CARPATHIAN J. MATH.
Volume **40** (2024), No. 2,
Pages 419 - 430

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.02.12

On approximating fixed points of strictly pseudocontractive mappings in metric spaces

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ABSTRACT. In this work, we analyse the class of strictly pseudocontractive mappings in general metric spaces by providing a comprehensive and appropriate definition of a strictly pseudocontractive mapping, which serves as a natural extension of the existing notion. Moreover, we establish its various characterizations and explore several significant properties of these mappings in relation to fixed point theory in CAT(0) spaces. Specifically, we establish that these mappings are Lipschitz continuous, satisfying the demiclosedness-type property, and possessing a closed convex fixed point set. Furthermore, we show that the fixed points of the mappings can be effectively approximated using an iterative scheme for fixed points of nonexpansive mappings. The results in this work contribute to a deeper understanding of strictly pseudocontractive mappings and their applicability in the context of fixed point theory in metric spaces.

1. INTRODUCTION

The class of strictly pseudocontractive mappings (or strict pseudo-contractions) plays an important role in the iterative approximation of fixed points of nonexpansive type mappings.

Strict pseudo-contractions have been introduced by Browder and Petryshyn in 1967 [8] in the setting of a Hilbert space and since then have been intensively studied by several authors in linear settings (Hilbert spaces, Banach spaces), see [5], [11], [12], [15], [17], [22], [24], [26], for a very selective list, and also the references therein.

Let $(E, \|\cdot\|)$ be a normed linear space and let C be a nonempty subset of E. A mapping $T : C \to E$ is said to be a *strict pseudocontraction* if there exists a nonnegative constant $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)(x) - (I - T)(y)\|^2, \tag{1.1}$$

for all $x, y \in C$. To specify the constant k in (1.1), we refer to the mapping as k-strictly pseudocontractive.

If in (1.1) we have $k = 1$, then T is called a pseudocontraction.

It is well-known that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings, that is, of those mappings satisfying

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

Construction of fixed points of strictly pseudocontractive mappings via iterative algorithms is an important research topic with applications in various current research fields like variational inequalities [25], [23], split variational inclusion [1], equilibrium problems [2], [18], [19], split feasibility problems [2] etc.

Starting from the fact that all previous contributions to the study of strictly pseudocontractive mappings are obtained in linear settings, our aim in this work is to analyse

Received: 19.09.2023. In revised form: 20.02.2024. Accepted: 27.02.2024
2020 *Mathematics Subject Classification.* 47H09, 47H10, 47J25.
Key words and phrases. *CAT(0) space, convergence, enriched nonexpansive, fixed point, metric space, strictly pseudocontractive.*
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the class of strictly pseudocontractive mappings in general metric spaces by providing a comprehensive and appropriate definition of strictly pseudocontractive mappings which is a natural extension from linear to nonlinear settings.

2. Strictly Pseudocontractive Mappings In metric spaces

In the general setting of metric spaces, $T : \mathcal{D} \to \mathcal{H}$ is known as a strictly pseudocontractive mapping, see [4], if there exists $\kappa \in [0, 1)$ such that

$$d^2(Tu, Tw) \leq d^2(u, w) + \kappa\Big[d(u, Tu) + d(w, Tw)\Big]^2, \quad \forall\, u, w \in \mathcal{D} \tag{2.2}$$

where $d^2(x, y) = \big[d(x, y)\big]^2$ for all $x, y \in \mathcal{H}$.

However, the definition above does not coincide with the definition of Browder and Petryshyn in [8] when $\mathcal{H}$ is a Hilbert space with the usual metric. For this sake, we recall that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x,\ y\rangle \tag{2.3}$$

for two points x and y in a real inner-product space. The identity (2.3) justifies that (1.1) is equivalent to the following:

$$\|Tu - Tw\|^2 \leq \|u - w\|^2 + \kappa\Big[\|u - w\|^2 + \|Tu - Tw\|^2 + 2\langle w - u, Tu - Tw\rangle\Big]. \tag{2.4}$$

Furthermore, in real inner-product spaces, the following identity holds:

$$\langle x - y,\ u - w\rangle = \frac{1}{2}\Big[\|x - w\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - w\|^2\Big], \tag{2.5}$$

where x, y, u, w are arbitrary points in the space.

It is natural to extend the identity (2.5) to metric setting by defining

$$\mathcal{Q}(x, y, u, w) := \frac{1}{2}\Big[d^2(x, w) + d^2(y, u) - d^2(x, u) - d^2(y, w)\Big]. \tag{2.6}$$

The quantity $\mathcal{Q}(x, y, u, w)$ is known as quasilinearization (see [6]).

Therefore, (2.6) together with (2.4) are yielding the appropriate definition of a strictly pseudocontractive mapping in general metric spaces as follows.

Definition 2.1. Let $(\mathcal{H}, d)$ be a metric space and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. A mapping $T : \mathcal{D} \to \mathcal{H}$ is said to be κ-strictly pseudocontractive if there exists $\kappa \in [0, 1)$ such that

$$d^2(Tu, Tw) \leq d^2(u, w) + \kappa\Big[d^2(u, w) + d^2(Tu, Tw) + 2\mathcal{Q}(w, u, Tu, Tw)\Big], \tag{2.7}$$

for all $u, w \in \mathcal{D}$.

Example 2.1. Let $\mathcal{H} = \mathcal{D} = \mathbb{R}^4$ be endowed with the metric d defined by

$$d(u, w) = \sqrt{\sum_{i=1}^{3}(u_i - w_i)^2 + (u_3^2 + w_4 - u_4 - w_3^2)^2},$$

for all $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ and $w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$. Then $(\mathcal{H}, d)$ is a metric space. Consider $T : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$Tu = -\ell(u_1, u_2, u_3, -\ell u_3^2),\ \forall u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4, \quad \text{where} \quad \ell \geq 2.$$

Clearly T is not nonexpansive mapping with respect to the both usual metric and $(\mathbb{R}^4, d)$ since for any $u = (u_1, 0, 0, 0)$, $u_1 \neq 0$ and $w = (0, 0, 0, 0)$, we have

$$d(Tu, Tw) = \ell|u_1| > |u_1| = d(u, w).$$

However, T is κ-strictly pseudocontractive mapping with respect to $(\mathbb{R}^4, d)$ for $\kappa \in \left[\frac{\ell-1}{\ell+1},\ 1\right)$. Indeed for all $u, w \in \mathbb{R}^4$, we have

$$\begin{aligned}
d^2(Tu, Tw) &= \ell^2 \sum_{i=1}^{3} (u_i - w_i)^2, \\
2\mathcal{Q}(u, w, Tu, Tw) &= d^2(u, Tw) + d^2(w, Tu) - d^2(u, Tu) - d^2(w, Tw) \\
&= \left[\sum_{i=1}^{3} (u_i + \ell w_i)^2 + (u_3^2 - u_4)^2\right] + \left[\sum_{i=1}^{3} (w_i + \ell u_i)^2 + (w_3^2 - w_4)^2\right] \\
&- \left[\sum_{i=1}^{3} (u_i + \ell u_i)^2 + (u_3^2 - u_4)^2\right] - \left[\sum_{i=1}^{3} (w_i + \ell w_i)^2 + (w_3^2 - w_4)^2\right] \\
&= \sum_{i=1}^{3} \left[(u_i + \ell w_i)^2 + (w_i + \ell u_i)^2 - (u_i + \ell u_i)^2 - (w_i + \ell w_i)^2\right] \\
&= -2\ell \sum_{i=1}^{3} (u_i - w_i)^2 .
\end{aligned}$$

So, we get

$$\begin{aligned}
&d^2(u, w) + d^2(Tu, Tw) + 2\mathcal{Q}(w, u, Tu, Tw) \\
&\qquad = d^2(u, w) + d^2(Tu, Tw) - 2\mathcal{Q}(u, w, Tu, Tw) \\
&\qquad = \sum_{i=1}^{3} (u_i - w_i)^2 + (u_3^2 + w_4 - u_4 - w_3^2)^2 \\
&\qquad \quad + \ell^2 \sum_{i=1}^{3} (u_i - w_i)^2 + 2\ell \sum_{i=1}^{3} (u_i - w_i)^2 \\
&\qquad = (1 + \ell)^2 \sum_{i=1}^{3} (u_i - w_i)^2 + (u_3^2 + w_4 - u_4 - w_3^2)^2.
\end{aligned}$$

Thus, for any κ with $\frac{\ell-1}{\ell+1} \leq \kappa < 1$, we have

$$\begin{aligned}
d^2(Tu, Tw) &= \ell^2 \sum_{i=1}^{3} (u_i - w_i)^2 = \sum_{i=1}^{3} (u_i - w_i)^2 + (\ell^2 - 1) \sum_{i=1}^{3} (u_i - w_i)^2 \\
&= \sum_{i=1}^{3} (u_i - w_i)^2 + \frac{\ell-1}{\ell+1} (\ell + 1)^2 \sum_{i=1}^{3} (u_i - w_i)^2 \\
&\leq \sum_{i=1}^{3} (u_i - w_i)^2 + \kappa (\ell + 1)^2 \sum_{i=1}^{3} (u_i - w_i)^2 \\
&\leq d^2(u, w) + \kappa (\ell + 1)^2 \sum_{i=1}^{3} (u_i - w_i)^2 \\
&\leq d^2(u, w) + \kappa \Big[d^2(u, w) + d^2(Tu, Tw) + 2\mathcal{Q}(w, u, Tu, Tw)\Big].
\end{aligned}$$

Following the results in [7], it is given in [21] that a mapping $T : \mathcal{D} \to \mathcal{H}$ is called α-enriched nonexpansive if there exists $\alpha \in [0, +\infty)$ such that

$$d^2(Tu, Tw) + \alpha^2 d^2(u, w) + 2\alpha \mathcal{Q}(u, w, Tu, Tw) \le (\alpha + 1)^2 d^2(u, w), \tag{2.8}$$

for all $u, w \in \mathcal{H}$. In the sequel, we discuss significant properties and inequalities associated with the class of strictly pseudocontractive mappings in connection to enriched nonexpansive mappings.

Theorem 2.1. *Let $(\mathcal{H}, d)$ be a metric space and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{H}$ is a mapping. Then T is κ-strictly pseudocontractive mapping if and only if T is α-enriched nonexpansive mapping, where $\alpha = \dfrac{\kappa}{1-\kappa}$.*

Proof. Let $u, w \in \mathcal{D}$. By (2.7), T is κ-strictly pseudocontractive mapping if and only if

$$(1-\kappa) d^2(Tu, Tw) \le (1+\kappa) d^2(u, w) + 2\kappa \mathcal{Q}(w, u, Tu, Tw),$$

which is equivalent to

$$d^2(Tu, Tw) \le \frac{1+\kappa}{1-\kappa} d^2(u, w) + 2\frac{\kappa}{1-\kappa} \mathcal{Q}(w, u, Tu, Tw).$$

This can be rewritten as

$$d^2(Tu, Tw) + 2\frac{\kappa}{1-\kappa} \mathcal{Q}(u, w, Tu, Tw) \le \frac{1+\kappa}{1-\kappa} d^2(u, w),$$

which is equivalent to

$$d^2(Tu, Tw) + \frac{\kappa^2}{(1-\kappa)^2} d^2(u, w) + 2\frac{\kappa}{1-\kappa} \mathcal{Q}(u, w, Tu, Tw) \le \left(\frac{1+\kappa}{1-\kappa} + \frac{\kappa^2}{(1-\kappa)^2} \right) d^2(u, w).$$

This means

$$d^2(Tu, Tw) + \frac{\kappa^2}{(1-\kappa)^2} d^2(u, w) + 2\frac{\kappa}{1-\kappa} \mathcal{Q}(u, w, Tu, Tw) \le \left(1 + \frac{\kappa}{1-\kappa} \right)^2 d^2(u, w),$$

which is equivalent to α-enriched nonexpansive mapping with $\alpha = \dfrac{\kappa}{1-\kappa}$. □

Remark 2.1. Theorem 2.1 signifies that the class of strictly pseudocontractive mappings coincides with the class of enriched nonexpansive mappings in the following sense:

(i) every α-enriched nonexpansive mapping is $\frac{\alpha}{1+\alpha}$-strictly pseudocontractive.
(ii) every κ-strictly pseudocontractive mapping is $\frac{\kappa}{1-\kappa}$-enriched nonexpansive.

Corollary 2.1. *Let $\mathcal{H}$ be a real Hilbert space endowed with the usual metric d and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{H}$ is a mapping. Then T is κ-strictly pseudocontractive mapping if and only if T is α-enriched nonexpansive mapping, where $\alpha = \frac{\kappa}{1-\kappa}$.*

In the sequel, we shall say that a metric space $(\mathcal{H}, d)$ satisfies the Cauchy-Schwarz inequality whenever

$$|\mathcal{Q}(u, w, x, y)| \le d(u, w) d(x, y), \quad \forall\, u, w, x, y \in \mathcal{H}. \tag{2.9}$$

In the next corollary, we shall assume this condition.

Corollary 2.2. *Let $(\mathcal{H}, d)$ be a metric space that satisfies Cauchy-Schwarz inequality and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-strictly pseudocontractive mapping, then T is ℓ-Lipschitz with $\ell = \dfrac{1+\kappa}{1-\kappa}$.*

Proof. Let $u, w \in \mathcal{D}$. Since T is κ-strictly pseudocontractive, we have the last inequality of the proof of Theorem 2.1, that is,

$$\begin{aligned} d^2(Tu, Tw) + \frac{\kappa^2}{(1-\kappa)^2} d^2(u, w) + 2\frac{\kappa}{1-\kappa} \mathcal{Q}(u, w, Tu, Tw) \\ \le \left(1 + \frac{\kappa}{1-\kappa}\right)^2 d^2(u, w). \end{aligned} \tag{2.10}$$

This and (2.9) yield that

$$\begin{aligned} \left[d(Tu, Tw) - \frac{\kappa}{1-\kappa} d(u, w)\right]^2 &= d^2(Tu, Tw) + \frac{\kappa^2}{(1-\kappa)^2} d^2(u, w) \\ &\quad - 2\frac{\kappa}{1-\kappa} d(u, w) d(Tu, Tw) \\ &\le d^2(Tu, Tw) + \frac{\kappa^2}{(1-\kappa)^2} d^2(u, w) \\ &\quad + 2\frac{\kappa}{1-\kappa} \mathcal{Q}(u, w, Tu, Tw) \\ &\le \left(1 + \frac{\kappa}{1-\kappa}\right)^2 d^2(u, w). \end{aligned}$$

Consequently, we get

$$\begin{aligned} d(Tu, Tw) &= d(Tu, Tw) - \frac{\kappa}{1-\kappa} d(u, w) + \frac{\kappa}{1-\kappa} d(u, w) \\ &\le \left| d(Tu, Tw) - \frac{\kappa}{1-\kappa} d(u, w) \right| + \frac{\kappa}{1-\kappa} d(u, w) \\ &\le \left(1 + \frac{\kappa}{1-\kappa}\right) d(u, w) + \frac{\kappa}{1-\kappa} d(u, w) \\ &= \frac{1+\kappa}{1-\kappa} d(u, w), \end{aligned}$$

as desired. □

Remark 2.2. It follows from (2.7) that for any $p \in \mathrm{Fix}(T)$, the following inequality holds:

$$d^2(Tu, p) \le d^2(u, p) + \kappa d^2(u, Tu), \tag{2.11}$$

for all $u \in \mathcal{D}$. This guarantees that every strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

3. The case of CAT(0) Spaces

A special metric space $(\mathcal{H}, d)$ has a convex structure in the sense that for every two points $u, w \in \mathcal{H}$, there exists a mapping $\phi_u^w : [0, 1] \subset \mathbb{R} \to \mathcal{H}$ satisfying the following:

- $\phi_u^w(0) = u$,
- $\phi_u^w(1) = w$,
- $d\left(\phi_u^w(t_1), \phi_u^w(t_2)\right) = |t_1 - t_2| d(u, w)$ for every $t_1, t_2 \in [0, 1]$.

This kind of metric space is referred to as a *geodesic space* and the image of ϕ_u^w is often called a *geodesic segment* connecting u and w. If every two points are connected by a unique segment, then the setting is called a *unique geodesic space*. Given $u, w \in \mathcal{H}$ and $t \in [0,1]$, this setting guarantees the existence of a unique y on the segment connecting u and w, denoted by $(1-t)u \oplus tw$, with the following conditions:

$$d(u,y) = td(u,w) \quad \text{and} \quad d(y,w) = (1-t)d(u,w). \tag{3.12}$$

For this metric, a set is convex if it contains the geodesic segment connecting any pair of its points. It is known that CAT(0) spaces are unique geodesic spaces. Moreover, we have the following inequalities (see, for example, [13]) for $u, v, w \in \mathcal{H}$ and $t \in [0,1]$:

$$d((1-t)u \oplus tv, w) \leq (1-t)d(u,w) + td(v,w); \tag{3.13}$$

$$d^2((1-t)u \oplus tv, w) \leq (1-t)d^2(u,w) + td^2(v,w) - t(1-t)d^2(u,v), \tag{3.14}$$

When $t = \frac{1}{2}$, inequality (3.14) reduces to the CN-inequality of Bruhat and Tits [10]. A complete CAT(0) space is called a Hadamard space. For further details on CAT(0) spaces, see, for example, [9] and [16].

Following the results in [14, 3] which are based on CAT(0) spaces, T is κ-strictly pseudocontractive if there exists $\kappa < 1$, such that

$$d^2(Tu, Tw) \leq d^2(u,w) + 4\kappa d^2\left(\frac{1}{2}u \oplus \frac{1}{2}Tw,\ \frac{1}{2}w \oplus \frac{1}{2}Tu\right), \forall\, u, w \ \in \mathcal{D}. \tag{3.15}$$

We now, discuss the relationship between (2.7) and (3.15).

Remark 3.3. It can be easily shown that in a CAT(0) space, if T satisfies (3.15), then it also satisfies (2.7). This can be justified by utilizing (3.14) twice. However, for the converse, an additional condition is required.

Recall that a CAT(0) space is said to be *flat* if the CN-inequality holds with equality. Here is an example of a non-linear flat CAT(0) space.

Example 3.2. Let $\mathcal{H} = \mathbb{R}^m$ $(m \geq 2)$ be endowed with the metric d defined by

$$d(x,y) = \sqrt{(x_1 + y_2^2 - y_1 - x_2^2)^2 + \sum_{i=2}^{m}(x_i - y_i)^2},$$

for all $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. It is clear that $(\mathbb{R}^m, d)$ is a metric space. Also, the mapping $\phi_x^y : [0,1] \to \mathbb{R}^2$ defined by

$$\phi_x^y(t) = \big(x_1 + t\,(y_1 - x_1) - t(1-t)(y_2 - x_2)^2, (1-t)x_2 + ty_2,\ \ldots,\\ (1-t)x_m + ty_m\big),$$

is the geodesic connecting x and y. It follows that,

$$\frac{1}{2}x \oplus \frac{1}{2}y = \left(\left(\frac{x_2 + y_2}{2}\right)^2 - \frac{x_2^2 - x_1}{2} - \frac{y_2^2 - y_1}{2},\ \frac{x_2 + y_2}{2},\ \ldots,\ \frac{x_m + y_m}{2}\right).$$

Consequently, we obtain

$$
\begin{aligned}
& d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y,\ u\right) \\
&= \left[\left(\frac{x_2+y_2}{2}\right)^2 - \left(\left(\frac{x_2+y_2}{2}\right)^2 - \frac{x_2^2-x_1}{2} + -\frac{y_2^2-y_1}{2}\right) - u_2^2 + u_1\right]^2 \\
&\quad \sum_{i=2}^{m}\left[\frac{x_i+y_i}{2} - u_i\right]^2 \\
&= \sum_{i=2}^{m}\left[\frac{x_i-u_i}{2} + \frac{y_i-u_i}{2}\right]^2 + \left[\frac{x_2^2-x_1-u_2^2+u_1}{2} + \frac{y_2^2-y_1-u_2^2+u_1}{2}\right]^2 \\
&= \sum_{i=2}^{m}\frac{(x_i-u_i)^2}{2} - \frac{(u_i-y_i)^2}{2} - \frac{(x_i-y_i)^2}{4} + \frac{(x_2^2-x_1-u_2^2+u_1)^2}{2} \\
&\quad + \frac{(u_2^2-u_1-y_2^2+y_1)^2}{2} - \frac{(x_2^2-x_1-y_2^2+y_1)^2}{4} \\
&= \frac{1}{2}\left[\sum_{i=2}^{m}(x_i-u_i)^2 + \left(x_2^2-x_1-u_2^2+u_1\right)^2\right] + \frac{1}{2}\left[\sum_{i=2}^{m}(u_i-y_i)^2\right. \\
&\quad \left. + \left(u_2^2-u_1-y_2^2+y_1\right)^2\right] - \frac{1}{4}\left[\sum_{i=2}^{m}(x_i-y_i)^2 + \left(x_2^2-x_1-y_2^2+y_1\right)^2\right] \\
&= \frac{1}{2}d^2(x,\ u) + \frac{1}{2}d^2(y,\ u) - \frac{1}{4}d^2(x,\ y).
\end{aligned}
$$

Thus $(\mathcal{H}, d)$ is a non-linear flat CAT(0) space.

Proposition 3.1. *Let $(\mathcal{H}, d)$ be a flat CAT(0) space and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{H}$ is a mapping. Then the inequality* (3.15) *coincides with the inequality* (2.7).

Proof. Let $u, w \in \mathcal{D}$. Since CN-inequality is equality in this case, then it follows that

$$
\begin{aligned}
d^2\left(\frac{1}{2}u \oplus \frac{1}{2}Tw,\ \frac{1}{2}w \oplus \frac{1}{2}Tu\right) &= \frac{1}{2}d^2\left(u,\ \frac{1}{2}w \oplus \frac{1}{2}Tu\right) \\
&\quad + \frac{1}{2}d^2\left(Tw,\ \frac{1}{2}w \oplus \frac{1}{2}Tu\right) \\
&\quad - \frac{1}{4}d^2\left(w,\ Tu\right) \\
&= \frac{1}{4}d^2\left(u,\ w\right) + \frac{1}{4}d^2\left(u,\ Tu\right) \\
&\quad - \frac{1}{4}d^2\left(w,\ Tu\right) + \frac{1}{4}d^2\left(Tw,\ w\right) \\
&\quad + \frac{1}{4}d^2\left(Tw,\ Tu\right) - \frac{1}{4}d^2\left(w,\ Tu\right) \\
&\quad - \frac{1}{4}d^2\left(w,\ Tu\right) \\
&= \frac{1}{4}\Big[d^2\left(u,\ w\right) + d^2\left(Tu,\ Tw\right) \\
&\quad + 2\mathcal{Q}(w, u, Tu, Tw)\Big].
\end{aligned}
$$

□

Thus we have the following consequent result.

Corollary 3.3. *Let $\mathcal{H}$ be a normed linear space endowed with the usual distance d, and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{H}$ is a mapping. The inequality* (3.15) *coincides with the inequality* (2.7)*, which can be further simplified to:*

$$\|Tu - Tw\|^2 \le \|u - w\|^2 + \kappa\|(I - T)u - (I - T)w\|^2, \ \forall \, u, w \ \in \mathcal{D}.$$

The results in Corollary 3.3 follows directly from Proposition 3.1 using the following facts:

- $d^2(x, y) = \|x - y\|^2$ and $\mathcal{Q}(x, y, v, w) = \langle x - y, \ v - w\rangle$ for all $x, y, v, w \in \mathcal{H}$.
- $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, \ y\rangle$ for all $x, y \in \mathcal{H}$.
- $\left\|\frac{1}{2}x + \frac{1}{2}y\right\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}\|x - y\|^2$ for all $x, y \in \mathcal{H}$.

It follows from [6, Corollary 3] that every CAT(0) space satisfies the Cauchy-Schwarz inequality. Consequently, we get the following result from Corollary 2.2.

Corollary 3.4. *Let $(\mathcal{H}, d)$ be a CAT(0) space and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-strictly pseudocontractive mapping, then T is ℓ-Lipschitz with $\ell = \dfrac{1+\kappa}{1-\kappa}$.*

Proposition 3.2. *Let $(\mathcal{H}, d)$ be a CAT(0) space and let $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-strictly pseudocontractive mapping and* $\mathrm{Fix}(T) \neq \emptyset$*, then* $\mathrm{Fix}(T)$ *is closed and convex.*

Proof. The proof of Proposition 3.2 follows similar lines with the proof [14, Theorem 2.3] using (2.11) and Corollary 3.4. Therefore, we skip it. □

Corollary 3.5. *Let $(\mathcal{H}, d)$ be a complete CAT(0) space and let $\mathcal{D}$ be a nonempty bounded closed convex subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{D}$ is κ-strictly pseudocontractive mapping, then* $\mathrm{Fix}(T)$ *is nonempty closed and convex.*

Proof. The proof of Corollary 3.5 follows from the fact in Remark 2.1(ii) and [21, Theorem 4.8(i)]. □

Since every real Hilbert space is a CAT(0) space, the following known result is a consequence of Corollary 3.4.

Corollary 3.6. *Let $\mathcal{H}$ be a real Hilbert space endowed with the usual distance d and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-strictly pseudocontractive mapping, then T is ℓ-Lipschitz with $\ell = \dfrac{1+\kappa}{1-\kappa}$.*

Let $(\mathcal{H}, d)$ be a metric space and $\{u_n\}$ be a bounded sequence in $\mathcal{H}$. The *asymptotic center* of $\{u_n\}$ is defined by

$$A(\{u_n\}) := \big\{u \in \mathcal{H} : \limsup_{n\to\infty} d(u, u_n) = \inf_{v\in\mathcal{H}} \limsup_{n\to\infty} d(v, u_n)\big\}.$$

The sequence $\{u_n\}$ Δ-*converges* to a point w in $\mathcal{H}$ if $\{w\}$ is the asymptotic center of every subsequence of $\{u_n\}$ and it *converges strongly* to w if $\lim_{n\to\infty} d(u_n, w) = 0$. We write $u_n \xrightarrow{\Delta} u$ to mean $\{u_n\}$ is Δ-convergent to u and $u_n \to u$ means $\{u_n\}$ converges strongly to u. Moreover, a map $T : \mathcal{D} \to \mathcal{H}$ is said to have a *demiclosedness-type property* if for any sequence $\{u_n\} \subseteq \mathcal{H}$,

$$\left.\begin{array}{r} u_n \xrightarrow{\Delta} u \\ d(u_n, Tu_n) \to 0 \end{array}\right\} \implies \quad u = Tu. \tag{3.16}$$

Corollary 3.7. *Let $\mathcal{D}$ be a nonempty closed convex subset of a complete CAT(0) space $(\mathcal{H}, d)$ and let $T : \mathcal{D} \to \mathcal{D}$ be κ-strictly pseudocontractive mapping. Then T satisfies demiclosedness-type property.*

The next result is an immediate consequence of Corollary 3.7.

Corollary 3.8. *Let $\mathcal{H}$ be a real Hilbert space endowed with the usual distance d and let $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-strictly pseudocontractive mapping. Then T satisfies demiclosedness-type property.*

Now, we recall a substantial result of [20] which we will use accordingly.

Lemma 3.1. [20, Theorem 3.3] *Let $\mathcal{D}$ be a nonempty closed convex subset of a complete CAT(0) space $(\mathcal{H}, d)$. Let $T : \mathcal{D} \to \mathcal{H}$ be a mapping with nonempty fixed point set and which satisfies the demiclosedness-type property* (3.16). *Suppose $\{u_n\}$ is a sequence in $\mathcal{D}$ such that*

(a) $d(u_n, Tu_n) \to 0$,

(b) $\{d(u_n, u^*)\}$ *converges in $\mathbb{R}$ for every $u^* \in \mathrm{Fix}(T)$,*

then $\{u_n\}$ Δ-converges to a fixed point of T.

The Krasnoselskii-Mann algorithm is one of the prominent algorithm for approximating fixed point of nonexpansive type mapping. In the setting of CAT(0) spaces, the algorithm is updated as follows:

$$u_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n T u_n,\ n \geq 1, \tag{3.17}$$

where, $\{\alpha_n\} \subseteq [0, 1]$. To obtain the convergence result, it is required that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. In the next theorem, we show that slight modification of the conditions on $\{\alpha_n\}$ will guarantee the convergence for the studied class of mappings.

Theorem 3.2. *Let $(\mathcal{H}, d)$ be a complete CAT(0) space and $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $\{u_n\}$ is a sequence generated by* (3.17) *with $0 < a \leq \alpha_n \leq b < 1-\kappa$ for all $n \geq N \in \mathbb{N}$. Then $\{u_n\}$ Δ-converges to a fixed point of T.*

Proof. Let $u^* \in \mathrm{Fix}(T)$. Using (3.14), (3.17) and (2.11), we have

$$\begin{aligned}
d^2\left(u_{n+1},\ u^*\right) &= d^2\left((1-\alpha_n)u_n \oplus \alpha_n T u_n,\ u^*\right) \\
&\leq (1-\alpha_n)d^2(u_n, u^*) + \alpha_n d^2(Tu_n, u^*) \\
&\qquad - \alpha_n(1-\alpha_n)d^2(u_n, Tu_n) \\
&\leq (1-\alpha_n)d^2(u_n, u^*) + \alpha_n\left[d^2(u_n, u^*) + \kappa d^2(u_n, Tu_n)\right] \\
&\qquad - \alpha_n(1-\alpha_n)d^2(u_n, Tu_n) \\
&= d^2(u_n, u^*) - \alpha_n(1-\alpha_n-\kappa)d^2(u_n, Tu_n).
\end{aligned}$$

This implies that for all $n \geq N$,

$$d^2\left(u_{n+1},\ u^*\right) \leq d^2(u_n, u^*) \tag{3.18}$$

and also

$$\begin{aligned}
d^2(u_n, Tu_n) &\leq \frac{1}{\alpha_n(1-\alpha_n-\kappa)}\left[d^2(u_n, u^*) - d^2\left(u_{n+1},\ u^*\right)\right] \\
&\leq \frac{1}{a(1-b-\kappa)}\left[d^2(u_n, u^*) - d^2\left(u_{n+1},\ u^*\right)\right].
\end{aligned} \tag{3.19}$$

Lemma 3.1(b) follows from (3.18), and Lemma 3.1(a) is achieved as consequence of (3.19) together with Lemma 3.1(b). Moreover, T satisfies demiclosedness-type property. Hence by Lemma 3.1, we obtain that $\{u_n\}$ Δ-converges to a fixed point of T. □

In a linear setting, Δ-convergence results yield weak convergence results. Consequently, we have the following corollary, which incorporates the result of Marino and Xu [17, Theorem 3.1].

Corollary 3.9. *Let $\mathcal{H}$ be a Hilbert space with the usual distance d and let $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $\{u_n\}$ is a sequence generated by*

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, \ n \geq 1,$$

with $0 < a \leq \alpha_n \leq b < 1 - \kappa$ for all $n \geq N \in \mathbb{N}$. Then $\{u_n\}$ converges weakly to a fixed point of T.

Several other results can be deduced from Theorem 3.2. For instance, dispensing with the condition $\mathrm{Fix}(T) \neq \emptyset$ yields the following corollary.

Corollary 3.10. *Let $(\mathcal{H}, d)$ be a complete CAT(0) space and $\mathcal{D}$ be a nonempty bounded closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-strictly pseudocontractive mapping and $\{u_n\}$ is a sequence generated by* (3.17) *with $0 < a \leq \alpha_n \leq b < 1 - \kappa$ for all $n \geq N \in \mathbb{N}$. Then $\{u_n\}$ Δ-converges to a fixed point of T.*

Corollary 3.10 follows from Theorem 3.2 using the fact stated in Theorem 2.1 and the result of [21, Theorem 4.8(i)]. Moreover, we have the following result in the case of Hilbert spaces.

Corollary 3.11. *Let $\mathcal{H}$ be a Hilbert space with the usual distance d and let $\mathcal{D}$ be a nonempty bounded closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-strictly pseudocontractive mapping and $\{u_n\}$ is a sequence generated by*

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, \ n \geq 1,$$

with $0 < a \leq \alpha_n \leq b < 1 - \kappa$ for all $n \geq N \in \mathbb{N}$. Then $\{u_n\}$ converges weakly to a fixed point of T.

For the case when $\{\alpha_n\}$ is a constant sequence, Corollary 3.11 yields the result in Browder and Petryshyn [8, Theorem 12].

Remark 3.4. In contrast to (2.2), (1.1) coincides with (2.7) and (3.15) in general flat CAT(0) spaces, including Hilbert spaces. Furthermore, (3.15) implies (2.7) in general CAT(0) spaces and (2.7) is easier to verify since it does not require computations of geodesic segments. Additionally, (2.7) applies to all metric spaces, unlike (3.15) and (1.1). Therefore, Definition 2.1 offers the most natural and appropriate definition of strict pseudocontraction in a general metric space, building on the work of Browder and Petryshyn in 1967 [8].

4. Conclusions

In this work, we have provided an appropriate definition of a strictly pseudocontractive mapping in general metric spaces. Based on this definition, we have established that the class of strictly pseudocontractions coincides with the class of enriched nonexpansive mappings. Furthermore, we have proved that this class of mappings is a subclass of Lipschitz mappings, and it incorporates existing notions of strictly pseudocontractive mappings in geodesic spaces. Moreover, we have analysed certain properties of strictly pseudocontractions in CAT(0) spaces. Specifically, we have shown that strictly pseudocontractions satisfy a demiclosedness-type property in CAT(0) spaces. Additionally, we have proven that the fixed point set of strictly pseudocontractions is closed and convex, provided it is nonempty. Finally, we have shown the applicability of the Krasnoselskii-Mann iteration in approximating fixed points of strictly pseudocontractive mappings.

Acknowledgments. The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research project was supported by Thailand Science Research and Innovation (TSRI), and National Science, Research and Innovation Fund (NSRF) Fiscal year 2024 Grant number FRB670073/0164. The first author was supported by the "Petchra Pra Jom Klao Ph.D. Research Scholarship" from "King Mongkut's University of Technology Thonburi" with contract number 67/2563. The work was initiated while the first author visited the North University Center at Baia Mare, Technical University of Cluj-Napoca, Romania, during April- June 2023.

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