CARPATHIAN J. MATH.
Volume **40** (2024), No. 2,
Pages 493 - 513

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.02.17

Characterizations of ε-Approximate Solutions for Robust Convex Semidefinite Programming Problems

RABIAN WANGKEEREE[1] and PAKKAPON PREECHASILP[2]

ABSTRACT. In this article, we explore the characterizations of ε-approximate solutions for convex semidefinite programming problems that involve uncertain data. It first reviews essential findings regarding the optimality condition and duality of robust convex semidefinite programming problems. Then, we establish the optimality and duality conditions concerning the problem by assuming specific constraint qualifications. The study investigates ε-Kuhn-Tucker vectors and their relationships with the optimal solutions, maximizers of the corresponding Lagrangian dual problem, saddle points of the Lagrangian, and Kuhn-Tucker vectors. Finally, the article establishes the characterization of ε-approximate solution sets for the problem by studying the connection among three sets: the set of Lagrange multipliers corresponding to ε-approximate solutions, the set of ε-Kuhn-Tucker vectors, and the set of approximate solutions for their Lagrangian dual problems. The characterization is illustrated with several examples.

1. INTRODUCTION

Robust convex semidefinite programming is a type of mathematical optimization that aims to solve semidefinite programming problems with uncertain data. The objective is to minimize a convex function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite, where these symmetric matrices are only known to belong to some uncertainty sets. The applications of semidefinite programming problems under data uncertainty are extensive, including fields like engineering, computer science, finance, control theory, signal processing, quantum information theory, and machine learning [3, 4, 19, 20]. Many studies have focused on robust semidefinite programming from both theoretical and practical perspectives, including works such as [1, 2, 28, 29] and their respective references.

Approximate solutions are widely studied in robust optimization because finding the exact solution to a problem may be challenging or impractical. Even though an approximate solution is not necessarily optimal, it can provide useful insights and inform decision-making in many situations. Approximate solutions can also strike a balance between computational cost and solution quality. For example, a good enough answer can be obtained in a shorter amount of time using an approximate solution, while a more precise solution may require more computing time and resources. Many papers have contributed to the development of approximate solutions for robust optimization, including [11, 13, 14, 15, 17, 22, 26, 27].

In the past decade, the characterization of robust optimal solutions has been one of the most interesting and intensively studied areas, as evidenced by [8, 10, 18, 23, 25, 27]. The reason for this interest is that the characterization of the solution set can aid in identifying the problem's structure, thereby leading to more efficient algorithm design. This

Received: 06.09.2023. In revised form: 18.02.2024. Accepted: 25.02.2024
2010 *Mathematics Subject Classification.* 90C22, 90C31, 90C46.
Key words and phrases. *robust convex semidefinite programming problem, ε-approximate solution set, ε-Kuhn Tucker vector.*
Corresponding author: Pakkapon Preechasilp; preechasilpp@gmail.com

can also help identify the limitations of existing methods and develop new approaches capable of addressing more complex and realistic problems. In recent years, there has been a significant amount of research focused on characterizing robust optimal solutions. The initial research on characterizing the solution sets of convex optimization problems with uncertain data was conducted by Jeyakumar et al. [10], who assumed the presence of the robust Slater constraint qualification. Li and Wang [18] provided a characterization of the features of the robust solution sets of convex optimization problems with data uncertainty by using the less restrictive robust Farkas-Minkowski constraint qualification. Sun et al. [24] introduced certain descriptions of robust optimal solutions for a convex optimization problem that experiences data uncertainty in both the objective function and constraints. Sun and colleagues [26] proposed novel characterizations of robust ε-quasi Pareto efficient solutions for semi-infinite multiobjective programming problems that are nonconvex and subject to data uncertainty in both the objective functions and constraints. For more related findings, refer to [12, 23, 25].

Our observation indicates that there are no existing findings that discuss the characterization of approximate solutions for convex semidefinite programming problems in the presence of data uncertainty. However, some initial outcomes have been reported that pertain to the identification of approximate solutions for general problems, including convex or semi-infinite problems, under uncertain data conditions. The importance of studying convex semidefinite programming problems with uncertain data lies in their optimization over positive semidefinite matrices, which leads to the introduction of unique structures and constraints that are absent in other convex optimization problems. Due to the focus on positive semidefinite matrices, there exist specialized techniques and algorithms that are specific to semidefinite programs with uncertain data. In what follows, we consider the following robust convex semidefinite programming problem.

$$\text{(RSDP)} \qquad \begin{aligned} &\text{Minimize } \varphi(x), \\ &\text{subject to } Q(x) := Q_0 + \sum_{i=1}^{m} x_i Q_i \succeq 0, \\ &\qquad\qquad \forall Q_i \in \mathcal{U}_i, i \in I := \{0, 1, 2, \dots, m\}, \\ &\qquad\qquad x := (x_1, x_2, \dots, x_m) \in K, \end{aligned}$$

where $\varphi : \mathbb{R}^m \to \mathbb{R}$ is a convex function, K is a nonempty closed convex subset of $\mathbb{R}^m$ and for each $i \in I$, Q_i is uncertain and belongs to the uncertainty set $\mathcal{U}_i$ which is closed and convex subset of $n \times n$ symmetric matrices.

We will use the notation $\mathcal{F}(K)$ to refer to the feasible set of problem RSDP, i.e.,

$$\mathcal{F}(K) = \mathcal{F} \cap K := \left\{ x \in \mathbb{R}^m : Q_0 + \sum_{i=1}^{m} x_i Q_i \succeq 0, \forall\, Q_i \in \mathcal{U}_i, i \in I \right\} \cap K,$$

and by $\inf$ (RSDP) the optimal valued of (RSDP). For any given $\varepsilon \geq 0$, $\tilde{x} \in \mathcal{F}(K)$ is said to be an ε-approximate solution of (RSDP), if $\varphi(\tilde{x}) \leq \varphi(y) + \varepsilon,\ \forall y \in \mathcal{F}(K)$. We will denote Sol_ε(RSDP) as the ε-approximate solution for RSDP.

The aim of this paper is to present characterizations of ε-approximate solutions for RSDP. To begin with, we will review some fundamental findings related to the optimality condition and duality of robust convex semidefinite programming problems. Specifically, we establish the optimality and duality conditions within the context of problem RSDP by assuming certain constraint qualifications. Taking inspiration from [21], the idea of ε-Kuhn-Tucker vectors is investigated for robust versions of convex semidefinite programming problems, and the relationships between the optimal solutions, maximizers of the corresponding Lagrangian dual problem, saddle points of the Lagrangian, and Kuhn-Tucker vectors are investigated. Our findings reveal that if strong duality holds, the collection of ε-Kuhn-Tucker vectors for RSDPs is equivalent to the assortment of ε-approximate solutions for their respective Lagrangian dual problems. Furthermore, we observe that there is a connection between the set of ε-solutions for RSDPs and the Lagrange multipliers that correspond to these ε-solutions, provided that the reviewed approximate optimality condition is satisfied. Using this observation, we use the optimality conditions for ε-approximate solutions of RSDPs to construct the set of all ε-solutions that correspond to Lagrange multipliers satisfying the particular optimality condition. To sum up, we establish the characterization of ε-approximate solution sets for RSDPs by above studying the relationships among three sets: the set of Lagrange multipliers corresponding to ε-approximate solutions for RSDPs, the set of ε-Kuhn-Tucker vectors for RSDPs, and the set of approximate solutions for their Lagrangian dual problems.

This paper is organized as follows: Section 2 provides basic notations and essential tools that will be used in this article. Additionally, we review some ε-approximate optimality and duality conditions. Section 3 characterizes the ε-approximate solution set of a convex semidefinite programming problem in terms of the ε-subdifferential of the convex objective and the structure of semidefinite matrices. We provide an example to illustrate these results. In Section 4, we discuss the robust semidefinite programming problem. As a practical application, we examine the characterization of ε-approximate solution sets for semidefinite linear programming problems with uncertain data. We provide several examples to illustrate our characterization results.

2. Preliminaries

In this article, we have gathered several notations and tools that will be used later on. The Euclidean space of n dimensions is denoted by $\mathbb{R}^n$, while the non-negative orthant is denoted by $\mathbb{R}^n_+$. The inner product of two vectors x and y belonging to $\mathbb{R}^n$ is represented by $\langle x, y\rangle$, which is equal to the summation of the products of their corresponding elements, i.e., $\sum_{i=1}^n x_i y_i$.

Let $\mathbb{S}^n$ be the set of all $n \times n$ symmetric matrices. For any $A, B \in \mathbb{S}^n$, their inner product is represented by $\langle A, B\rangle$, which is defined as the trace of their matrix product, i.e., $\mathrm{tr}[AB]$. Here, $\mathrm{tr}[M]$ is the sum of diagonal entries of the matrix $M = [m_{ij}]_{n\times n}$. A matrix A belonging to $\mathbb{S}^n$ is said to be positive semidefinite (denoted by $A \succeq 0$) if for any $x \in \mathbb{R}^n$, $x^t Ax \geq 0$. A matrix A is said to be positive definite (denoted by $A \succ 0$) if for any $0 \neq x \in \mathbb{R}^n$, $x^t Ax > 0$. The set of all $n \times n$ positive semidefinite (positive definite) symmetric matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$), respectively. The linear operator $\widehat{Q}(x) : \mathbb{R}^m \to \mathbb{S}^n$ is defined by

$$\widehat{Q}(x) = \sum_{i=1}^{m} x_i Q_i.$$

A subset $K \subseteq \mathbb{R}^n$ is convex if for every $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in K$ for all $x, y \in K$. Let $\mathrm{cl}(K)$ and $\mathrm{conv(K)}$ denote the closure of subset K and convex hull of subset K of $\mathbb{R}^n$. A function $\phi : K \to \mathbb{R}$ is said to be convex if for each $\alpha \in [0, 1]$, $\phi(\alpha x + (1 - \alpha)y) \leq \alpha\phi(x) + (1 - \alpha)\phi(y)$ for all $x, y \in K$. For a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, the subdifferential of a function ϕ at $u \in \mathrm{dom}(\phi)$ is defined by

$$\partial\phi(u) := \{v \in \mathbb{R}^n : \phi(y) \geq \phi(u) + \langle v, y - u\rangle,\ \forall y \in \mathbb{R}^n\}.$$

For $\varepsilon \geq 0$, the ε-subdifferential of a function ϕ at $u \in \mathrm{dom}(\phi)$ is defined by

$$\partial_\varepsilon\phi(u) := \{v \in \mathbb{R}^n : \phi(y) \geq \phi(u) + \langle v, y - u\rangle - \varepsilon,\ \forall y \in \mathbb{R}^n\}.$$

It is conventionally agreed that $\partial_\varepsilon \phi(a) = \emptyset$ if $a \notin \mathrm{dom}(\phi)$. For a function $\phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$, we define its conjugate function $\phi^* : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ as $\phi^*(v) = \sup_{y \in \mathbb{R}^n} \{\langle v, y\rangle - \phi(y)\}$. Furthermore, we define the epigraph of ϕ as

$$\mathrm{epi}\,\phi := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) \le \alpha\}.$$

It should be noted that the epigraph of a positively homogeneous function is a cone, and the epigraph of a convex and lower semi-continuous function is a closed and convex set.

The indicator function $\mathcal{I}_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined for a nonempty subset K of $\mathbb{R}^n$ as follows:

$$\mathcal{I}_K(z) = \begin{cases} 0 & \text{if } z \in K, \\ +\infty & \text{if } z \notin K. \end{cases}$$

Similarly, for the subset K, the support function $\mathcal{S}_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $\mathcal{S}_K(w) = \sup\{\langle w, z\rangle : z \in K\}$. It is evident that the support function is the conjugate of the indicator function, i.e., $\mathcal{I}_K^* = \mathcal{S}_K$. Moreover, the epigraph of the support function, $\mathrm{epi}\,\mathcal{S}_K$, is a closed convex cone.

Let $K \subseteq \mathbb{R}^n$ be a closed convex set. The normal cone to K at u, $N(K, u)$, is defined by $N(K, u) = \{v \in \mathbb{R}^n : \langle v, y - u\rangle \le 0 \text{ for all } y \in K\}$. For $\varepsilon \ge 0$, the ε-normal set to K at u is defined as $N_\varepsilon(K, u) = \{v \in \mathbb{R}^n : \langle v, y - u\rangle \le \varepsilon \text{ for all } y \in K\}$.

The proposition below explains how the epigraph of a conjugate of a sum of functions is related to the sum of the epigraphs of conjugate functions.

Proposition 2.1. [7] *Let $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. If $\mathrm{dom}\,\phi_1 \cap \mathrm{dom}\,\phi_2 \neq \emptyset$, then $\mathrm{epi}(\phi_1 + \phi_2)^* = \mathrm{cl}(\mathrm{epi}\,\phi_1^* + \mathrm{epi}\,\phi_2^*)$. Moreover, if one of the functions ϕ_1 and ϕ_2 is continuous, then $\mathrm{epi}(\phi_1 + \phi_2)^* = \mathrm{epi}\,\phi_1^* + \mathrm{epi}\,\phi_2^*$.*

We are now prepared to provide the following sum rule for approximating subdifferentials of convex functions, which is used in the optimality theorem.

Proposition 2.2. [5, 6] *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $K \subseteq \mathbb{R}^n$ be a closed convex set. Then,*

$$\partial_\varepsilon(\phi + \mathcal{I}_K)(z) = \bigcup_{\substack{\varepsilon_0 \ge 0, \varepsilon_1 \ge 0 \\ \varepsilon_0 + \varepsilon_1 = \varepsilon}} \{\partial_{\varepsilon_0}\phi(z) + \partial_{\varepsilon_1}\mathcal{I}_K(z)\}.$$

2.1. **ε-approximate optimality theorem.** In this section, we recall some approximate optimality and duality conditions for RSDP. Before presenting the approximate optimality theorem for the solution of (RSDP), it is important to review the conditions for constraint qualification that are utilized to characterize ε-approximate solutions.

For the problem (RSDP), the robust characteristic cone is defined as follows:

$$D(K) := \bigcup_{\substack{Q_i \in \mathcal{U}_i, i \in I \\ (\Lambda, \delta) \in \mathbb{S}^n_+ \times \mathbb{R}_+}} \left\{ \begin{pmatrix} (\mathrm{tr}[Q_1\Lambda], \mathrm{tr}[Q_2\Lambda], \dots, \mathrm{tr}[Q_m\Lambda]) \\ -\mathrm{tr}[Q_0\Lambda] - \delta \end{pmatrix} - \mathrm{epi}\,\mathcal{S}_K \right\}.$$

Remark 2.1. In [11, 17], the authors investigated the problem (RSDP) in the case where $K = \mathbb{R}^m$. They introduced the following characteristic cone D and gave the optimality and duality theory for (quasi) approximate solution in (RSDP);

$$D := \bigcup_{\substack{Q_i \in \mathcal{U}_i, i \in I \\ (\Lambda, \delta) \in \mathbb{S}^n_+ \times \mathbb{R}_+}} \left\{ \begin{pmatrix} (\mathrm{tr}[Q_1\Lambda], \mathrm{tr}[Q_2\Lambda], \dots, \mathrm{tr}[Q_m\Lambda]) \\ -\mathrm{tr}[Q_0\Lambda] - \delta \end{pmatrix} \right\}.$$

According to Jeyakumar and Li's findings in [9], the set D forms a cone provided that $\mathcal{U}_i$ is a closed and convex set for all $i \in I$. Additionally, they establish the closed and convexity of the characteristic cone D under appropriate conditions as follows.

Proposition 2.3. [9] *For each $i \in I$, let $\mathcal{U}_i \subseteq \mathbb{S}^n$ be compact and convex. Assume that $\{x \in \mathbb{R}^m : Q_0 + \sum_{i=1}^m x_i Q_i \succ 0, \forall Q_i \in \mathcal{U}_i, i \in I\} \neq \emptyset$. Then the robust characteristic cone D is closed.*

Proposition 2.4. [9] *For any $i \in I$, let $Q_i \in \mathcal{U}_i := \left\{A_i^0 + \sum_{j=1}^l v_i^j A_i^j, (v_i^1, \ldots, v_i^l) \in \mathcal{V}_i\right\}$, where $\mathcal{V}_i$ is compact convex set in $\mathbb{R}^l$, $A_0^j \in \mathbb{S}^n$ and $A_i^j \in \mathbb{S}_+^n$, $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, l\}$. Then the robust characteristic cone D is a convex subset of $\mathbb{R}^{m+1}$.*

Moreover, if for any, $i \in I, \mathcal{U}_i$ is compact and convex, then $cl(conv(D)) = -\operatorname{epi} \mathcal{I}_{\mathcal{F}}^* = -\operatorname{epi} \mathcal{S}_{\mathcal{F}}$, (see [17]).

The following robust version of Farkas's Lemma for convex semidefinite programming is now gathered.

Lemma 2.1. *Suppose that $\mathcal{F}(K) \neq \emptyset$ and $\mathcal{U}_i$ is a compact and convex subset of $\mathbb{S}^n$ for all $i \in I$. Then, the following are equivalent:*

(i) $\mathcal{F}(K) \subset \{x \in \mathbb{R}^m : \langle c, x \rangle \geq \gamma\}$.

(ii) $\begin{pmatrix} c \\ \gamma \end{pmatrix} \in \operatorname{cl}\left(\operatorname{conv}\left(\bigcup_{\substack{Q_i \in \mathcal{U}_i, i \in I \\ (\Lambda, \alpha) \in \mathbb{S}_+^n \times \mathbb{R}_+}} \left\{\begin{pmatrix} (\operatorname{tr}[Q_1\Lambda], \operatorname{tr}[Q_2\Lambda], \ldots, -\operatorname{tr}[Q_m\Lambda]) \\ -\operatorname{tr}[Q_0\Lambda] - \delta \end{pmatrix}\right\} - \operatorname{epi} \mathcal{S}_K\right)\right)$.

Proof. ($\Longrightarrow$) We see that $\mathcal{F}(K) \subset \{x \in \mathbb{R}^m : \langle c, x \rangle - \gamma \geq 0\}$. We define real-valued function $\Gamma : \mathbb{R}^m \to \mathbb{R}$ as $\Gamma(x) = \langle c, x \rangle - \gamma$. Thanks to continuity of Γ and proposition 2.1, one has

$$(0,0) \in \operatorname{epi}(\Gamma + \mathcal{I}_{\mathcal{F}(K)})^* = \operatorname{epi} \Gamma^* + \operatorname{epi} \mathcal{I}_{\mathcal{F}(K)}^* = (c, \gamma) + \{0\} \times \mathbb{R}_+ + \operatorname{epi} \mathcal{I}_{\mathcal{F}(K)}^*.$$

This implies that

$$(c, \gamma) \in -\operatorname{epi} \mathcal{I}_{\mathcal{F}(K)}^* - \{0\} \times \mathbb{R}_+. \tag{2.1}$$

It follows from $cl(conv(D)) = -\operatorname{epi} \mathcal{I}_{\mathcal{F}}^*$ that

$$\begin{aligned} -\operatorname{epi} \mathcal{I}_{\mathcal{F}(K)}^* &= -\operatorname{epi} \mathcal{S}_{\mathcal{F} \cap K} = -\operatorname{cl}(\operatorname{epi} \mathcal{S}_{\mathcal{F}} + \operatorname{epi} \mathcal{S}_K) \\ &= \operatorname{cl}(-\operatorname{epi} \mathcal{I}_{\mathcal{F}}^* - \operatorname{epi} \mathcal{S}_K) = \operatorname{cl}\left(\operatorname{cl}(\operatorname{conv}(D)) - \operatorname{epi} \mathcal{S}_K\right) \\ &= \operatorname{cl}\left(\operatorname{conv}(D) - \operatorname{epi} \mathcal{S}_K\right) = \operatorname{cl}\left(\operatorname{conv}(D - \operatorname{epi} \mathcal{S}_K)\right) \\ &= \operatorname{cl}(\operatorname{conv}(D(K))). \end{aligned}$$

Applying (2.1), this gives that

$$(c, \gamma) \in D(K) := \operatorname{cl}\left(\operatorname{conv}(D - \operatorname{epi} \mathcal{S}_K)\right).$$

($\Longleftarrow$) Let $z \in \mathcal{F}(K)$. Since $(c, \gamma) \in \operatorname{cl}\left(\operatorname{conv}(D - \operatorname{epi} \mathcal{S}_K)\right) = -\operatorname{epi} \mathcal{S}_{\mathcal{F}(K)}$, one has

$$\begin{aligned} (-c, -\gamma) \in \operatorname{epi} \mathcal{S}_{\mathcal{F}(K)} &\Longrightarrow \sup_{y \in \mathcal{F}(K)} \langle -c, y \rangle \leq -\gamma \\ &\Longrightarrow \langle c, z \rangle \geq \gamma. \end{aligned}$$

□

Remark 2.2. If $K = \mathbb{R}^m$ in Lemma 2.1, then $\operatorname{epi} \mathcal{S}_K = \{0\} \times \mathbb{R}_+$. Thus, Lemma 2.1 reduces to Lemma 2.1 in [17].

Using Lemma 2.1, we can derive the ε-approximate optimality condition for robust convex semidefinite programming, given that $D(K)$ is a closed and convex cone.

Lemma 2.2. [Optimality Theorem for ε-Approximation Robust Solutions] *Let* $\tilde{x} \in \mathcal{F}(K)$ *and* $\varepsilon > 0$. *We suppose further that* $D(K)$ *is closed and convex cone. Then,* $\tilde{x} \in Sol_\varepsilon$(RSDP) *if and only if there exist* $\varepsilon_0 \geq 0, \varepsilon_1 \geq 0$, $\tilde{v} \in \partial_{\epsilon_0}\varphi(\tilde{x}) + N_{\varepsilon_1}(K, \tilde{x})$, $\widetilde{\Lambda} \in \mathbb{S}^n_+$ *and* $\widetilde{Q}_i \in \mathcal{U}_i$, $\forall i \in I$ *such that*

(Condition R)
$$\tilde{v} = \left(\operatorname{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \operatorname{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \operatorname{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and}$$
$$\varepsilon_0 + \varepsilon_1 + \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

Proof. Let $\varepsilon \geq 0$. Since $\tilde{x} \in Sol_\varepsilon$(RSDP), one has
$$\varphi(\tilde{x}) \leq \varphi(y) + \varepsilon, \text{ for all } y \in \mathcal{F}(K).$$

That is, $\varphi(\tilde{x}) + \mathcal{I}_{\mathcal{F}(K)}(x) \leq \varphi(x) + \mathcal{I}_{\mathcal{F}(K)}(x) + \varepsilon$, for all $x \in \mathbb{R}^m$. By the concept of ε-subdifferential, it can be rewritten as, $0 \in \partial_\varepsilon\left(\varphi + \mathcal{I}_{\mathcal{F}(K)}\right)(\tilde{x})$. Then, by Proposition 2.2, there exist non-negative real numbers $\varepsilon_0, \varepsilon'$ with $\varepsilon = \varepsilon_0 + \varepsilon'$, $\tilde{v} \in \partial_{\varepsilon_0}\varphi(\tilde{x})$ and $-\tilde{v} \in \partial_{\varepsilon'}\mathcal{I}_{\mathcal{F}(K)}(\tilde{x})$. Then, by Lemma 2.1 and $D(K)$ is closed and convex cone one has
$$\begin{pmatrix}\tilde{v}\\ \langle\tilde{v}, \tilde{x}\rangle - \varepsilon'\end{pmatrix} \in \bigcup_{\substack{Q_i\in\mathcal{U}_i, i\in I\\ (\Lambda,\alpha)\in\mathbb{S}^n_+\times\mathbb{R}_+}} \left\{\begin{pmatrix}(\operatorname{tr}[Q_1\Lambda], \operatorname{tr}[Q_2\Lambda], \ldots, \operatorname{tr}[Q_m\Lambda])\\ -\operatorname{tr}[Q_0\Lambda] - \delta\end{pmatrix}\right\} - \operatorname{epi}\mathcal{S}_K.$$

Thus there exists $(\widetilde{\Lambda}, \tilde{\delta}) \in \mathbb{S}^n_+ \times \mathbb{R}_+$, $\widetilde{Q}_i \in \mathcal{U}_i, i \in I$ and $\tilde{c} \in \operatorname{dom}\mathcal{S}_K$ such that
$$\tilde{v} = \left(\operatorname{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \operatorname{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \operatorname{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) - \tilde{c}$$
$$\langle\tilde{v}, \tilde{x}\rangle - \varepsilon' = -\operatorname{tr}[\widetilde{Q}_0\widetilde{\Lambda}] - \tilde{\delta} - \mathcal{S}_K(\tilde{c}).$$

We then have that
$$\begin{aligned}\varepsilon' &= \langle\tilde{v}, \tilde{x}\rangle + \operatorname{tr}[\widetilde{Q}_0\widetilde{\Lambda}] + \tilde{\delta} + \mathcal{S}_K(\tilde{c})\\ &= \left\langle\left(\operatorname{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \operatorname{tr}[\widehat{Q}_2\widetilde{\Lambda}], \ldots, \operatorname{tr}[\widehat{Q}_m\widetilde{\Lambda}]\right), \tilde{x}\right\rangle - \langle\tilde{c}, \tilde{x}\rangle\\ &\quad + \operatorname{tr}[\widetilde{Q}_0\widetilde{\Lambda}] + \tilde{\delta} + \mathcal{S}_K(\tilde{c})\\ &= \operatorname{tr}\left[\left(\sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] + \operatorname{tr}[\widetilde{Q}_0\widetilde{\Lambda}] - \langle\tilde{c}, \tilde{x}\rangle + \tilde{\delta} + \mathcal{S}_K(\tilde{c})\\ &= \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] + \varepsilon_1,\end{aligned}$$

where $\varepsilon_1 = \mathcal{S}_K(\tilde{c}) - \langle\tilde{c}, \tilde{x}\rangle + \tilde{\delta} \geq 0$. It means that, for any $y \in K$
$$\langle\tilde{c}, y - \tilde{x}\rangle = \langle\tilde{c}, y\rangle - \langle\tilde{c}, \tilde{x}\rangle = \langle\tilde{c}, y\rangle + \varepsilon_1 - \mathcal{S}_K(\tilde{c}) - \tilde{\delta} \leq \varepsilon_1.$$

This implies that $\tilde{c} \in N_{\varepsilon_1}(K, \tilde{x})$, hence we have desired.

On the other hand, we assume there exist $\varepsilon_0 \geq 0, \varepsilon_1 \geq 0$, $\tilde{v} \in \partial_{\epsilon_0}\varphi(\tilde{x}) + N_{\varepsilon_1}(K, \tilde{x})$, $\widetilde{\Lambda} \in \mathbb{S}^n_+$, and $\widetilde{Q}_i \in \mathcal{U}_i$, $\forall i \in I$ such that
$$\tilde{v} = \left(\operatorname{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \operatorname{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \operatorname{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and}$$
$$\varepsilon_0 + \varepsilon_1 + \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

Then $\tilde{v} = u + w$ for some $u \in \partial_{\varepsilon_0}\varphi(\tilde{x})$ and $w \in N_{\varepsilon_1}(K, \tilde{x})$. We have that

$$\begin{aligned} \varphi(y) - \varphi(\tilde{x}) &\geq \langle u, y - \tilde{x}\rangle - \varepsilon_0, \ \forall y \in \mathbb{R}^m, \\ \langle w, y - \tilde{x}\rangle &\leq \varepsilon_1, \ \forall y \in K. \end{aligned}$$

Thus, for any $y \in K$,

$$\begin{aligned}
&\varphi(y) - \varphi(\tilde{x}) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \\
&\geq \langle u, y - \tilde{x}\rangle - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 \\
&= \left\langle -w + \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), y - \tilde{x}\right\rangle - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 \\
&= \langle -w, y - \tilde{x}\rangle + \left\langle\left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), y - \tilde{x}\right\rangle \\
&\quad - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 \\
&\geq -\varepsilon_1 + \left\langle\left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), y - \tilde{x}\right\rangle - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 \\
&= \left\langle\left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), y\right\rangle - \varepsilon_0 - \varepsilon_1 \\
&\quad - \left\langle\left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), \tilde{x}\right\rangle - \mathrm{tr}\left[\widetilde{Q}_0\widetilde{\Lambda}\right] - \mathrm{tr}\left[\left(\sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \\
&= \mathrm{tr}\left[\left(\sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \mathrm{tr}\left[\left(\sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \\
&\quad - \mathrm{tr}[\widetilde{Q}_0\widetilde{\Lambda}] - \mathrm{tr}\left[\left(\sum_{i=1}^m y_iQ_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 - \varepsilon_1 \\
&= -\,\mathrm{tr}\left[\left(\widetilde{Q}_0 + \widetilde{\Lambda}\sum_{i=1}^m \tilde{x}_iQ_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 - \varepsilon_1 = -\varepsilon.
\end{aligned} \tag{2.2}$$

Thus

$$\varphi(\tilde{x}) + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varphi(y) + \varepsilon \ \text{ for all } y \in K.$$

Thanks to $\mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \geq 0, \ \forall y \in \mathcal{F}(K)$, Hence,

$$\varphi(\tilde{x}) \leq \varphi(y) + \varepsilon, \ \forall y \in \mathcal{F}(K).$$

Hence, $\tilde{x}$ is a minimizer of ε-approximate solution of (RSDP). □

Remark 2.3. (1) When $K = \mathbb{R}^m$ in Lemma 2.2, the set $\mathrm{epi}\,\mathcal{S}_K = \{0\} \times \mathbb{R}_+$ and for any $\varepsilon \geq 0$ and $u \in \mathbb{R}^m$, the set $N_\varepsilon(K, u) = \{0\}$. Hence, Lemma 2.2 covers Lemma 3.1 from [17].

(2) For any $i \in I$, suppose $\mathcal{U}_i$ is a singleton set contained in $\mathbb{S}^n$. Problem (RSDP) covers convex semidefinite programming, as studied by Lee and Lee in [16], which examines approximate duality conditions.

2.2. **ε-approximate duality for RSDP.** We provide a robust strong duality theorem for the primal problem RSDP and its Lagrangian dual problem, which is formulated in terms of the robust characteristic cone $D(K)$. Specifically, The concept of strong duality asserts that the optimal solutions for both the primal and dual problems are equivalent in value. Let $\mathcal{U} := \mathcal{U}_0 \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_m = \prod_{i=0}^{m} \mathcal{U}_i$ and $Q \in \mathcal{U}$ means that for each $i \in I$, $Q_i \in \mathcal{U}_i$, $Q = (Q_1, Q_2, \ldots, Q_m)$. We define mapping $\Phi : \mathbb{S}^n_+ \times \mathcal{U} \to \mathbb{R}$ as

$$\Phi(\Lambda, Q) = \inf\left\{\varphi(z) - \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} z_i Q_i\right)\Lambda\right] : z \in K\right\}.$$

The Lagrangian dual problem of RSDP is given as follows.

$$\text{(RSDD)} \qquad \max\left\{\Phi(\Lambda, Q) : (\Lambda, Q) \in \mathbb{S}^n_+ \times \mathcal{U}\right\},$$

The notation sup (RSDD) represents the maximum achievable value of problem RSDDs. We say that strong duality holds for problem (RSDP) if inf (RSDP) = sup (RSDD)

The following lemma demonstrates that when the characteristic cone $D(K)$ satisfies the conditions of being a closed and convex cone, the strong duality between RSDP and RSDD is valid.

Proposition 2.5. [Strong duality theorem] *Assume that $\mathcal{F}(K) \neq \emptyset$ and $D(K)$ is a closed and convex cone. Then,* inf (RSDP) = sup (RSDD).

Proof. [Weak duality] Let $x := (x_1, \ldots, x_m) \in \mathcal{F}(K)$. Then

$$Q_0 + \sum_{i=1}^{m} x_i Q_i \succeq 0, \ \forall\, Q_i \in \mathcal{U}_i, i \in I.$$

For each $(\Lambda, Q) \in \mathbb{S}^n_+ \times \mathcal{U}$, we see that $\operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} x_i Q_i\right)\Lambda\right] \geq 0$. This implies that

$$\varphi(x) - \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} x_i Q_i\right)\Lambda\right] \leq \varphi(x).$$

Thus,

$$\Phi(\Lambda, Q) = \inf_{x \in K}\left\{\varphi(x) - \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} x_i Q_i\right)\Lambda\right]\right\} \leq \inf_{x \in K} \varphi(x).$$

Hence, sup (RSDD) $\leq$ inf (RSDP).

[Strong duality] If inf (RSDP) $= -\infty$, then inf (RSDP) $\leq$ sup (RSDD) holds. Since $\mathcal{F}(K) \neq \emptyset$, inf (RSDP) $< \infty$. Let $\Phi(x) = \varphi(x) -$ inf (RSDP) for all $x \in \mathbb{R}^m$. This means that $x \in \mathcal{F}(K) \Rightarrow \mathit{\Phi}(x) \geq 0$. From our hypothesis,

$$\begin{aligned}(0,0) &\in \operatorname{epi} \mathit{\Phi}^* - \operatorname{cl}(\operatorname{conv}(D(K)))\\ &= (0, \inf(\text{RSDP})) + \operatorname{epi}\varphi^* - \operatorname{cl}(\operatorname{conv}(D(K)))\\ &= (0, \inf(\text{RSDP})) + \operatorname{epi}\varphi^* - D(K)\\ &= (0, \inf(\text{RSDP})) + \operatorname{epi}\varphi^* + \operatorname{epi}\mathcal{I}^*_{\mathcal{F}(K)}.\end{aligned}$$

Thus, there exist $(z,\delta) \in \operatorname{epi}\varphi^*$ and $(-z,-\delta-\inf(\text{RSDP})) \in \operatorname{epi}\mathcal{I}^*_{\mathcal{F}(K)}$. This gives us that, there exist $(\Lambda, Q) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that, for each $y \in K$,

$$\begin{aligned}
-\varphi(y) + \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^m y_i Q_i\right)\Lambda\right] &= \langle z, y\rangle - \varphi(y) + \langle -z, y\rangle + \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^m y_i Q_i\right)\Lambda\right] \\
&= \langle z, y\rangle - \varphi(y) + \langle -z, y\rangle - \operatorname{tr}\left[-\left(Q_0 + \sum_{i=1}^m y_i Q_i\right)\Lambda\right] \\
&\le \varphi^*(z) + \mathcal{I}^*_{\mathcal{F}(K)}(-z) \\
&\le \delta + (-\delta - \inf(\text{RSDP})) = -\inf(\text{RSDP}).
\end{aligned}$$

This implies that

$$\max_{(\Lambda,Q)\in\mathbb{S}^n_+\times\mathcal{U}}\left\{\inf_{y\in K}\left\{\varphi(y) - \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^m y_i Q_i\right)\Lambda\right]\right\}\right\} \ge \inf(\text{RSDP}).$$

Therefore, $\sup(\text{RSDD}) \ge \inf(\text{RSDP})$. □

Before we move to the next section let's recall some concepts of ε-Kuhn-Tucker vector which will be used in our main result. Building upon the concept of ε-Kuhn-Tucker vector introduced by Scovel et al. [21] for convex optimization problems, we adapt its use to the case of robust convex semidefinite programming problems. We also introduce the concept of ε-approximate solutions for RSDD and ε-saddle points for RSDP, and investigate the relationship between these solution sets.

Let $\varepsilon > 0$. An array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ is said to be ε-Kuhn-Tucker vector for (RSDP) if

$$\Phi(\widetilde{\Lambda}, \widetilde{Q}) \ge \inf(\text{RSDP}) - \varepsilon.$$

An array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ is said to be ε-approximate solution of (RSDD) if

$$\Phi(\widetilde{\Lambda}, \widetilde{Q}) \ge \sup(\text{RSDD}) - \varepsilon.$$

An array $(\tilde{x}, \widetilde{\Lambda}, \widetilde{Q}) \in K \times \mathbb{S}^n_+ \times \mathcal{U}$ is said to be ε-saddle point for (RSDP) if

$$\begin{aligned}
\varphi(\tilde{x}) - \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^m \tilde{x}_i Q_i\right)\Lambda\right] - \varepsilon &\le \varphi(\tilde{x}) - \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \\
&\le \varphi(y) - \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] + \varepsilon, \qquad (\text{RSP}_\varepsilon)
\end{aligned}$$

for all $(y, \Lambda, Q) \in K \times \mathbb{S}^n_+ \times \mathcal{U}$.

The set of ϵ-approximate solutions for (RSDD), the set of all robust ε-Kuhn-Tucker vectors for (RSDP), and the set of all robust ε-saddle points are denoted by Sol_ε(RSDD), RKT_ε, and RSad_ε, respectively.

Remark 2.4. Verifying that $\text{RKT}_\varepsilon \subseteq Sol_\varepsilon$(RSDD) is not difficult, as weak duality holds.

The following lemma establishes an equality relationship between Sol_ε(RSDD) and RKT_ε, assuming that strong duality holds.

Lemma 2.3. *Assume that strong duality for RSDP and RSDD holds. Given $\varepsilon > 0$. Then,*

$$Sol_\varepsilon(\text{RSDD}) = RKT_\varepsilon.$$

Proof. It suffices to show that Sol_ε(RSDD) $\subseteq$ RKT$_\varepsilon$. Let $(\Lambda, Q) \in Sol_\varepsilon$(RSDD). Then we have

$$\sup \text{(RSDD)} \leq \Phi(\Lambda, Q) + \varepsilon.$$

Strong duality implies that $\inf$ (RSDP) $=$ $\sup$ (RSDD) $\leq \Phi(\Lambda, Q) + \varepsilon$. Therefore, $(\Lambda, Q) \in$ RKT$_\varepsilon$. □

Proposition 2.6. *Assume that* $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ *satisfies the Condition* R *corresponding to the* ε*-approximate solution* $\tilde{x}$ *of* (RSDP). *Suppose further that strong duality holds. Then,* $(\widetilde{\Lambda}, \widetilde{Q}) \in$ *RKT*$_\varepsilon$.

Proof. Since $(\widetilde{\Lambda}, \widetilde{Q})$ satisfies the Condition R corresponding $\tilde{x}$, there exist $\varepsilon_0 \geq 0, \varepsilon_1 \geq 0$ and $\tilde{v} \in \partial_{\epsilon_0}(\tilde{x}) + N_{\varepsilon_1}(K, \tilde{x})$ such that

$$\tilde{v} = \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and}$$

$$\varepsilon_0 + \varepsilon_1 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

Thus, there exists $u + \tilde{v} \in \partial_{\epsilon_0}\varphi(\tilde{x})$ such that $-u \in N_{\varepsilon_1}(C, \tilde{x})$. We get

$$\varphi(y) - \varphi(\tilde{x}) \geq \langle u + \tilde{v}, y - \tilde{x}\rangle - \varepsilon_0, \ \forall y \in K.$$

This argues that

$$\varphi(y) - \varphi(\hat{x}) - \langle \tilde{v}, y - \tilde{x}\rangle \geq \langle u, y - \tilde{x}\rangle - \varepsilon_0, \ \forall y \in K.$$

Since $-u \in N_{\varepsilon_1}(K, \tilde{x})$, we have that

$$\varphi(y) - \varphi(\tilde{x}) - \langle \tilde{v}, y - \tilde{x}\rangle \geq -\varepsilon_0 - \varepsilon_1, \ \forall y \in K.$$

Hence,

$$\begin{aligned}\varphi(y) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varphi(\tilde{x}) &\geq -\,\mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 - \varepsilon_1 \\ &= -\varepsilon, \ \forall y \in K.\end{aligned}$$

Therefore,

$$\varphi(\tilde{x}) \leq \varphi(y) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] + \varepsilon \ \text{ for all } y \in K.$$

By the virtue of weak duality, one has

$$\Phi\,(\Lambda, Q) \leq \inf_{x\in K} \varphi(x) = \varphi(\tilde{x}), \text{ for all } (\Lambda, Q) \in \mathbb{S}^n_+ \times \mathcal{U}.$$

This gives that

$$\Phi\,(\Lambda, Q) - \varepsilon \leq \varphi(\tilde{x}) \leq \varphi(y) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \ \text{ for all } y \in K.$$

Then, for all $(\Lambda, Q) \in \mathbb{S}^n_+ \times \mathcal{U}$,

$$\Phi(\widetilde{\Lambda}, \widetilde{Q}) = \inf_{y\in K}\left\{\varphi(y) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right]\right\} \geq \Phi(\Lambda, Q) - \varepsilon.$$

Thus, $(\widetilde{\Lambda}, \widetilde{Q})$ is an ε-solution of (RSDD). It follows from Lemma 2.3 that $(\widetilde{\Lambda}, \widetilde{Q})$ is also a point in RKT$_\varepsilon$. □

3. CHARACTERIZATION FOR ε-APPROXIMATE SOLUTION SETS

This section focuses on characterizing ε-approximate solutions for robust convex semidefinite programming problems. To achieve this, we introduce a following lemma that will be instrumental in the characterization process.

Lemma 3.4. *Let $\varepsilon \geq 0$ and $\tilde{x}$ be an ε-solution of* (RSDP). *Assume that Condition R holds. Then, there exists an array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that*

$$\varphi(\tilde{x}) + \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} y_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varphi(y) + \varepsilon \text{ for all } y \in K. \tag{3.3}$$

Proof. We assume that the Condition R holds, there exist $\varepsilon_0 \geq 0, \varepsilon_1 \geq 0$ and array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$\left(\operatorname{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \operatorname{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \operatorname{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \in \partial_{\varepsilon_0}\varphi(\tilde{x}) + N_{\varepsilon_1}(K, \tilde{x}),$$

$$\varepsilon_0 + \varepsilon_1 + \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

Applying the equation (2.2) of Lemma 2.2, the result is complete. □

Remark 3.5. (i) It follows from inequality (3.3) that

$$0 \leq \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varepsilon.$$

(ii) For any $\tilde{x} \neq \hat{x} \in Sol_\varepsilon$(RSDP), inequality (3.3) implies

$$\varphi(\tilde{x}) - \varphi(\hat{x}) + \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \hat{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varepsilon.$$

Therefore, $0 \leq \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \hat{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq 2\varepsilon$.

Lemma 3.5. *Let $\tilde{x} \in \mathcal{F}(K)$. If there exists an array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that*

$$\varphi(\tilde{x}) + \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} y_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varphi(y) + \varepsilon, \ \forall y \in K,$$

then $\tilde{x} \in Sol_\varepsilon$(RSDP).

Proof. For feasible solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \in \mathcal{F}(K) \subseteq K$, we get

$$\widetilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i \succeq 0 \ \Rightarrow \ \operatorname{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \geq 0.$$

Thus, for any $y \in \mathcal{F}(K) \subseteq K$,

$$\varphi(\tilde{x}) \leq \varphi(\tilde{x}) + \operatorname{tr}\left[\left(\widetilde{F}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{F}_i\right)\widetilde{\Lambda}\right] \leq \varphi(y) + \varepsilon.$$

Hence, $\tilde{x} \in \mathrm{Sol}_\varepsilon$(RSDP). □

It's important to note that the array $(\widetilde{\Lambda}, \widetilde{Q})$ mentioned above is not the only possible array for an ε-approximate solution $\tilde{x}$. The collection of all arrays $(\widetilde{\Lambda}, \widetilde{Q})$ that correspond to $\tilde{x}$ and satisfy the Condition R will be referred to as $RM_\varepsilon(\tilde{x})$.

Lemma 3.6. *Assume that* $\tilde{x} \in Sol_\varepsilon$(RSDP) *and* $(\widetilde{\Lambda}, \widetilde{Q}) \in RM_\varepsilon(\tilde{x})$*. Then an array* $(\tilde{x}, \widetilde{\Lambda}, \widetilde{Q})$ *is a solution of* (RSP$_\varepsilon$)*.*

Proof. Let $(\widetilde{\Lambda}, \widetilde{Q}) \in RM_\varepsilon(\tilde{x})$. It follows from Lemma 3.4 that

$$\varphi(\tilde{x}) + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} y_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varphi(y) + \varepsilon, \ \forall y \in K. \tag{3.4}$$

Since $\left\{\left(\tilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_1\right), \widetilde{\Lambda}\right\} \subset \mathbb{S}^n_+$, one has $\mathrm{tr}\left[\left(\tilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_1\right)\widetilde{\Lambda}\right] \geq 0$.
Thus, for any $y \in K$,

$$\varphi(\tilde{x}) + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} y_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varphi(y) + \varepsilon + \mathrm{tr}\left[\left(\tilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right]. \tag{3.5}$$

Remark 3.5 (i) gives that, $0 \leq \mathrm{tr}\left[\left(\tilde{Q}_0 + \tilde{x}_i \sum_{i=1}^{m} \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq \varepsilon$. Combining with (3.4) and (3.5), one has for each $(y, \Lambda, Q) \in K \times \mathbb{S}^n_+ \times \mathcal{U}$,

$$\begin{aligned}
\varphi(\tilde{x}) - \mathrm{tr}\left[\left(Q_0 + \sum_{i=1}^{m} \tilde{x}_i Q_i\right)\Lambda\right] - \varepsilon &\leq \varphi(\tilde{x}) - \varepsilon \\
&\leq \varphi(\tilde{x}) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] \\
&\leq \varphi(y) - \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} \tilde{x}_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right] + \varepsilon.
\end{aligned}$$

□

We will introduce a technique for deriving the set of ε-approximate solutions of (RSDP). To define the set of ε-approximate solutions for the positive semidefinite matrix Λ, we will utilize the following lemmas.

Lemma 3.7. *Given* $\varepsilon > 0$*.*

(i) Let $\tilde{x} \in Sol_\varepsilon$(RSDP)*. Then, there exist* $\varepsilon_0 \in [0, \varepsilon]$*,* $\tilde{u} \in \partial_{\varepsilon_0} f(\tilde{x})$ *and array* $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ *such that*

$$\langle \tilde{u}, y - \tilde{x} \rangle \geq \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} y_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right], \ \forall y \in K.$$

(ii) For $\varepsilon_0 \in [0, \varepsilon]$ *and* $\tilde{x} \in \mathcal{F}(K)$*. If there exist* $\tilde{u} \in \partial_{\varepsilon_0} f(\tilde{x})$ *and array* $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ *such that*

$$\langle \tilde{u}, y - \tilde{x} \rangle \geq \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^{m} y_i \widetilde{Q}_i\right)\widetilde{\Lambda}\right], \ \forall y \in K,$$

then $\tilde{x} \in Sol_\varepsilon$(RSDP)*.*

Proof. (i) Let $\tilde{x} \in \mathrm{Sol}_\varepsilon$(RSDP). It follows from Condition R, there exist $\varepsilon_0 \geq 0, \varepsilon_1 \geq 0$ and array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$\tilde{v} := \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \in \partial_{\varepsilon_0}\varphi(\tilde{x}) + N_{\varepsilon_1}(K, \tilde{x}) \text{ and}$$

$$\varepsilon_0 + \varepsilon_1 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

So, there exist $\tilde{u} \in \partial_{\varepsilon_0} f(\tilde{x})$ with $-\tilde{u} + \tilde{v} \in N_{\varepsilon_1}(K, \tilde{x})$. This implies that

$$\langle \tilde{u} - \tilde{v}, y - \tilde{x}\rangle \geq -\varepsilon_1 = \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right],$$

for all $y \in K$. Since $\tilde{v} := \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right)$, the above inequality is equivalent to

$$\langle \tilde{u}, y - \tilde{x}\rangle \geq \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right], \quad \text{for all } y \in K.$$

(ii) Let $\varepsilon_0 \in [0, \varepsilon]$ and $\tilde{x} \in A$. By our hypothesis, there exist $\tilde{u} \in \partial_{\varepsilon_0} f(\tilde{x})$ and array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$\langle \tilde{u}, y - \tilde{x}\rangle \geq \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right], \quad \text{for all } y \in K.$$

Since $\tilde{u} \in \partial_{\varepsilon_0}\varphi(\tilde{x})$, one has

$$\varphi(y) - \varphi(\tilde{x}) \geq \langle \tilde{u}, y - \tilde{x}\rangle - \varepsilon_0, \quad \text{for all } y \in K.$$

Notice that $\mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \geq 0$, for each $y = (y_1, y_2, \ldots, y_m) \in \mathcal{F}(K)$. This gives that

$$\begin{aligned}
\varphi(y) - \varphi(\tilde{x}) &\geq \langle \tilde{u}, y - \tilde{x}\rangle - \varepsilon_0 \\
&\geq \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] - \varepsilon_0 \\
&= -\varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \\
&\geq -\varepsilon, \quad \text{for all } y \in \mathcal{F}(K).
\end{aligned}$$

Therefore, $\tilde{x} \in \mathrm{Sol}_\varepsilon$(RSDP).

□

Given $\varepsilon > 0$ and $\varepsilon' \in [0, \varepsilon]$. The ε-approximate solution set of (RSDP) corresponding to $(\widetilde{\Lambda}, \widetilde{Q}, \varepsilon')$ is denoted $Sol(\widetilde{\Lambda}, \widetilde{Q}, \varepsilon')$. That is,

$$Sol(\widetilde{\Lambda}, \widetilde{Q}, \varepsilon') = \left\{ \tilde{x} \in \mathcal{F}(K) \;\middle|\; \begin{array}{l} u \in \partial_{\varepsilon'} f(\tilde{x}), \\ \langle u, y - \tilde{x}\rangle \geq \varepsilon_0 - \varepsilon + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right], \ \forall y \in \mathcal{F}(K), \\ 0 \leq \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \leq 2\varepsilon \end{array} \right\}.$$

Set

$$Sol_\varepsilon(\widetilde{\Lambda},\widetilde{Q}) := \bigcup_{\substack{\varepsilon'\geq 0\\ 0\leq\varepsilon'\leq\varepsilon}} Sol(\widetilde{\Lambda},\widetilde{Q},\varepsilon').$$

It is clear that $Sol_\varepsilon(\widetilde{\Lambda},\widetilde{Q}) \subseteq Sol_\varepsilon$(RSDP).

Theorem 3.1. *Assume that strong duality between RSDP and RSDD holds. Let $\varepsilon > 0$, then*

$$Sol_\varepsilon(\text{RSDP}) = \bigcup_{(\Lambda,Q)\in RKT_\varepsilon} Sol_\varepsilon\left(\Lambda, Q\right),$$

where

$$Sol_\varepsilon\left(\Lambda, Q\right) = \bigcup_{0\leq\varepsilon'\leq\varepsilon} Sol_\varepsilon\left(\Lambda, Q, \varepsilon'\right).$$

That is,

(3.6)

$$Sol_\varepsilon(\text{RSDP}) = \bigcup_{(\Lambda,Q)\in RKT_\varepsilon}\ \bigcup_{0\leq\varepsilon'\leq\varepsilon} \left\{ z\in\mathcal{F}(K) \;\middle|\; \begin{array}{l} v\in\partial_{\varepsilon'} f(z),\\ \langle v, y-z\rangle \geq \varepsilon_0-\varepsilon+\operatorname{tr}\left[\left(Q_0+\sum_{i=1}^m y_iQ_i\right)\Lambda\right],\ \forall y\in\mathcal{F}(K),\\ 0\leq \operatorname{tr}\left[\left(Q_0+\sum_{i=1}^m z_iQ_i\right)\Lambda\right]\leq 2\varepsilon \end{array} \right\}.$$

Proof. Let $\tilde{x}\in Sol_\varepsilon$(RSDP). It follows from Lemma 3.7 (i) that there exist $\varepsilon_0\in[0,\varepsilon]$, $\tilde{v}\in\partial_{\varepsilon_0} f(\tilde{x})$ and $(\widetilde{\Lambda},\widetilde{Q})\in\mathbb{S}^n_+\times\mathcal{U}$ such that

$$\langle\tilde{v}, y-\tilde{x}\rangle \geq \varepsilon_0-\varepsilon+\operatorname{tr}\left[\left(\widetilde{Q}_0+\sum_{i=1}^m y_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right],\ \forall y\in K.$$

Thus, $\tilde{x}\in Sol(\widetilde{\Lambda},\widetilde{Q},\varepsilon_0)$ and then $Sol_\varepsilon(\widetilde{\Lambda},\widetilde{Q})$. By Proposition 2.6, one has $(\widetilde{\Lambda},\widetilde{Q})\in \text{RKT}_\varepsilon$. This implies that

$$Sol_\varepsilon(\text{RSDP}) \subset \bigcup_{(\Lambda,Q)\in\text{RKT}_\varepsilon} Sol_\varepsilon\left(\Lambda, Q\right).$$

Conversely, let $\tilde{x}\in\bigcup_{(\Lambda,Q)\in\text{RKT}_\varepsilon} Sol_\varepsilon\left(\Lambda,Q\right)$. Then, then there exist $(\widetilde{\Lambda},\widetilde{Q})\in\text{RKT}_\varepsilon$ and $0\leq\varepsilon'\leq\varepsilon$ such that $\hat{x}\in Sol(\widetilde{\Lambda},\widetilde{Q},\varepsilon')$. Since $(\widetilde{\Lambda},\widetilde{Q})\in\text{RKT}_\varepsilon = Sol_\varepsilon$(RSDD), one has

$$(\widetilde{\Lambda},\widetilde{Q})\in\mathbb{S}^n_+\times\mathcal{U}.$$

By Lemma 3.7 (ii), $\hat{x}$ is an ε-solution of (RSDP). □

Remark 3.6. Since the problem (RSDP) is finite, for any minimizing sequence $\{x_k\}\in\mathcal{F}(K)$ such that $\varphi(x_k)\to\inf$(RSDP), the formula (3.6) can be rewritten by the following formula.

$$Sol_{\varepsilon}(\text{RSDP}) = \bigcup_{(\Lambda,Q)\in \text{RKT}_{\varepsilon}} \bigcup_{0\le\varepsilon'\le\varepsilon} \left\{ z \in \mathcal{F}(K) \;\middle|\; \begin{array}{l} v \in \partial_{\varepsilon'} f(z), \\ \langle v, x_k - z\rangle \ge \varepsilon_0 - \varepsilon + \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^m x_{(i,k)} Q_i\right)\Lambda\right], \forall k \in \mathbb{N}, \\ 0 \le \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^m z_i Q_i\right)\Lambda\right] \le 2\varepsilon \end{array} \right\}.$$

Example 3.1. Consider the following robust convex semidefinite programming:

$$(P_1) \qquad \begin{array}{ll} \text{Minimize} & x_1 + x_2^2, \\ \text{subject to} & Q_0 + x_1Q_1 + x_2Q_2 \succeq 0, \forall Q_i \in \mathcal{U}_i, i = 0,1,2, \\ & (x_1,x_2) \in \mathbb{R}^2_+, \end{array}$$

where

$$\mathcal{U}_0 = \left\{ \begin{pmatrix} u_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : u_0 = 1 \right\},\ \mathcal{U}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_1 \\ 0 & u_1 & 0 \end{pmatrix} : u_1 \in [-1,1] \right\} \text{ and}$$

$$\mathcal{U}_2 = \left\{ \begin{pmatrix} u_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : u_2 \in [0,1] \right\}.$$

Thus, we have that

$$Q_0 + x_1Q_1 + x_2Q_2 = \begin{pmatrix} 1 + u_2x_2 & 0 & 0 \\ 0 & 0 & u_1x_1 \\ 0 & u_1x_1 & 0 \end{pmatrix}.$$

We observe that the set of robust feasible solutions for the convex semidefinite problem (P_1) is given by

$$\{x := (x_1,x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \ge 0\},$$

from which we can immediately conclude that $\inf(P_1) = 0$. Furthermore, we have that

$$Sol_{\varepsilon}(P_1) := \left\{(0,x_2) : 0 \le x_2 \le \sqrt{\varepsilon}\right\}.$$

We can express the dual problem of (P_1) in the following manner, where $\Lambda := [\lambda_{ij}] \in \mathbb{S}^3_+$ and $Q_i \in \mathcal{U}_i$ for $i = 0,1,2$.

$$\begin{aligned}
&\Phi(\Lambda, Q_0, Q_1, Q_2) \\
&\quad = \inf_{(x_1,x_2)\in\mathbb{R}^2_+} \{f(x) - \operatorname{tr}[(Q_0 + x_1Q_1 + x_2Q_2)\Lambda]\} \\
&\quad = \inf_{(x_1,x_2)\in\mathbb{R}^2_+} \left\{x_1 + x_2^2 - [\lambda_{11}(1 + u_2x_2) + \lambda_{23}u_1x_1 + \lambda_{32}u_1x_1]\right\} \\
&\quad = \inf_{(x_1,x_2)\in\mathbb{R}^2_+} \left\{x_1 + x_2^2 - \lambda_{11}(1 + u_2x_2) - 2\lambda_{23}u_1x_1\right\} \\
&\quad = \inf_{(x_1,x_2)\in\mathbb{R}^2_+} \left\{(1 - 2\lambda_{23}u_1)x_1 + x_2^2 - \lambda_{11}u_2x_2 + \frac{(\lambda_{11}u_2)^2}{4} - \frac{(\lambda_{11}u_2)^2}{4} - \lambda_{11}\right\} \\
&\quad = \inf_{(x_1,x_2)\in\mathbb{R}^2_+} \left\{(1 - 2\lambda_{23}u_1)x_1 + \left[x_2 - \frac{(\lambda_{11}u_2)^2}{2}\right]^2 - \frac{(\lambda_{11}u_2)^2}{4} - \lambda_{11}\right\}.
\end{aligned}$$

This implies that

$$\Phi(\Lambda, Q_0, Q_1, Q_2) = \begin{cases} -\frac{(\lambda_{11}u_2)^2}{4} - \lambda_{11}, & \text{if } \lambda_{11} \geq 0 \text{ and } 0 \leq \lambda_{23} \leq \frac{1}{2}, \\ -\infty, & \text{if otherwise.} \end{cases}$$

The following Lagrangian dual problem of (P_1) is

$$(D_1) \qquad \max_{(\Lambda, Q_0, Q_1, Q_2) \in \mathbb{S}^3_+ \times \prod_{i=0}^{2} \mathcal{U}_i} \Phi(\Lambda, Q_0, Q_1, Q_2).$$

Hence, $\sup(D_1) = 0$, so we have that the strong duality holds. Moreover,

$$Sol(D_1) = \left\{(\Lambda, Q_0, Q_1, Q_2) \in \mathbb{S}^3_+ \times \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2 : \lambda_{11} = 0 \text{ and } 0 \leq \lambda_{23} \leq \frac{1}{2}\right\},$$

where $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}$.

Let $\varepsilon = \frac{1}{4}$, the ε-appproximate solution set of (D_1) is

$$Sol_{\frac{1}{4}}(D_1) = \left\{ (\Lambda, Q_0, Q_1, Q_2) \in \mathbb{S}^3_+ \times \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2 \;\middle|\; \begin{array}{l} 0 \leq \lambda_{23} \leq \frac{1}{2} \text{ and} \\ \lambda_{11} \in \left[0, \frac{-2+2\sqrt{1+\frac{u_2^2}{4}}}{u_2^2}\right], \text{ where } u_2 \in (0,1] \end{array} \right\}.$$

Lemma (2.3) gives that $Sol_{\frac{1}{4}}(D_1) = RKT_{\frac{1}{4}}$. For minimizing sequence $\{(x_{(1,k)}, x_{(2,k)})\} := \{(0, \frac{1}{k})\}$, one has

(3.7)

$$Sol_{\frac{1}{4}}(P_1) =$$

$$\bigcup_{(\Lambda, Q_0, Q_1, Q_2) \in RKT_{\frac{1}{4}}} \bigcup_{0 \leq \varepsilon' \leq \frac{1}{4}} \left\{ z \in \mathcal{F}(K) \;\middle|\; \begin{array}{l} v \in \partial_{\varepsilon'} f(z), \\ \langle v, x_k - z \rangle \geq \varepsilon' - \varepsilon + \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} x_{(i,k)} Q_i\right)\Lambda\right] \forall k \in \mathbb{N}, \\ 0 \leq \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} z_i Q_i\right)\Lambda\right] \leq \frac{1}{2} \end{array} \right\}.$$

For $z = (0, z_2) \in \mathcal{F}(K)$, we notice that

$$\partial_{\varepsilon'} f(z_1, z_2) = \{1\} \times \left[2z_2 - 2\sqrt{\varepsilon'}, 2z_2 + 2\sqrt{\varepsilon'}\right].$$

If we take any $v = (1, 2z_2 + 2t\sqrt{\varepsilon'}) \in \partial_{\varepsilon'} f(z_1, z_2)$, where $t \in [-1, 1]$, then the first inequality in (3.7) can be inferred

$$\begin{aligned} (2z_2 + 2t\sqrt{\varepsilon'})(x_{(2,k)} - z_2) &= (1, v_2) \begin{pmatrix} 0 - 0 \\ x_{(2,k)} - z_2 \end{pmatrix} \\ &= \langle v, x_k - z \rangle \\ &\geq \varepsilon' - \varepsilon + \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{2} x_{(i,k)} Q_i\right)\Lambda\right] \\ &= \varepsilon' - \varepsilon + 2\lambda_{23} u_1 x_{(1,k)} + \lambda_{11}(1 + u_2 x_{(2,k)}) \\ &= \varepsilon' - \varepsilon + \lambda_{11}(1 + u_2 x_{(2,k)}). \end{aligned}$$

Consequently,

$$(2z_2 + 2t\sqrt{\varepsilon'})(x_{(2,k)} - z_2) \geq \varepsilon' - \frac{1}{4} + \lambda_{11}(1 + u_2 x_{(2,k)}).$$

For minimizing $\left\{(0, \frac{1}{k})\right\}$ give that

$$(2z_2 + 2t\sqrt{\varepsilon'})\left(\frac{1}{k} - z_2\right) \geq \varepsilon' - \frac{1}{4} + \lambda_{11}\left(1 + u_2\frac{1}{k}\right).$$

Taking $k \to +\infty$ gives us

$$(2z_2 + 2t\sqrt{\varepsilon'})(-z_2) \geq \varepsilon' - \frac{1}{4} + \lambda_{11}.$$

It implies that

$$2z_2^2 + 2t\sqrt{\varepsilon'}z_2 - \varepsilon' + \frac{1}{4} - \lambda_{11} = (2z_2 + 2t\sqrt{\varepsilon'})(z_2) - \varepsilon' + \frac{1}{4} - \lambda_{11} \leq 0.$$

From above inequality, one has

$$z_2 \in \left[\frac{-2t\sqrt{\varepsilon'} - \alpha}{4}, \frac{-2t\sqrt{\varepsilon'} + \alpha}{4}\right], \text{ where } \alpha = \sqrt{(2t\sqrt{\varepsilon'})^2 - 4(2)(-\varepsilon' + \frac{1}{4} - \lambda_{11})}.$$

So that

$$z_2 \in \left[0, \frac{-4t\sqrt{\varepsilon'} + \sqrt{16t^2\varepsilon' - 8 + 32\varepsilon' + 32\lambda_{11})}}{8}\right]. \tag{3.8}$$

The second inequality in (3.7) implies that

$$0 \leq \operatorname{tr}\left[\left(Q_0 + \sum_{i=1}^{m} z_i Q_i\right)\Lambda\right] = 2\lambda_{23}u_1 z_1 + \lambda_{11}(1 + u_2 z_2) \leq \frac{1}{2},$$

and so

$$0 \leq \lambda_{11}(1 + u_2 z_2) \leq \frac{1}{2} \tag{3.9}$$

For any $\varepsilon' \in \left[0, \frac{1}{4}\right]$, $\lambda_{11} \in \left[0, \frac{-2+2\sqrt{1+\frac{u_2^2}{4}}}{u_2^2}\right]$ and $u_2 \in (0, 1]$, the union of sets satisfying (3.8) and (3.9) is $\left[0, \frac{1}{2}\right]$.

4. Robust Semidefinite Linear Programming

The following represents a mathematical formulation of the robust semidefinite linear programming problem [9].

$$\text{(RSDLP)} \qquad \begin{aligned} &\text{Minimize } c^T x, \\ &\text{subject to } Q(x) := Q_0 + \sum_{i=1}^{m} x_i Q_i \succeq 0, \\ &\qquad \forall Q_i \in \mathcal{U}_i, i \in I. \end{aligned}$$

This model is a special case of RSDP, where $K = \mathbb{R}^m$ and $\varphi(x) = c^T(x)$ for all $x \in \mathbb{R}^m$. Given $\varepsilon \geq 0$, $\tilde{x} \in \mathcal{F}$ is said to be an ε-approximate solution of RSDLP if $\langle c, \hat{x} - y\rangle \leq \varepsilon$ for all $y \in \mathcal{F}$. Let Sol_ε(RSDLP) denote ε-approximate solution set of RSDLP. The Lagrangian dual problem for RSDLPs is defined as follows:

$$\max_{(\Lambda, Q) \in \mathbb{S}_+^n \times \mathcal{U}} \left\{-\operatorname{tr}[Q_0\Lambda] : (\operatorname{tr}[Q_1\Lambda], \operatorname{tr}[Q_2\Lambda], \ldots, \operatorname{tr}[Q_m\Lambda]) = c\right\}.$$

We assume D is closed and convex cone for strong duality of RSDLP and its Lagrangian dual problem.

Corollary 4.1. *Given $\varepsilon > 0$. Suppose further that Condition R holds for problem RSDLP.*

(i) If $\tilde{x} \in Sol_\varepsilon$(RSDLP), then there exist $\varepsilon_0 \in [0, \varepsilon]$ and array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$c = \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and } \varepsilon_0 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

(ii) For $\varepsilon_0 \in [0, \varepsilon]$ and $\tilde{x} \in \mathcal{F}$, if there exist an array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$c = \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and } \varepsilon_0 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon,$$

then $\tilde{x} \in Sol_\varepsilon$(RSDLP).

Proof. (i) Since $\tilde{x}$ is an ε-approximate solution of (RSDLP), for any $y \in \mathcal{F}$

$$\langle c, \tilde{x}\rangle \leq \langle c, y\rangle + \varepsilon.$$

It follows from Lemma 2.2 that there exist $\varepsilon_0 \geq 0, \varepsilon_1 \geq 0$, $\tilde{v} \in \partial_{\epsilon_0} c\tilde{x} + N_{\varepsilon_1}(\mathbb{R}^m, \tilde{x})$, and $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$\tilde{v} = \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}\left[\widetilde{Q}_2\widetilde{\Lambda}\right], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and}$$
$$\varepsilon_0 + \varepsilon_1 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

This means that there exist $\varepsilon_0 \geq 0$ and $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$c = \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and } \varepsilon_0 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

(ii) Let $\varepsilon_0 \in [0, \varepsilon]$ and $\tilde{x} \in \mathcal{F}$. Suppose that there exist an array $(\widetilde{\Lambda}, \widetilde{Q}) \in \mathbb{S}^n_+ \times \mathcal{U}$ such that

$$c = \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right) \text{ and } \varepsilon_0 + \mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] = \varepsilon.$$

Since $\mathrm{tr}\left[\left(\widetilde{Q}_0 + \sum_{i=1}^m x_i\widetilde{Q}_i\right)\widetilde{\Lambda}\right] \geq 0$, for every $y \in \mathcal{F}$, one has

$$\begin{aligned}
\langle c, y\rangle - \langle c, \tilde{x}\rangle &= \left\langle \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), x\right\rangle \\
&\quad - \left\langle \left(\mathrm{tr}[\widetilde{Q}_1\widetilde{\Lambda}], \mathrm{tr}[\widetilde{Q}_2\widetilde{\Lambda}], \ldots, \mathrm{tr}[\widetilde{Q}_m\widetilde{\Lambda}]\right), \tilde{x}\right\rangle \\
&= \left\langle \sum_{i=1}^m y_i\widetilde{Q}_i, \widetilde{\Lambda}\right\rangle + \left\langle \widetilde{Q}_0, \widetilde{\Lambda}\right\rangle - \left\langle \widetilde{Q}_0, \widetilde{\Lambda}\right\rangle - \left\langle \sum_{i=1}^m y_i\widetilde{Q}_i, \widetilde{\Lambda}\right\rangle \\
&= \left\langle \left(\widetilde{Q}_0 + \sum_{i=1}^m y_i\widetilde{Q}_i\right), \widetilde{\Lambda}\right\rangle - \mathrm{tr}\left\langle \left(\widetilde{Q}_0 + \sum_{i=1}^m \tilde{x}_i\widetilde{Q}_i\right), \widetilde{\Lambda}\right\rangle \\
&\geq -\varepsilon + \varepsilon_0 \geq -\varepsilon.
\end{aligned}$$

Thus,

$$\langle c, \tilde{x}\rangle \leq \langle c, y\rangle + \varepsilon, \ \forall y \in \mathcal{F}.$$

That is, $\tilde{x} \in \mathrm{Sol}_\varepsilon$(RSDLP).

□

From Corollary 4.1, we immediately obtain the following result for characterization of ε-approximate solution set for RSDLPs.

Corollary 4.2. *Given $\varepsilon \geq 0$, one has*

$$Sol_\varepsilon(\text{RSDLP}) = \bigcup_{(\Lambda,Q)\in RKT_\varepsilon} \bigcup_{0\leq\varepsilon'\leq\varepsilon} \left\{ z \in \mathcal{F} \;\middle|\; \begin{array}{l} c = (\text{tr}[Q_1\Lambda], \text{tr}[Q_2\Lambda], \ldots, \text{tr}[Q_m\Lambda]), \\ \text{tr}\left[\left(Q_0 + \sum_{i=1}^m z_i Q_i\right)\Lambda\right] = \varepsilon - \varepsilon' \end{array} \right\}.$$

We now give the following example to illustrate Corollary 4.2.

Example 4.2. Consider the following robust semidefinite linear programming with uncertainty data:

$$(P_2) \qquad \begin{array}{ll} \text{Minimize} & x_1 + x_2, \\ \text{subject to} & \begin{pmatrix} u_2x_2 & u_1x_1 & 0 \\ u_1x_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0, \end{array}$$

where

$$\mathcal{U}_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_0 \end{pmatrix} : u_0 = 1 \right\}, \; \mathcal{U}_1 = \left\{ \begin{pmatrix} 0 & u_1 & 0 \\ u_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : u_1 \in [-1, 1] \right\}$$

$$\text{and } \mathcal{U}_2 = \left\{ \begin{pmatrix} u_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : u_2 \in [0, 1] \right\}.$$

Thus, the feasible solution for problem (P_2) is $\{(0, x_2) : x_2 \geq 0\}$, and so the optimal valued is 0.

The Lagrangian dual problem for problem (P_2) is defined as follows:

$$\begin{aligned} (D_2) \qquad & \max_{(\Lambda,Q_0,Q_1,Q_2)\in\mathbb{S}^3_+\times\prod_{i=0}^2\mathcal{U}_i} \{-\text{tr}\,[Q_0\Lambda] : (\text{tr}[Q_1\Lambda], \text{tr}[Q_2\Lambda]) = (1, 1)\} \\ &= \max_{(\Lambda,Q_0,Q_1,Q_2)\in\mathbb{S}^3_+\times\prod_{i=0}^2\mathcal{U}_i} \{-\lambda_{33} : (2u_1\lambda_{12}, u_2\lambda_{11}) = (1, 1)\} \\ &= 0, \end{aligned}$$

where $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}$. Hence, strong duality holds. For any $\varepsilon \geq 0$, the ε-approximate solution is $\{(0, z_2) : 0 \leq z_2 \leq \varepsilon\}$. If $\varepsilon = 2$, then

$$Sol_2(D_2) = \left\{ (\Lambda, Q_0, Q_1, Q_2) \in \mathbb{S}^3_+ \times \prod_{i=0}^2 \mathcal{U}_i : (2u_1\lambda_{12}, u_2\lambda_{11}) = (1, 1) \text{ and } \lambda_{33} \in [0, 2] \right\},$$

and

$$RKT_2(D_2) = \left\{ (\Lambda, Q_0, Q_1, Q_2) \in \mathbb{S}^3_+ \times \prod_{i=0}^2 \mathcal{U}_i : (2u_1\lambda_{12}, u_2\lambda_{11}) = (1, 1) \text{ and } \lambda_{33} \in [0, 2] \right\}.$$

Then,

$$\begin{aligned} Sol_2(P_2) &= \bigcup_{(\Lambda,Q_0,Q_1,Q_2)\in RKT_2} \bigcup_{0\le\varepsilon'\le 2} \left\{ z\in\mathcal{F} \,\middle|\, \begin{array}{l} (1,1)=(2u_1\lambda_{12},u_2\lambda_{11}) \text{ and} \\ u_2\lambda_{11}z_2+2u_1\lambda_{12}z_1+\lambda_{33}=2-\varepsilon' \end{array} \right\} \\ &= \bigcup_{(\Lambda,Q_0,Q_1,Q_2)\in RKT_2} \bigcup_{0\le\varepsilon'\le 2} \left\{ (0,z_2)\in\mathcal{F} \,\middle|\, \begin{array}{l} (1,1)=(2u_1\lambda_{12},u_2\lambda_{11}) \text{ and} \\ 0\le z_2=2-\varepsilon'-\lambda_{33} \end{array} \right\} \\ &= \{(0,z_2) : 0\le z_2\le 2\}. \end{aligned}$$

5. CONCLUSIONS

This article is focused on the characterization of ε-approximate solutions for convex semidefinite programming problems that involve uncertainty data. The paper begins by reviewing essential findings related to the optimality condition and duality of robust convex semidefinite programming problems. It then establishes the optimality and duality conditions for the problem by assuming specific constraint qualifications. The study investigates ε-Kuhn-Tucker vectors and their relationships with optimal solutions, maximizers of the corresponding Lagrangian dual problem, saddle points of the Lagrangian, and Kuhn-Tucker vectors. Finally, the article describes the characterization of ε-approximate solution sets for the problem, analyzing the connection between three sets: the set of Lagrange multipliers corresponding to ε-approximate solutions, the set of ε-Kuhn-Tucker vectors, and the set of approximate solutions for their Lagrangian dual problems. This special characterization is seen due to the semidefinite structure of robust convex semidefinite programming problems. Additionally, the article examines the practical application of the characterization of ε-approximate solution sets for semidefinite linear programming problems with uncertain data. The characterization is illustrated using several examples.

Ackhowledgement. This research was partially supported by the National Science, Research and Innovation Fund (NSRF) via the Program Management Unit for Human Resources and Institutional Development, Research and Innovation, Thailand, Grant No. B05F640180.

REFERENCES

[1] Ben-Tal, A.; El Ghaoui, L.; Nemirovski, A. Robustness in *Handbook of Semidefinite Programming*(Wolkowicz, H., Saigal, R. and Vandenberghe, L. Ed.), Springer, Boston, 2000, 139-162.

[2] Ben-Tal, A.; Nemirovski, A. Robust solutions of uncertain linear programs. *Oper. Res. Lett.* **25** (1999), no. 1, 1–13.

[3] Boyd, S.; El Ghaoui, L.; Feron, E.; Balakrishnan, V. *Linear matrix inequalities in system and control theory*. SIAM, 1994.

[4] Bujarbaruah, M.; Nair, S. H.; Borrelli, F. A semi-definite programming approach to robust adaptive MPC under state dependent uncertainty in *European Control Conference (ECC)*, IEEE, 2020, 960–965.

[5] Hiriart-Urruty, J.-B. and Lemaréchal, C. *Convex analysis and minimization algorithms I: Fundamentals*. (Vol. 305), Springer science & business media, 1993.

[6] Hiriart-Urruty, J.-B.; Lemaréchal, C. *Convex analysis and minimization algorithms I: Advanced Theory and Bundle Methods*. (Vol. 306). Springer science & business media, 1993.

[7] Jeyakumar, V.; Li, G. Y. Strong duality in robust convex programming: complete characterizations. *SIAM J. Optim.* **20** (2010), no. 6, 3384–3407.

[8] Jeyakumar, V.; Li, G. Y. Characterizing robust set containments and solutions of uncertain linear programs without qualifications. *Oper. Res. Lett.* **38** (2010), no. 3, 188–194.

[9] Jeyakumar, V.; Li, G. Y. Strong duality in robust semi-definite linear programming under data uncertainty. *Optimization* **63** (2014), no. 5, 713–733.

[10] Jeyakumar, V.; Lee, G. M.; Li, G. Y. Characterizing robust solution sets of convex programs under data uncertainty. *J. Optim. Theory Appl.* **164** (2015), 407–435.

[11] Jiao, L. and Lee, J. H. Approximate optimality and approximate duality for quasi approximate solutions in robust convex semidefinite programs. *J. Optim. Theory Appl.* **176** (2018), 74–93.

[12] Kerdkaew, J.; Wangkeeree, R. Characterizing robust weak sharp solution sets of convex optimization problems with uncertainty. *J. Ind. Manag. Optim.* **16** (2020), no. 6, 2651–2673.
[13] Khantree, C.; Wangkeeree, R. On quasi approximate solutions for nonsmooth robust semi-infinite optimization problems. *Carpathian J. Math.* **35** (2019), no. 3, 417–426.
[14] Khantree, C.; Wangkeeree, R. Approximate quasi solutions of multiobjective optimization problems. *Thai J. Math.* **19**, (2021), no. 1, 233–249.
[15] Kerdkaew, J.; Wangkeeree, R.; Lee, G. M. Approximate optimality for quasi approximate solutions in nonsmooth semi-infinite programming problems, using ε-upper semi-regular semi-convexificators. Filomat **34** (2020), no. 6, 2073–2089.
[16] Lee, G.; Lee, J. ε-Duality theorems for convex semidefinite optimization problems with conic constraints. *J. Inequal. Appl.* (2010), 1–13.
[17] Lee, J. H.; Lee, G. M. On approximate solutions for robust convex semidefinite optimization problems. *Positivity*, **22** (2018), no. 3, 845–857.
[18] Li, X.-B.; Wang, S. Characterizations of robust solution set of convex programs with uncertain data. *Optim. Lett.* **12** (2018), 1387–1402.
[19] Lui, K. W. K.; Ma, W.-K.; So, H.-C.; Chan, F. K. W. Semi-definite programming algorithms for sensor network node localization with uncertainties in anchor positions and/or propagation speed. *IEEE Trans. Signal Process.*, **57** (2008), no. 2, 752–763.
[20] Pınar, M. C.; Tütüncü, R. H. Robust profit opportunities in risky financial portfolios. *Oper. Res. Lett.* **33** (2005), no. 4, 331–340.
[21] Scovel, C.; Hush, D.; Steinwart, I. Approximate duality. *J. Optim. Theory Appl.* **135** (2007), 429–443.
[22] Sirichunwijit, T.; Wangkeeree, R. Approximate Optimality Conditions and Approximate Duality Conditions for Robust Multiobjective Optimization Problems. *Thai J. Math.* **20** (2022), no. 1, 121–140.
[23] Sisarat, N.; Wangkeeree, R.; Lee, G. M. Some characterizations of robust solution sets for uncertain convex optimization problems with locally Lipschitz inequality constraints. *J. Ind. Manag. Optim* **16** (2020), 469–493.
[24] Sun, X.-K.; Peng, Z.-Y.; Guo, X.-L. Some characterizations of robust optimal solutions for uncertain convex optimization problems. *Optim. Lett.* **10** (2016), 1463–1478.
[25] Sun, X.-K.; Teo, K. L.; Tang, L. Dual approaches to characterize robust optimal solution sets for a class of uncertain optimization problems. *J. Optim. Theory Appl.* **182** (2019), 984–1000.
[26] Sun, X.-K.; Teo, K. L.; Long, X.-J. Some characterizations of approximate solutions for robust semi-infinite optimization problems. *J. Optim. Theory Appl.* **191** (2021), 281–310.
[27] Tuyen, N. V.; Wen, C.-F.; Son, T. Q. An approach to characterizing ε-solution sets of convex programs. *TOP*, **30** (2022), no. 2, 249–269.
[28] Vandenberghe, L.; Boyd, S. Semidefinite programming. *SIAM Rev.* **38** (1996), no. 1, 49–95.
[29] Wolkowicz, H.; Saigal, R.; Vandenberghe, L. *Handbook of semidefinite programming: theory, algorithms, and applications*. **27**, Springer Science & Business Media, 2012.

[1] Research center for Academic Excellence in Mathematics
Department of mathematics
Faculty of Science
Naresuan University
Phitsanulok, 65000, Thailand
Email address: rabianw@nu.ac.th

[2] Program in mathematics, Faculty of Education
Pibulsongkram Rajabhat University
Phitsanulok, 65000, Thailand
Email address: preechasilpp@gmail.com, pakkapon.p@psru.ac.th