|^2+\alpha_n\big(2M_1\|x_n-p\|+\alpha_n M_1^2\big) \\
&= \|x_n-p\|^2+\alpha_n M_3, \qquad (3.28)
\end{aligned}
$$

for each $n \geq n_0$, where $M_3 = \sup_n \left\{2M_1\|x_n - p\| + \alpha_n M_1^2\right\} > 0$.

By using (2.11), (3.14) and (3.28), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + \delta_n y_n - p\right\|^2 \\
&= \left\|\alpha_n\big(f(x_n) - p\big) + \beta_n(x_n - p) + \delta_n(y_n - p)\right\|^2 \\
&= \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\quad -\alpha_n\beta_n\left\|f(x_n) - x_n\right\|^2 - \alpha_n\delta_n\left\|f(x_n) - y_n\right\|^2 - \beta_n\delta_n\|x_n - y_n\|^2 \\
&\leq \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\leq \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad +\delta_n\left(\|z_n - p\|^2 - \left(1 - \theta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2\right) \\
&\leq \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad +\delta_n\left(\|x_n - p\|^2 + \alpha_n M_3 - \left(1 - \theta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2\right) \\
&\leq \|x_n - p\|^2 + \alpha_n\left(\left\|f(x_n) - p\right\|^2 + M_3\right) - \delta_n\left(1 - \theta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2,
\end{aligned}
$$

for each $n \geq n_0$. This implies

$$
\begin{aligned}
\delta_n\left(1 - \theta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|z_n - w_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +\alpha_n\left(\left\|f(x_n) - p\right\|^2 + M_3\right),
\end{aligned} \tag{3.29}
$$

for each $n \geq n_0$. Thus, by condition (C1), (C2) and (3.21), we get

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.30}$$

Using this one together with (3.15), we also get

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.31}$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\quad -\alpha_n\beta_n\|f(x_n) - x_n\|^2 - \alpha_n\delta_n\|f(x_n) - y_n\|^2 - \beta_n\delta_n\|x_n - y_n\|^2,
\end{aligned}
$$

for each $n \geq n_0$. This is,

$$
\begin{aligned}
\beta_n\delta_n\|x_n - y_n\|^2 &\leq \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\quad -\|x_{n+1} - p\|^2.
\end{aligned} \tag{3.32}
$$

Moreover, we know that

$$\|y_n - p\|^2 \leq \|z_n - p\|^2 \leq \|x_n - p\|^2 + \alpha_n M_3.$$

Then, from (3.32), we obtain

$$\begin{aligned}\beta_n\delta_n\|x_n-y_n\|^2 &\le \alpha_n\big\|f(x_n)-p\big\|^2+\beta_n\|x_n-p\|^2+\delta_n\|x_n-p\|^2+\alpha_n\delta_n M_3\\ &\quad -\|x_{n+1}-p\|^2\\ &\le \alpha_n\left(\big\|f(x_n)-p\big\|^2+M_3\right)+\|x_n-p\|^2-\|x_{n+1}-p\|^2. \qquad (3.33)\end{aligned}$$

By conditions (C1) and (C2), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \qquad (3.34)$$

Next, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Using the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \big\|\alpha_n f(x_n)+\beta_n x_n+\delta_n y_n-x_n\big\|\\ &\le \alpha_n\big\|f(x_n)-x_n\big\|+\delta_n\|y_n-x_n\|,\end{aligned}$$

for each $n\ge n_0$. By using (3.34) and condition (C1), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.35)$$

Now, consider

$$\begin{aligned}\|z_n-x_n\| &= \big\|x_n+\mu_n(x_n-x_{n-1})-x_n\big\|\\ &\le \alpha_n\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|,\end{aligned}$$

for each $n\ge n_0$. By conditions (C1) and (C3), we get

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \qquad (3.36)$$

On the other hand, since $\{x_n\}$ is bounded on H, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to $x^*\in H$. Next, we will show that $x^*\in\Omega$. Consider

$$\begin{aligned}\big\|x^*-J^B_{\lambda_n}(I-\lambda_nA)x^*\big\|^2 &\le \big\langle x^*-J^B_{\lambda_n}(I-\lambda_nA)x^*,x^*-z_{n_j}\big\rangle\\ &\quad+\big\langle x^*-J^B_{\lambda_n}(I-\lambda_nA)x^*,z_{n_j}-J^B_{\lambda_n}(I-\lambda_nA)z_{n_j}\big\rangle\\ &\quad+\big\langle x^*-J^B_{\lambda_n}(I-\lambda_nA)x^*,J^B_{\lambda_n}(I-\lambda_nA)z_{n_j}-J^B_{\lambda_n}(I-\lambda_nA)x^*\big\rangle,\end{aligned}$$

for each $j\in\mathbb{N}$. By using (3.30) and (3.36), we obtain

$$\lim_{j\to\infty}\big\|x^*-J^B_{\lambda_n}(I-\lambda_nA)x^*\big\|=0.$$

It follows that, $x^*=J^B_{\lambda_n}(I-\lambda_nA)x^*$ and hence $x^*\in\Omega$.

Finally, we prove that the sequence $\{x_n\}$ converges strongly to $p=P_\Omega f(p)$. Now, we know that $\{x_n\}$ is bounded, and we have from (3.35) that $\|x_{n+1}-x_n\|\to 0$, as $n\to\infty$. With loss of generality, we may assume that a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ converges weakly to $x^*\in H$. Thus, we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{2}{1-\kappa^2}\big\langle f(p)-p,x_{n+1}-p\big\rangle &= \lim_{j\to\infty}\frac{2}{1-\kappa^2}\big\langle f(p)-p,x_{n_j+1}-p\big\rangle\\ &= \frac{2}{1-\kappa^2}\big\langle f(p)-p,x^*-p\big\rangle\le 0. \qquad (3.37)\end{aligned}$$

By using (3.37) and together with the condition (C3), we obtain

$$\limsup_{n\to\infty}\left(\frac{M_2}{1-\kappa^2}\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|+\frac{2}{1-\kappa^2}\big\langle f(p)-p,x_{n+1}-p\big\rangle\right)\le 0. \qquad (3.38)$$

From (3.27), by using Lemma 2.3, we can conclude that $x_n\to p$, as $n\to\infty$.

Case 2: Suppose that $\{\|x_n - p\|\}$ is not monotonically non-increasing for all $n \geq n_0$. By the setting $\Gamma_n = \|x_n - p\|, \forall n \in \mathbb{N}$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \quad \Gamma_k \leq \Gamma_{k+1}\},$$

for all $n \geq n_0$. Then, we have $\{\tau(n)\}$ is a nondecreasing sequence, with $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0,$$

see [17]. And, it follows that $\|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \leq 0$, for each $n \geq n_0$. From (3.29), we obtain the following relation

$$\begin{aligned}
&\delta_{\tau(n)}\left(1 - \theta^2 \frac{\lambda^2_{\tau(n)}}{\lambda^2_{\tau(n)+1}}\right)\|z_{\tau(n)} - w_{\tau(n)}\|^2 \\
&\leq \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 + \alpha_{\tau(n)}\big(\|f(x_{\tau(n)}) - p\|^2 + M_3\big) \\
&\leq \alpha_{\tau(n)}\big(\|f(x_{\tau(n)}) - p\|^2 + M_3\big),
\end{aligned}$$

for each $n \geq n_0$. Following the line proof as in the **Case 1**, we can get

$$\lim_{n\to\infty} \|z_{\tau(n)} - w_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|y_{\tau(n)} - w_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0,$$

and

$$\limsup_{n\to\infty}\left(\frac{M_2}{1-\kappa^2}\frac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| + \frac{2}{1-\kappa^2}\langle f(p) - p, x_{\tau(n)+1} - p\rangle\right) \leq 0.$$

Since the sequence $\{x_{\tau(n)}\}$ is bounded, we can find a subsequence of $\{x_{\tau(n)}\}$, and for the sake of simplicity we will still denote it by $\{x_{\tau(n)}\}$, which converges weakly to $x^* \in (A+B)^{-1}0$. From the relation in (3.27), we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - p\|^2 &\leq \big(1 - \alpha_{\tau(n)}(1-\kappa^2)\big)\|x_{\tau(n)} - p\|^2 \\
&\quad + \alpha_{\tau(n)}(1-\kappa^2)T_{\tau(n)},
\end{aligned}$$

where $T_{\tau(n)} = \dfrac{M_2}{1-\kappa^2}\dfrac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| + \dfrac{2}{1-\kappa^2}\langle f(p) - p, x_{\tau(n)+1} - p\rangle$, for each $n \geq n_0$. Consequently, we have

$$\begin{aligned}
\alpha_{\tau(n)}(1-\kappa^2)\|x_{\tau(n)} - p\|^2 &\leq \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \\
&\quad + \alpha_{\tau(n)}(1-\kappa^2)T_{\tau(n)} \\
&\leq \alpha_{\tau(n)}(1-\kappa^2)T_{\tau(n)}. \qquad (3.39)
\end{aligned}$$

Since $\alpha_{\tau(n)}(1-\kappa^2) > 0$, it follows from (3.39) that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - p\|^2 \leq 0.$$

This implies

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\|^2 = 0,$$

and also

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0. \qquad (3.40)$$

By using $\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\|=0$ and (3.40), then we have

$$\|x_{\tau(n)+1}-p\| \le \|x_{\tau(n)+1}-x_{\tau(n)}\| + \|x_{\tau(n)}-p\| \to 0, \tag{3.41}$$

as $n\to\infty$. Moreover, if $\tau(n)<n$, we also have $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, because $\Gamma_{\tau(n)+1} \ge \Gamma_j$ for $\tau(n)+1 \le j \le n$. Consequently, we obtain

$$0 \le \Gamma_n \le \max\left\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\right\} = \Gamma_{\tau(n)+1},$$

for each $n \ge n_0$. By using (3.41), we have that $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, we can conclude that $\{x_n\}$ converges strongly to p. This completes the proof. ☐

Remark 3.1. Observe that if $\beta_n = 0$, Algorithm 3.1 is reduced to Algorithm (1.9). However, according to condition (C2), we can not use the Theorem 3.1 to guarantee the convergence of sequence $\{x_n\}$ to a solution point in this situation.

Remark 3.2. (a) [27] The condition (C3) is easily implemented in numerical computation because we can find the value of $\|x_n - x_{n-1}\|$ before choosing μ_n. Indeed, we can choose the parameter μ_n such that $0 \le \mu_n \le \bar{\mu}_n$, where

$$\bar{\mu}_n = \begin{cases} \min\left\{\mu, \dfrac{\omega_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \ne x_{n-1}; \\ \mu, & \text{otherwise,} \end{cases}$$

where ω_n is a positive sequence such that $\omega_n = o(\alpha_n)$.

(b) The following choice is the special case of (a); we choose $\alpha_n = \dfrac{1}{n+1}$, $\omega_n = \dfrac{1}{(n+1)^2}$ and $\mu = \dfrac{n-1}{n+\kappa-1} \in [0,1)$. Then, we have

$$\bar{\mu}_n = \begin{cases} \min\left\{\dfrac{n-1}{n+\kappa-1}, \dfrac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \ne x_{n-1}; \\ \dfrac{n-1}{n+\kappa-1}, & \text{otherwise.} \end{cases}$$

(c) If $f := \nabla h$, where $h : H \to \mathbb{R}$ is a continuous differentiable function, then in Theorem 3.1 we have $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega \nabla h(p)$, which is the optimality condition for the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\|x\|^2 - h(x).$$

4. Applications

In this section, we will show some applications of the problem (1.3) via Theorem 3.1.

4.1. **Variational inequality problem.**

Recall that the normal cone to C at $u \in C$ is defined as

$$N_C(u) = \left\{z \in H : \langle z, y-u\rangle \le 0, \quad \forall y \in C\right\}. \tag{4.42}$$

It is well known that N_C is a maximal monotone operator. In the case $B := N_C : H \to 2^H$, we can verify that the problem (1.3) is reduced to the variational inequality problem: the problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \ge 0, \qquad \forall x \in C. \tag{4.43}$$

We will denote $VIP(C, A)$ for the solution set of the problem (4.43). Also, in this case, we have $J_\lambda^B =: P_C$ (the metric projection of H onto C). By the above setting, the problem (1.3) is reduced to a problem of finding a point

$$x^* \in VIP(C, A) =: \Omega_{A,C}. \tag{4.44}$$

Thus, by applying Theorem 3.1, we obtain the following result.

Algorithm 4.1. *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ with $\alpha_n + \beta_n + \delta_n = 1$ and the initial $x_0, x_1 \in H$ be arbitraries, define*

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ w_n &= P_C(I - \lambda_n A)z_n, \\ y_n &= w_n - \lambda_n(Aw_n - Az_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \delta_n y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{4.45}$$

where $\{\mu_n\} \subset [0, \mu)$ with $\mu \in [0,1)$ and $\{\lambda_n\}$ is defined by

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\theta\|z_n - w_n\|}{\|Az_n - Aw_n\|}, \lambda_n \right\}, & \textit{if } Az_n - Aw_n \neq 0; \\ \lambda_n, & \textit{otherwise}, \end{cases}$$

when $\lambda_1 > 0$ and $\theta \in (0,1)$.

Theorem 4.2. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $\{x_n\}$ be generated by Algorithm 4.1. Suppose that the assumptions (A1) and (A3) hold, $\Omega_{A,C} \neq \emptyset$ and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) There exist a positive real number a with $0 < a \leq \beta_n$ and $0 < a \leq \delta_n$, for each $n \in \mathbb{N}$;

(C3) $\lim_{n\to\infty} \dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $p \in \Omega_{A,C}$, where $p = P_{\Omega_{A,C}} f(p)$.

4.2. **Convex minimization problem.** We will consider a convex function $g : H \to R$, which is Fréchet differentiable. Let C be a given closed convex subset of H. In this case, by setting $A := \nabla g$ (the gradient of g) and $B := N_C$, the problem of finding $x^* \in (A+B)^{-1}0$ is equivalent to find a point $x^* \in C$ such that

$$\left\langle \nabla g(x^*), x - x^* \right\rangle \geq 0, \qquad \forall x \in C. \tag{4.46}$$

Note that (4.46) is equivalent to the following minimization problem: find $x^* \in C$ such that

$$x^* \in \arg\min_{x \in C} g(x).$$

Thus, in this situation, the problem (1.3) is reduced to a problem of finding a point

$$x^* \in \arg\min_{x \in C} g(x) =: \Omega_{g,C}. \tag{4.47}$$

Subsequently, we obtain the following result.

Algorithm 4.2. *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ with $\alpha_n+\beta_n+\delta_n=1$ and the initial $x_0, x_1 \in H$ be arbitraries, define*

$$\begin{aligned} z_n &= x_n+\mu_n(x_n-x_{n-1}), \\ w_n &= P_C(I-\lambda_n\nabla g)z_n, \\ y_n &= w_n-\lambda_n\big(\nabla g(w_n)-\nabla g(z_n)\big), \\ x_{n+1} &= \alpha_n f(x_n)+\beta_n x_n+\delta_n y_n, \qquad \forall n\in\mathbb{N}, \end{aligned} \tag{4.48}$$

where $\{\mu_n\}\subset[0,\mu)$ with $\mu\in[0,1)$ and $\{\lambda_n\}$ is defined by

$$\lambda_{n+1}=\begin{cases}\min\left\{\frac{\theta\|z_n-w_n\|}{\|\nabla g(z_n)-\nabla g(w_n)\|},\lambda_n\right\}, & \text{if } \nabla g(z_n)-\nabla g(w_n)\neq 0;\\ \lambda_n, & \text{otherwise,}\end{cases}$$

when $\lambda_1>0$ and $\theta\in(0,1)$.

Theorem 4.3. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $g: H\to\mathbb{R}$ be convex and Fréchet differentiable, ∇g be η-Lipschitz. Let $\{x_n\}$ be generated by Algorithm 4.2. Suppose that the assumptions (A4) hold, $\Omega_{g,C}\neq\emptyset$ and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(C2) There exist a positive real number a with $0<a\le\beta_n$ and $0<a\le\delta_n$, for each $n\in\mathbb{N}$;
(C3) $\lim_{n\to\infty}\dfrac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|=0$.

Then, $\{x_n\}$ converges strongly to $p\in\Omega_{g,C}$, where $p=P_{\Omega_{g,C}}f(p)$.

4.3. **Split feasibility problem.** Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively, and let $L: H_1\to H_2$ be a bounded linear operator. We set $B:=N_C: H_1\to 2^{H_1}$, then $J_{\lambda_n}^B=: P_C$. It follows that $F(J_{\lambda_n}^B)=F(P_C)=C$. Now, we note that $L^*(I-P_Q)L$ is $\frac{1}{2\|L\|^2}-ism$; see [30]. By the setting $A=: L^*(I-P_Q)L$, then it is $2\|L\|^2$-Lipschitz. From the above setting, we can verify that the problem (1.3) is reduced to the following split feasibility problem; the problem of finding a point

$$x^*\in C\cap L^{-1}Q=:\Omega_{C,Q}; \tag{4.49}$$

see [7, 35] for more information. Then, by applying Theorem 3.1, we obtain the following result.

Algorithm 4.3. *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ with $\alpha_n+\beta_n+\delta_n=1$ and the initial $x_0, x_1 \in H$ be arbitraries, define*

$$\begin{aligned} z_n &= x_n+\mu_n(x_n-x_{n-1}), \\ w_n &= P_C\big(I-\lambda_n L^*(I-P_Q)L\big)z_n, \\ y_n &= w_n-\lambda_n\big(L^*(I-P_Q)Lw_n-L^*(I-P_Q)Lz_n\big), \\ x_{n+1} &= \alpha_n f(x_n)+\beta_n x_n+\delta_n y_n, \qquad \forall n\in\mathbb{N}, \end{aligned} \tag{4.50}$$

where $\{\mu_n\}\subset[0,\mu)$ with $\mu\in[0,1)$ and $\{\lambda_n\}$ is defined by

$$\lambda_{n+1}=\begin{cases}\min\left\{\frac{\theta\|z_n-w_n\|}{\|L^*(I-P_Q)Lz_n-L^*(I-P_Q)Lw_n\|},\lambda_n\right\}, & \text{if } L^*(I-P_Q)Lz_n-L^*(I-P_Q)Lw_n\neq 0;\\ \lambda_n, & \text{otherwise,}\end{cases}$$

when $\lambda_1>0$ and $\theta\in(0,1)$.

Theorem 4.4. *Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively, and let $L : H_1 \to H_2$ be a bounded linear operator. Let $\{x_n\}$ be generated by Algorithm 4.3. Suppose that the assumptions (A3) hold, $\Omega_{C,Q} \neq \emptyset$ and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) There exist a positive real number a with $0 < a \leq \beta_n$ and $0 < a \leq \delta_n$, for each $n \in \mathbb{N}$;

(C3) $\lim_{n\to\infty} \dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $p \in \Omega_{C,Q}$, where $p = P_{\Omega_{C,Q}} f(p)$.

5. Numerical experiments

In this section, we will consider some numerical experiments to illustrate the use of Theorem 3.1.

Example 5.1. Let $H = \mathbb{R}^2$ be equipped with the Euclidean norm. For each $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H$, we consider the following two norms:

$$\|x\|_1 = |x_1| + |x_2| \quad \text{and} \quad \|x\|_\infty = \max\big\{|x_1|, |x_2|\big\}.$$

Let a function $g : H \to \mathbb{R}$, which is defined by

$$g(x) = \|x\|_1, \qquad \forall x \in H.$$

Now, we have a subdifferential operator of g is

$$\partial g(x) = \big\{z \in H : \langle x, z\rangle = \|x\|_1, \|z\|_\infty \leq 1\big\}, \qquad \forall x \in H.$$

Since g is a convex function, then $\partial g(\cdot)$ is a maximal monotone operator. Moreover, for each $\lambda > 0$, we have

$$J_\lambda^{\partial g}(x) = \left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H : u_i = x_i - \big(\min\{|x_i|, \lambda\}\big) sgn(x_i), \quad \text{for} \quad i = 1, 2\right\},$$

where $sgn(\cdot)$ stands for the signum function.

Next, let $\bar{x} := \begin{pmatrix} 5 \\ 4 \end{pmatrix} \in H$ be fixed vector. We consider 1-Lipschitz operator P_Q, where

$$Q := \big\{u \in H : \langle \bar{x}, u\rangle \leq -9\big\}.$$

Furthermore, we consider a contraction mapping $f := \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$.

Under the above settings, we will consider the problem to find a point

$$x^* \in (P_Q + \partial g)^{-1} 0. \tag{5.51}$$

Notice that the solution set of problem (5.51) is $\Omega := \left\{\begin{pmatrix} x \\ \frac{4x-1}{5} \end{pmatrix} \in H : x \geq 0.25\right\}$. Moreover, we can check that $p = f_\Omega(p)$, when $p = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$. We determine the results using the stopping criterion by $\dfrac{\|x_{n+1} - x_n\|}{\max\{1, \|x_n\|\}} \leq 1.0e^{-06}$.

We first consider Algorithm 3.1 with four cases of the step size parameters α_n, β_n and δ_n:

Case 1. $\alpha_n = \dfrac{1}{n+10}$, $\beta_n = 0.9$, $\delta_n = 0.1 - \dfrac{1}{n+10}$;

Case 2. $\alpha_n = \dfrac{1}{n+10}$, $\beta_n = 0.5$, $\delta_n = 0.5 - \dfrac{1}{n+10}$;

Case 3. $\alpha_n = \dfrac{1}{n+10}$, $\beta_n = 0.1$, $\delta_n = 0.9 - \dfrac{1}{n+10}$;

Case 4. $\alpha_n = \dfrac{1}{n+10}$, $\beta_n = 0$, $\delta_n = 1 - \dfrac{1}{n+10}$.

We consider seven different initial points as follows:

IP 1. $x_0 = (0,0)^\top, x_1 = (1,1)^\top$;
IP 2. $x_0 = (-1,-1)^\top, x_1 = (1,1)^\top$;
IP 3. $x_0 = (1,-1)^\top, x_1 = (-1,1)^\top$;
IP 4. $x_0 = (1,10)^\top, x_1 = (1,-10)^\top$;
IP 5. $x_0 = (1,10)^\top, x_1 = (-1,10)^\top$;
IP 6. $x_0 = (1,-10)^\top, x_1 = (1,10)^\top$;
IP 7. $x_0 = (-10,10)^\top, x_1 = (10,-10)^\top$.

We choose $\mu = 0.5$ and let $\omega_n = \dfrac{1}{(n+10)^2}$, and define μ_n by following Remark 3.2(a). Under the different initial points, the results are shown in Table 1, with fixed values of $\theta = 0.5$ and $\lambda_1 = 0.5$. From Table 1, we compute the average of iterations for different four cases of parameters α_n, β_n and δ_n, and show in Figure 1. The figure shows that the average iteration number seems to decrease when the parameter β_n is decreased (consider Case 1 with Case 2 and Case 3). However, when we consider Case 3 and Case 4, we find that the number of iteration numbers is increased, but β_n is decreased from 0.1 to 0.

In Table 2, we consider the numerical experiments by focusing to the parameter θ. We consider the initial point IP 2 with the four cases of parameters α_n, β_n and δ_n as above. We will consider different three values of θ that are $\theta = 0.1$, $\theta = 0.5$, and $\theta = 0.9$. From the presented results in Table 2, we found that the Case 3 of parameters α_n, β_n and δ_n with $\theta = 0.5$, and $\theta = 0.9$ show the superiority result of the iteration number. Indeed, we may observe that the larger values of parameter θ provide faster convergence.

In conclusion, as we can see that when we consider the Case 4 of parameters α_n, β_n and δ_n, the Algorithm 3.1 is nothing but Algorithm (1.9). Thus, the above results and experiment show that more choices on the parameters based on Algorithm 3.1 will provide more chances for getting better convergence results.

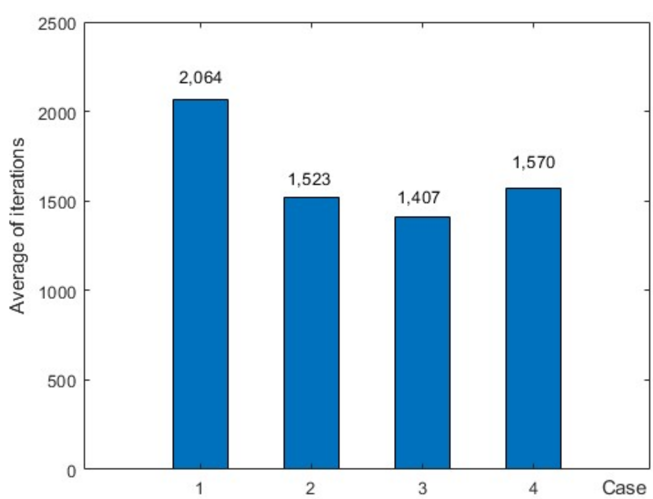

FIGURE 1. The average of iterations for different cases of parameters α_n, β_n and δ_n.

TABLE 1. Numerical experiments for the different stepsize parameters of α_n, β_n and δ_n to Algorithm 3.1 with some initial points.

Case →	**Case 1**		**Case 2**		**Case 3**		**Case 4**	
# x_0, x_1 ↓	**Iters**	**Sol**	**Iters**	**Sol**	**Iters**	**Sol**	**Iters**	**Sol**
IP 1	399	$\begin{pmatrix}0.231091\\0.001355\end{pmatrix}$	318	$\begin{pmatrix}0.245085\\0.000364\end{pmatrix}$	286	$\begin{pmatrix}0.246947\\0.000227\end{pmatrix}$	896	$\begin{pmatrix}0.249100\\0.000067\end{pmatrix}$
IP 2	399	$\begin{pmatrix}0.231091\\0.001355\end{pmatrix}$	318	$\begin{pmatrix}0.245085\\0.000364\end{pmatrix}$	286	$\begin{pmatrix}0.246947\\0.000227\end{pmatrix}$	896	$\begin{pmatrix}0.249100\\0.000067\end{pmatrix}$
IP 3	2,892	$\begin{pmatrix}0.247152\\0.000213\end{pmatrix}$	1,275	$\begin{pmatrix}0.248727\\0.000095\end{pmatrix}$	946	$\begin{pmatrix}0.249052\\0.000071\end{pmatrix}$	896	$\begin{pmatrix}0.249100\\0.000067\end{pmatrix}$
IP 4	2,892	$\begin{pmatrix}0.247152\\0.000213\end{pmatrix}$	1,275	$\begin{pmatrix}0.248727\\0.000095\end{pmatrix}$	946	$\begin{pmatrix}0.249052\\0.000071\end{pmatrix}$	896	$\begin{pmatrix}0.249100\\0.000067\end{pmatrix}$
IP 5	1,339	$\begin{pmatrix}0.244117\\0.000433\end{pmatrix}$	2,101	$\begin{pmatrix}0.249240\\0.000057\end{pmatrix}$	2,228	$\begin{pmatrix}0.249600\\0.000030\end{pmatrix}$	2,240	$\begin{pmatrix}0.249642\\0.000027\end{pmatrix}$
IP 6	2,698	$\begin{pmatrix}0.247079\\0.000216\end{pmatrix}$	3,392	$\begin{pmatrix}0.249532\\0.000035\end{pmatrix}$	3,496	$\begin{pmatrix}0.249747\\0.000019\end{pmatrix}$	3,504	$\begin{pmatrix}0.249773\\0.000017\end{pmatrix}$
IP 7	3,828	$\begin{pmatrix}0.247953\\0.000151\end{pmatrix}$	1,985	$\begin{pmatrix}0.249196\\0.000060\end{pmatrix}$	1,661	$\begin{pmatrix}0.249462\\0.000040\end{pmatrix}$	1,660	$\begin{pmatrix}0.249515\\0.000036\end{pmatrix}$

TABLE 2. Influence of the parameter θ of Algorithm 3.1 for different cases of parameters α_n, β_n and δ_n with the initial point $x_0 = (-1,-1)^\top, x_1 = (1,1)^\top$.

Case →	**Case 1**		**Case 2**		**Case 3**		**Case 4**	
# θ ↓	**Iters**	**Sol**	**Iters**	**Sol**	**Iters**	**Sol**	**Iters**	**Sol**
$\theta = 0.1$	562	$\begin{pmatrix}0.202179\\0.000095\end{pmatrix}$	380	$\begin{pmatrix}0.233877\\0.000033\end{pmatrix}$	327	$\begin{pmatrix}0.239381\\0.000022\end{pmatrix}$	317	$\begin{pmatrix}0.240120\\0.000020\end{pmatrix}$
$\theta = 0.5$	399	$\begin{pmatrix}0.231091\\0.001355\end{pmatrix}$	318	$\begin{pmatrix}0.245085\\0.000364\end{pmatrix}$	286	$\begin{pmatrix}0.246947\\0.000227\end{pmatrix}$	896	$\begin{pmatrix}0.249100\\0.000067\end{pmatrix}$
$\theta = 0.9$	399	$\begin{pmatrix}0.231091\\0.001355\end{pmatrix}$	318	$\begin{pmatrix}0.245085\\0.000364\end{pmatrix}$	286	$\begin{pmatrix}0.246947\\0.000227\end{pmatrix}$	896	$\begin{pmatrix}0.249100\\0.000067\end{pmatrix}$

6. CONCLUSIONS

In this work, we present a new algorithm for finding a solution of monotone inclusion problems in Hilbert spaces, problem (1.3). The suggestion algorithm is modified by including the inertial method and Tseng type algorithm, Algorithm 3.1. By providing suitable control conditions to the process, we obtain the strong convergence theorem of the proposed algorithm (Theorem 3.1). In applications, we apply the theorem to the variational inequality problem, convex minimization problem, and split feasibility problem. Finally, the numerical experiments of the suggested algorithm are shown, and it is found that considerations on the choice of parameters are needed in order to get better convergence results.

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