CARPATHIAN J. MATH.
Volume **40** (2024), No. 2,
Pages 559 - 568

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.02.20

Levitin-Polyak well-posedness for generalized (η, g, φ)-mixed vector variational-type inequality

PANU YIMMUANG and PATCHARAPA SRICHOK

ABSTRACT. This study delves into the concept of Levitin-Polyak well-posedness in the context of a generalized vector variational-type inequality problem with parameters (η, g, φ). Our primary objective is to establish a comprehensive set of conditions that can be employed to rigorously assess the attributes of Levitin-Polyak well-posedness, both in its standard form and as a generalized concept. To enhance the clarity and applicability of these conditions, we provide an instructive example that elucidates the underlying assumptions and demonstrates the practical implications of our research. Through this work, we contribute to the understanding and practical implementation of well-posedness in variational inequality problems.

1. INTRODUCTION

In the rich and intricate landscape of variational inequality theory, as well as its extensions to diverse mathematical and optimization problems, this research embarks on a journey to make a substantial and innovative contribution to the realm of well-posedness. Variational inequalities have long been the subject of rigorous examination by a multitude of researchers, as evidenced by the extensive body of work found in the literature [1, 5, 8, 13, 15, 25]. The allure of these inequalities lies in their profound relationship with mathematical programming problems, even under conditions that can be deemed relatively mild. This relationship, in turn, has prompted the natural progression of the concept of Tykhonov well-posedness, evolving it to encompass a wider spectrum of variational inequalities [6, 7, 8, 9, 10, 12], equilibrium problems, fixed point problems, optimization problems, mixed quasivariational-like inequalities with constraints, and numerous other problem classes [14, 16, 19, 24, 26].

Hadamard [11] was a pioneer in the field of optimization, introducing the concept of well-posedness based on the existence and uniqueness of an optimal solution, coupled with the essential notion of the continuous dependence of this optimal solution and the corresponding optimal value on the problem's data. His groundbreaking work laid the foundation for a deeper understanding of optimization problems.

Building upon Hadamard's insights, Tykhonov [23] further expanded the notion of well-posedness in the context of minimization problems. His approach was inspired by numerical methods and emphasized two fundamental aspects. The first was the insistence on the existence and uniqueness of an optimal solution, which served as a cornerstone for well-posedness. The second aspect focused on the convergence of any sequence that seeks to minimize the objective function, ensuring that it ultimately converges to the unique solution. Tykhonov's contributions added rigor and practicality to the concept of well-posedness.

Received: 31.10.2023. In revised form: 24.03.2024. Accepted: 25.03.2024
2020 *Mathematics Subject Classification.* 49K40, 54C60, 90C33, 47H09, 47J20, 54H25.
Key words and phrases. *Generalized (η, g, φ)-mixed vector variational-type inequality problems, Well-posedness, Relaxed η-α_g-P-monotonicity.*
Corresponding author: Panu Yimmuang; panu.y@sut.ac.th

However, it was Levitin and Polyak [18] who introduced a new and more refined perspective on well-posedness. Their innovation strengthened Tykhonov's concept by requiring not only the convergence of minimizing sequences to the optimal solution but also that this convergence holds true for a more extensive set of minimizing sequences. This elevated criterion offered a more robust and versatile framework for understanding well-posedness in optimization.

Additionally, Konsulova and Revalski [17] delved into the realm of Levitin-Polyak (LP) well-posedness, specifically focusing on convex scalar optimization problems with functional constraints. Their work extended the applicability of LP well-posedness, shedding light on its relevance in more complex optimization scenarios.

In summary, the concept of well-posedness in optimization has evolved over time, with Hadamard's initial insights, Tykhonov's numerical emphasis, and Levitin-Polyak's refined definition, each contributing to a deeper and more comprehensive understanding of this crucial notion. Konsulova and Revalski's research further extended the reach of LP well-posedness, making it applicable to a broader range of optimization problems, particularly those involving convex scalar optimization with functional constraints.

In the annals of this field, a significant milestone was reached in the year 2000 when Lignola and Morgan [21] introduced the innovative concept of parametric well-posedness. This concept is tailored to optimization problems burdened with variational inequality constraints and relies on the elegant idea of approximating sequences. Subsequently, Lignola [20] embarked on a comprehensive exploration of well-posedness, L-well-posedness, and metric characterizations of well-posedness, a vital contribution that unlocked new insights into the subtleties of quasi-variational-inequality problems.

The journey of extending these conceptual boundaries did not halt there. Ceng and Yao [3] further extended these pioneering concepts to unveil the conditions under which generalized mixed variational inequality problems, a class of problems with broad applications, could be considered well-posed. Their work was instrumental in expanding the frontiers of well-posedness theory.

Within the broader panorama, Lin and Chuang [22] made notable strides by establishing well-posedness not only for variational inclusion problems but also for optimization problems that are graced with variational inclusion and scalar equilibrium constraints. Their work introduced a generalized sense of well-posedness, elucidating key facets of these interconnected problems.

In the year 2010, Fang and his collaborators [10] introduced yet another dimension to the concept of well-posedness. Their work was notable for extending the notion of well-posedness by perturbations, addressing a mixed variational inequality problem within the sophisticated setting of a Banach space. These developments marked significant progress in the quest to understand the depth and breadth of well-posedness within variational inequality theory.

Most recently, Ceng and his research team [2] have presented invaluable contributions by elucidating the conditions necessary for well-posedness in hemivariational inequality problems. Their work encompasses the involvement of Clarkes generalized directional derivative and various types of monotonicity assumptions, providing a sophisticated lens through which well-posedness can be understood in the context of variational inequalities.

Very Recently, Chang et al. [4] focused on the well-posedness for a generalized (η, g, φ)-mixed vector variational-type inequality and optimization problems with a constraint. They established a metric characterization of well-posedness in terms of an approximate solution set. Also they proved that well-posedness of optimization problem was closely related to that of generalized (η, g, φ)-mixed vector variational-type inequality problems

In light of this dynamic and evolving landscape, our research endeavors to contribute a novel perspective by investigating the Levitin-Polyak well-posedness of a particular problem. This problem has previously undergone rigorous examination and scrutiny in the context of well-posedness, as explored in another publication. By exploring Levitin-Polyak well-posedness, our work builds upon the foundations laid by these esteemed researchers, aiming to advance the understanding of well-posedness within variational inequality problems and bring new insights to this ongoing discourse.

2. Preliminaries

Consider X and Y as two real Banach spaces. Assume that D is a non-empty closed convex subset of X, and P is a closed convex cone with non-empty interior in Y while also being proper. Throughout this paper, we will employ the following inequalities. For all $x, y \in Y$:

(i) $x \leq_P y \Leftrightarrow y - x \in P$;
(ii) $x \nleq_P y \Leftrightarrow y - x \notin P$;
(iii) $x \leq_{P^0} y \Leftrightarrow y - x \in P^0$;

where P^0 denotes the interior of P.

If $\leq_P$ is a partial order, then $(Y, \leq_P)$ is called an ordered Banach space ordered by P. Let $T : X \to 2^{L(X,Y)}$ be a set-valued mapping where $L(X,Y)$ denotes the space of all continuous linear mappings from X into Y. Assume that $Q : L(X,Y) \times D \to L(X,Y), \varphi : D \times D \to Y, \eta : X \times X \to X$ are bi-mappings and $g : D \to D$ is single-valued mapping.

We consider the following generalized (η, g, φ)-mixed vector variational-type inequality problem for finding $x \in D$ and $u \in T(x)$ such that

$$\big\langle Q(u,x), \eta\big(y, g(x)\big)\big\rangle + \varphi\big(g(x), y\big) \nleq_{P^0} 0, \forall y \in D. \tag{2.1}$$

Denote the solution set of the problem (2.1) by

$$\Omega = \{x \in D : \exists u \in T(x) \text{ such that } \big\langle Q(u,x), \eta\big(y, g(x)\big)\big\rangle + \varphi\big(g(x), y\big) \nleq_{P^0} 0, \forall y \in D\}.$$

Definition 2.1. A set-valued mapping $T : D \to 2^{L(X,Y)}$ is said to be monotone with respect to the first variable of Q, if

$$\big\langle Q(u,\cdot) - Q(v,\cdot), x - y\big\rangle \geq_p 0, \ \forall x, y \in D, u \in T(x), v \in T(y).$$

Definition 2.2. Let $g : D \to D$ be a single-valued mapping. A set-valued mapping $T : D \to 2^{L(X,Y)}$ is said to be relaxed η-α_g-P-monotone with respect to the first variable of Q and g, if

$$\big\langle Q(u,\cdot) - Q(v,\cdot), \eta\big(g(x), y\big)\big\rangle - \alpha_g(x - y) \geq_p 0, \ \forall x, y \in D, u \in T(x), v \in T(y),$$

where $\alpha_g : X \to Y$ is a mapping such that $\alpha_g(tz) = t^p \alpha_g(z), \forall t > 0, z \in X,$ and $p > 1$ is a constant.

Definition 2.3. A mapping $\gamma : X \times X \to X$ is said to be affine with respect to the first variable if, for any $x_i \in D$ and $\lambda_i \geq 0 (1 \leq i \leq n)$ with $\sum_{i=0}^{n} \lambda_i = 1$ and for any $y \in D$,

$$\gamma\left(\sum_{i=0}^{n} \lambda_i x_i, y\right) = \sum_{i=1}^{n} \lambda_i \gamma(x_i, y).$$

Lemma 2.1. *Let (Y, P) be an ordered Banach space with closed convex pointed cone P and $P^0 \neq \emptyset$. Then, for all $x, y, z \in Y$, we have*

(i) $z \nleq_{P^0} x, x \geq_P y \Rightarrow z \nleq_{P^0} y$;

(ii) $z \not\geq_{P^0} x, x \leq_P y \Rightarrow z \not\geq_{P^0} y$.

Lemma 2.2. *Let $(X, \|\cdot\|)$ be a normed linear space and $\mathfrak{H}$ be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X induced by metric*

$$d(u,v) = \|u - v\|,$$

which is defined by

$$\mathfrak{H}(A,B) = max\left\{\sup_{u\in A}\inf_{v\in B}\|u-v\|, \sup_{v\in B}\inf_{u\in A}\|u-v\|\right\}, \forall A, B \in CB(X).$$

If A, B are compact sets in X, then for each $u \in A$ there exists $v \in B$ such that

$$\|u - v\| \leq \mathfrak{H}(A,B).$$

Definition 2.4. A set-valued mapping $T : D \to 2^{L(X,Y)}$ is said to be $\mathfrak{H}$-hemicontinuous, if

$$\mathfrak{H}\left(T\big(x + \tau(y-x)\big), T(x)\right) \to 0 \text{ as } \tau \to 0^+, \forall x, y \in D, \tau \in (0,1),$$

where $\mathfrak{H}$ is a Hausdorff metric defined on $CB(L(X,Y))$.

In the context of real Banach spaces, the following lemma explores a scenario where D is a closed convex subset of a Banach space X, and Y is another Banach space equipped with a nonempty closed convex pointed cone P having its apex at the origin, with $P^0 \neq \emptyset$.

Lemma 2.3. [4] *Consider a closed convex subset D of a real Banach space X, where Y is another real Banach space equipped with a nonempty closed convex pointed cone P having its apex at the origin, and $P^0 \neq \emptyset$. Let $Q : L(X,Y) \to L(X,Y)$ be a continuous mapping, and $T : D \to 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Provided the subsequent conditions hold:*

- *(i) $\varphi : D \times D \to Y$ is a P-convex in the second variable with $\varphi(x,x) = 0, \forall x \in D$;*
- *(ii) $\eta : X \times X \to X$ is an affine mapping in the first variable with $\eta(x,x) = 0, \forall x \in D$;*
- *(iii) $T : D \to 2^{l(X,Y)}$ is $\mathfrak{H}$-hemicontinuous and relaxed η-α-P-monotone with respect to Q;*

then the following two problems are equivalent:

- *(a) there exist $x_0 \in D$ and $u_0 \in T(x_0)$ such that*
  $\langle Q(u_0), \eta(y, x_0)\rangle + \varphi(x_0, y) \not\leq_{P^0} 0,\ \forall y \in D.$
- *(b) there exists $x_0 \in D$ such that*
  $\big\langle Q(v), \eta(y, x_0)\big\rangle + \varphi(x_0, y) - \alpha(x_0 - y) \not\leq_{P^0} 0,\ \forall y \in D, v \in T(y).$

3. Main results

Definition 3.5. A sequence x_n is said to be a Levitin-Polyak approximating sequence for problem (2.1) if, there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\varepsilon_n \to 0$ such that $d(x_n, D) \leq \varepsilon_n$ and

$$\Big\langle Q(u_n, x_n), \eta\big(y, g(x_n)\big)\Big\rangle + \varphi\big(g(x_n), y\big) + \varepsilon_n e \not\leq_{P^0} 0, \forall y \in D, e \in \text{int}P.$$

Definition 3.6. The generalized (η, g, φ)-mixed vector variational-type inequality and optimization problems is said to be Levitin-Polyak well-posed if

- (i) there exists a unique solution x_0 of problem (2.1);
- (ii) every Levitin-Polyak approximating sequence of problem (2.1) converges to x_0.

Definition 3.7. The generalized (η, g, φ)-mixed vector variational-type inequality and optimization problems is said to be generalized Levitin-Polyak well-posed if

- (i) the solution set Ω of problem (2.1) is nonempty;
- (ii) every Levitin-Polyak approximating sequence has a subsequence that converges to some point of Ω.

We denote the Levitin-Polyak approximate solution set of problem 2.1 by

$$\Omega_\varepsilon = \left\{ \begin{array}{l} x : \exists u \in T(x) \text{ such that } d(x, D) \leq \varepsilon \text{ and} \\ \big\langle Q(u,x), \eta\big(y, g(x)\big)\big\rangle + \varphi\big(g(x), y\big) + \varepsilon e \nleq_{P^0} 0, \forall y \in D, \varepsilon \geq 0. \end{array} \right\}$$

Within the framework of continuous mappings and continuous functions, this theorem addresses a key property related to problem (2.1) when specific conditions laid out in Lemma 2.3 are satisfied. The theorem establishes a fundamental characterization, demonstrating that problem (2.1) exhibits Levitin-Polyak well-posedness if and only if two critical conditions hold: firstly, the set Ω_ε is nonempty for all $\varepsilon > 0$, and secondly, the diameter of Ω_ε approaches zero as ε tends to zero.

Theorem 3.1. *Let $g : D \to D$ and $Q : L(X,Y) \times D \to L(X,Y)$ be two continuous mappings. Let $\varphi(\cdot, y), \eta(y, \cdot)$ and α_g be continuous functions for all $y \in D$. If the conditions in Lemma 2.3 are satisfied, then problem (2.1) is Levitin-Polyak well-posed if and only if*

$$\Omega_\varepsilon \neq \emptyset,\ \forall \varepsilon > 0$$

and

$$\text{diam}\, \Omega_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Proof. Let problem (2.1) is Levitin-Polyak well-posed, then it has a unique solution $x_0 \in \Omega$. Since $\Omega \subseteq \Omega_\varepsilon$, for all $\varepsilon > 0$, This implies that

$$\Omega_\varepsilon \neq \emptyset, \ \ \forall \varepsilon > 0.$$

On the contrary, if

$$\text{diam}\Omega_\varepsilon \not\to 0 \text{ as } \varepsilon \to 0,$$

then there exist $r > 0, m$(a positive integer), and a sequence $\{\varepsilon_n > 0\}$ with $\varepsilon_n \to 0$ and $x_n, x_n^{'} \in \Omega_{\varepsilon_n}$ such that

$$\|x_n - x_n^{'}\| > r,\ \forall n \geq m. \tag{3.2}$$

Since $x_n, x_n^{'} \in \Omega_{\varepsilon_n}$, there exist $u_n \in T(x_n)$ and $u_n^{'} \in T(x_n^{'})$ such that

$$d(x_n, D) \leq \varepsilon_n \text{ and } \big\langle Q(u_n, x_n), \eta\big(y, g(x_n)\big)\big\rangle + \varphi\big(g(x_n), y\big) + \varepsilon_n e \nleq_{P^0} 0,\ \forall y \in D,$$
$$d(x_n^{'}, D) \leq \varepsilon_n \text{ and } \big\langle Q(u_n^{'}, x_n^{'}), \eta\big(y, g(x_n^{'})\big)\big\rangle + \varphi\big(g(x_n^{'}), y\big) + \varepsilon_n e \nleq_{P^0} 0,\ \forall y \in D.$$

So $\{x_n\}$ and $\{x_n^{'}\}$ are Levitin-Polyak approximating sequences of problem (2.1).
Since the problem is Levitin-Polyak well-posed, the Levitin-Polyak approximating sequences $\{x_n\}$ and $\{x_n^{'}\}$ of problem (2.1) converge to x_0. Therefore we have

$$\|x_n - x_n^{'}\| = \|x_n - x_0 + x_0 - x_n^{'}\| \leq \|x_n - x_0\| + \|x_n^{'} - x_0\| \leq \varepsilon,$$

which contradicts to (3.2), for some $\varepsilon = r$.

Conversely, let $\Omega_\varepsilon \neq \emptyset, \forall \varepsilon > 0$ and diam $\Omega_\varepsilon \to 0$ as $\varepsilon \to 0$. Assume that $\{x_n\}$ is a Levitin-Polyak approximating sequence of problem (2.1). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\varepsilon_n \to 0$ such that $d(x_n, D) \leq \varepsilon_n$ and

$$\big\langle Q(u_n, x_n), \eta\big(y, g(x_n)\big)\big\rangle + \varphi\big(g(x_n), y\big) + \varepsilon_n e \nleq_{P^0} 0,\ \forall y \in D, \tag{3.3}$$

which implies that $x_n \in \Omega_{\varepsilon_n}$. Since diam $\Omega_{\varepsilon_n} \to 0$ as $\varepsilon_n \to 0$, so $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, we have $\{x_n\}$ converges to some $x_0 \in X$. Since $d(x_n, D) \leq \varepsilon_n$, we can choose $x_n^{'} \in D$ so that

$$\|x_n - x_n^{'}\| \leq \varepsilon_n$$

which implies that $x_n^{'} \to x_0$. Since D is closed, so $x_0 \in D$.

Again since T is relaxed η-α_g-P-monotone with respect to the first variable of Q and g on D, it follows from Definition 2.2, for any $y \in D$ and $u \in T(y)$, we have

$$\begin{aligned}\big\langle Q(u_n,x_n),\eta\big(y,g(x_n)\big)\big\rangle &+ \varphi\big(g(x_n),y\big)\\ \le_P \big\langle Q(u,x_n),\eta\big(y,g(x_n)\big)\big\rangle &+ \varphi\big(g(x_n),y\big) - \alpha_g(y-x_n).\end{aligned} \tag{3.4}$$

From the continuity of g, φ, η and α_g, we have

$$\begin{aligned}&\big\langle Q(u,x_0),\eta\big(y,g(x_0)\big)\big\rangle + \varphi\big(g(x_0),y\big) - \alpha_g(y-x_0)\\ &= \lim_{n\to\infty}\{\big\langle Q(u,x_n),\eta\big(y,g(x_n)\big)\big\rangle + \varphi\big(g(x_n),y\big) - \alpha_g(y-x_n)\}.\end{aligned}$$

This together with (3.4) shows that

$$\begin{aligned}&\big\langle Q(u,x_0),\eta\big(y,g(x_0)\big)\big\rangle + \varphi\big(g(x_0),y\big) - \alpha_g(y-x_0)\\ &\ge_P \lim_{n\to\infty}\{\big\langle Q(u_n,x_n),\eta\big(y,g(x_n)\big)\big\rangle + \varphi\big(g(x_n),y\big)\}.\end{aligned} \tag{3.5}$$

Taking the limit in (3.3), we have

$$\lim_{n\to\infty}\{\big\langle Q(u_n,x_n),\eta\big(y,g(x_n)\big)\big\rangle + \varphi\big(g(x_n),y\big)\} \not\le_{P^0} 0. \tag{3.6}$$

Combining (3.5) and (3.6) and using Lemma 2.1(ii), we get

$$\big\langle Q(u,x_0),\eta\big(y,g(x_0)\big)\big\rangle + \varphi\big(g(x_0),y\big) - \alpha_g(y-x_0) \not\le_{P^0} 0.$$

By Lemma 2.3, there exist $x_0 \in D$ and $u_0 \in T(x_0)$ such that

$$\big\langle Q(u_0,x_0),\eta\big(y,g(x_0)\big)\big\rangle + \varphi\big(g(x_0),y\big) \not\le_{P^0} 0,\ \forall y \in D,$$

which implies that $x_0 \in \Omega$. It remains to prove that x_0 is a unique solution of the problem (2.1). Contrary, let x_1 and x_2 are two distinct solutions of problem (2.1).
By

$$\text{diam}\Omega_\varepsilon = \sup_{x_1,x_2\in\Omega_\varepsilon}\|x_1-x_2\| \ge \|x_1-x_2\|$$

and diam $\Omega_\varepsilon \to 0$ as $\varepsilon \to 0$, we can get that

$$0 < \|x_1-x_2\| \le \text{diam}\Omega_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

So x_0 is a unique solution of problem (2.1). This is the proof is completed. □

In the context of continuous mappings and continuous functions, the following theorem addresses the well-posedness of problem (2.1) when the specified conditions outlined in Lemma 2.3 are met. This theorem establishes a fundamental equivalence, demonstrating that problem (2.1) exhibits Levitin-Polyak well-posedness if and only if it possesses a unique solution.

Theorem 3.2. *Let $g : D \to D$ and $Q : L(X,Y) \times D \to L(X,Y)$ be two continuous mappings. Let $\varphi(\cdot,y), \eta(y,\cdot)$ and α_g be continuous functions for all $y \in D$. If the conditions in Lemma 2.3 are satisfied. Then problem (2.1) is Levitin-Polyak well-posed if and only if it has a unique solution.*

Proof. By the definition, we know that Levitin-Polyak well-posedness for problem (2.1) implies that it has a unique solution.

Conversely, suppose that the problem (2.1) has a unique solution x_0. Let $\{\lambda_n\}$ be a sequence in X which converges to $\bar{\lambda}$. Let $\{x_n\}$ be an Levitin-Polyak approximating sequence with respect to $\{\lambda_n\}$. Then there exist $u_n \in T(x_n)$ and a sequence of positive real number $\varepsilon_n \to 0$ suct that

$$d(x_n, D) \le \varepsilon_n \tag{3.7}$$

and

$$\big\langle Q(u_n,x_n),\eta\big(y,g(x_n)\big)\big\rangle + \varphi\big(g(x_n),y\big) + \varepsilon_n e \not\le_{P^0} 0,\ \forall y \in D. \tag{3.8}$$

Using (3.7) and the closedness of D in X, for each positive integer n, we can choose $x_n^{'} \in D$ so that

$$\|x_n - x_n^{'}\| \leq \varepsilon_n \tag{3.9}$$

Since X is a compact set, the sequence $\{x_n^{'}\}$ has a subsequence $\{x_{n_l}^{'}\}$ which converges to a point $\bar{x} \in X$. Using (3.9), we conclude that the corresponding subsequence $\{x_{n_l}\}$ of $\{x_n\}$ converges to $\bar{x}$.
Again D is closed set, it follows that $\bar{x} \in D$. Proceeding along the lines of converse part in the proof of Theorem 3.1, we can show that $\bar{x} \in \Omega$.
Consequently, $\bar{x}$ coincides with x_0 ($\bar{x} = x_0$). Again, by the uniqueness of the solution, it is obvious that every possible subsequence converges to the unique solution x_0 and hence the whole sequence $\{x_n\}$ converges to x_0.
So the Levitin-Polyak well-posedness of problem (2.1) is satisfied. □

To shed further light on the concepts explored in our research, we present a practical example that underscores the significance of well-posedness within the framework of variational inequalities. In this example, we define and analyze the key components of the problem, its constraints, and the associated functions. This specific example serves as a tangible demonstration of the application of our findings and showcases how the notion of Levitin-Polyak well-posedness can be illustrated in a real-world context. Through this example, we aim to highlight the vital role of well-posedness in guaranteeing unique and meaningful solutions to mathematical and optimization problems.

Example 3.1. Let $X = Y = \mathbb{R}, D = [0, 1]$ and $P = [0, \infty)$. Let us define the mappings $T : D \to 2^{L(X,Y)}, \varphi : D \times D \to Y, \eta : X \times X \to X,$ and $Q : L(X, Y) \times D \to L(X, Y)$ as follows:

$$\begin{cases} T(x) = \{u : \mathbb{R} \to \mathbb{R} | u \text{ is a continuous linear mapping such that } u(x) = x\}; \\ g(x) = x; \\ \varphi\big(g(x), y\big) = y - x; \\ \eta\big(y, g(x)\big) = \frac{1}{2}(x - y); \\ Q(v, y) = v; \\ \alpha_g = -x^2. \end{cases}$$

In this case, the generalized (η, g, φ)- mixed vector variational-type inequality problem (2.1) is to find $x \in D$ and $u \in T(x)$ such that

$$\left\langle u, \frac{1}{2}(x - y) \right\rangle + y - x \nleq_{P^0} 0, \ \forall y \in D. \tag{3.10}$$

It easy to see that $\Omega = \{0\}$. Again since T is relaxed η-α_g-P-monotone with respect to the first variable of Q and g, and all conditions in Theorem 3.2 are satisfied. Therefore the problem (3.10) is Levitin-Polyak well-posed.

Theorem 3.3. *Suppose that all the conditions in Lemma 2.3 are satisfied. Further, assume that D is a compact set and $g, \varphi(\cdot, y), \eta(y, \cdot)$, α_g are continuous functions for all $y \in D$. Then problem (2.1) is generalized Levitin-Polyak well-posed if and only if the solution set Ω is nonempty.*

Proof. Suppose that problem (2.1) is generalized Levitin-Polyak well-posed it follows that $\Omega \neq \emptyset$. Conversely, let $\{\lambda_n\}$ be a sequence in X converging to $\bar{\lambda}$ and $\{x_n\}$ be an Levitin-Polyak approximating sequence with respect to $\{\lambda_n\}$. Then there exist $u_n \in T(x_n)$ and a sequence of positive real number $\varepsilon_n \to 0$ suct that

$$d(x_n, D) \leq \varepsilon_n \tag{3.11}$$

and

$$\big\langle Q(u_n, x_n), \eta\big(y, g(x_n)\big)\big\rangle + \varphi\big(g(x_n), y\big) + \varepsilon_n e \nleq_{P^0} 0, \ \forall y \in D. \tag{3.12}$$

Using (3.11) and the closedness of D in X, for each positive integer n, we can choose $x_n^{'} \in D$ so that

$$\|x_n - x_n^{'}\| \leq \varepsilon_n \tag{3.13}$$

Since D is a compact set, there exists a subsequence $\{x_{n_l}^{'}\}$ of $\{x_n^{'}\}$ converging to $\bar{x} \in D$. From (3.13), we conclude that the corresponding subsequence $\{x_{n_l}\}$ of $\{x_n\}$ converges to $\bar{x} \in D$.

Proceeding along the lines of converse part in the proof of Theorem 3.1, we can show that $\bar{x} \in \Omega$. The proof is completed. □

As we delve deeper into the study of well-posedness in the context of variational inequalities and their extensions, it is often insightful to examine specific instances and cases. In this regard, we present a practical example that illustrates the concepts discussed in our research. This example provides a concrete application of the generalized (η, g, φ)-mixed vector variational-type inequality problem (2.1). We define the problem, its constraints, and relevant functions to demonstrate the application of our findings. Through this example, we aim to showcase the effectiveness and relevance of our well-posedness criteria in real-world scenarios.

Example 3.2. Let $X = Y = \mathbb{R}^2, D = [0,1] \times [0,1]$ and $P = [0, \infty) \times [0, \infty)$. Let us define the mappings $T : D \to 2^{L(X,Y)}, \varphi : D \times D \to Y, \eta : X \times X \to X,$ and $Q : L(X,Y) \times D \to L(X,Y)$ as follows:

$$\begin{cases} T(x) = \{w, z : \mathbb{R}^2 \to \mathbb{R} | \text{w,z are a continuous linear mapping such that} \\ \quad w(x_1, x_2) = x_1, z(x_1, x_2) = x_2\}; \\ g(x) = x; \\ \varphi\big(g(x), y\big) = x - y; \\ \eta\big(y, g(x)\big) = y - x; \\ Q(u, x) = u; \\ \alpha_g = 0. \end{cases}$$

In this case, the generalized (η, g, φ)- mixed vector variational-type inequality problem (2.1) is to find $x \in D$ and $u \in T(x)$ such that

$$\big\langle u, y - x\big\rangle + x - y \nleq_{P^0} 0, \forall y \in D. \tag{3.14}$$

Clearly, $\Omega = [0,1] \times [0,1]$. It can be easily verified that T is relaxed η-α_g-P-monotone with respect to the first variable of Q and g, and all conditions in Theorem 3.3 are satisfied. Hence, problem (3.14) is generalized Levitin-Polyak well-posed.

4. CONCLUSIONS

In conclusion, our study has introduced and thoroughly examined the concept of Levitin-Polyak well-posedness within the context of a generalized vector variational-type inequality problem with parameters (η, g, φ). We have successfully established a set of sufficient conditions that allow for the assessment of both Levitin-Polyak well-posedness and generalized Levitin-Polyak well-posedness in this problem domain. Through an illustrative example, we have clarified the practical implications of these conditions and their role in understanding the underlying assumptions. This research contributes to the field by providing a framework for analyzing and characterizing well-posedness in vector variational-type inequality problems, with potential applications in optimization and decision-making processes.

ACKNOWLEDGMENTS

We would like to give manythanks for support from the School of Mathematics at SUT and Department of Mathematics at Naresuan University.

REFERENCES

[1] Ceng, L. C.; Hadjisavvas, N.; Schaible, S.; Yao, J. C. Well-posedness for mixed quasivariational-like inequalities. *J. Optim.Theory Appl.* **139** (2008), 109–125.

[2] Ceng, L. C.; Liou, Y. C.; Wen, C. F. Some equivalence results for well-posedness of generalized hemivariational inequalities with Clarke's generalized directional derivative. *J. Nonlinear Sci. Appl.* **9** (2016), 2798–2812.

[3] Ceng, L. C.; Yao, J. C. Well-posedness of generalized mixed variational inequalities, inclusion problems and fixed-point problems. *Nonlinear Anal.* **69** (2008), 4585–4603.

[4] Chang, S. S.; Salahuddin, Wang, L.; Wang, X. R.; Zhao, L. C. Well-posedness for generalized (η, g, φ)-mixed vector variational-type inequality and optimization problems. *J. Inequal. Appl.* (2019), 1–16.

[5] Chang, S. S.; Wen, C. F.; Wang, X. R. On the existence problem of solutions to a class of fuzzy mixed exponential vector variational inequalities. *J. Nonlinear Sci. Appl.* **11** (2018), 916–926.

[6] Fang, Y. P.; Hu, R. Parametric well-posedness for variational inequalities defined by bifunctions. *Comput. Math. Appl.* **53** (2007), 1306–1316.

[7] Fang, Y. P.; Hu, R.; Huang, N. J. Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints. *Comput. Math. Appl.* **55** (2008), 89–100.

[8] Fang, Y. P.; Huang, N. J. Variational-like inequalities with generalized monotone mappings in Banach spaces. *J. Optim. Theory Appl.* **118** (2003), 327–338.

[9] Fang, Y. P.; Huang, N. J.; Yao, J. C. (2008). Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems. *J. Global Optim.* **41** (2008), 117–133.

[10] Fang, Y. P.; Huang, N. J.; Yao, J. C. Well-posedness by perturbations of mixed variational inequalities in Banach spaces. *European J. Oper. Res.* **201** (2010), 682–692.

[11] Hadamard, J.; Sur les problèmes aux dérivées partielles et leur signification physique. *Princet. Univ. Bull.* (1902), 49–52.

[12] Huang, X. X.; Yang, X. Q.; Zhu, D. L. Levitin–Polyak well-posedness of variational inequality problems with functional constraints. *J. Global Optim.* **44** (2009), 159–174.

[13] Kim, J. K.; Salahuddin, G. Existence of solutions for multi-valued equilibrium problems. *Nonlinear Funct. Anal. Appl.* **23** (2018), 779–795.

[14] Kim, J. K.; Salahuddin, G.; Hyun, H. G. Well-posedness for parametric generalized vector equilibrium problems. *Far East J. Math. Sci.* **101** (2017), 2245–2269.

[15] Kim, S. H.; Lee, B. S. Salahuddin. Fuzzy variational inclusions with (H, ϕ, ψ)-η-monotone mappings in Banach Spaces. *J. Adv. Research Appl. Math.* (2012), 10–22.

[16] Kimura, K.; Liou, Y.; Wu, S.; Yao, J. Well-posedness for parametric vector equilibrium problems with applications. *J. Ind. Manag. Optim.* **4** (2008), 313–327.

[17] Konsulova, A.S.; Revalski, J.P. Constrained convex optimization problems-well-posedness and stability. *Numer. Funct. Anal. Optim.* **15** (1994), 889–907.

[18] Levitin, E. S.; Polyak, B. T. Convergence of minimizing sequences in conditional extremum problems. *Soviet Math. Dokl.* (1966), no. 5, 764–767.

[19] Li, X. B.; Agarwal, R. P.; Cho, Y. J.; Huang, N. J. The well-posedness for a system of generalized quasi-variational inclusion problems. *J. Inequal. Appl.* **2014** (2014), 1–15.

[20] Lignola, M. B. Well-posedness and L-well-posedness for quasivariational inequalities. *J. Optim. Theory Appl.* **128** (2006), 119–138.

[21] Lignola, M. B.; Morgan, J. Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution. *J. Global Optim.* **16** (2000), 57–67.

[22] Lin, L. J.; Chuang, C. S. Well-posedness in the generalized sense for variational inclusion and disclusion problems and well-posedness for optimization problems with constraint. *Nonlinear Anal.* **70** (2009), 3609–3617.

[23] Tykhonov, A. N. On the stability of functional optimization problems. *USSR Comput. Math. Math. Phys.* (1966), 28–33.

[24] Verma, R. U.; Salahuddin. Wel lposed generalized vector quasi equilibrium problems. *Commun. Appl. NonlinearAnal.* (2015), 90–102.

[25] Zeng, L. C.; Yao, J. C. Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces. *J. Global Optim.* **36** (2006), 483–497.

[26] Zolezzi, T. Extended well-posedness of optimization problems. *J. Optim. Theory Appl.* **91** (1996), 257–266.

School of Mathematics
Institute of Science
Suranaree University of Technology
Nakhon Ratchasima 30000,Thailand
Email address: panu.y@sut.ac.th