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Oscillation of second-order functional differential equations with mixed argument

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ABSTRACT. In this paper we introduce new effective technique for investigation of oscillation for second-order differential equation with mixed argument

$$y''(t) = p(t)y(\tau(t)). \tag{E}$$

Our criteria improves the existing ones and the progress is illustrated via several examples.

1. INTRODUCTION

This paper is concerned with oscillatory behavior of linear functional differential equation of the form

$$y''(t) = p(t)y(\tau(t)), \tag{E}$$

where the following conditions are assumed to hold

(H_1) $p(t) \in C([t_0, \infty))$, $p(t) > 0$,
(H_2) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

As usually, by a proper solution of Eq. (E) we mean a function $y : [T_y, \infty) \to R$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (E) does possess proper solutions.

The oscillatory character of the solutions is understood in the standard way, that is, a proper solution is termed oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros. There are numerous papers devoted to oscillation of differential equations, see e.g. [1]–[20].

If $y(t)$ is a nonoscillatory solution of (E), then there exist a number $\ell \in \{0, 2\}$ such that

$$\begin{aligned} y(t)y^{(i)}(t) &> 0 \quad \text{for } 0 \leq i \leq \ell, \\ (-1)^i y(t)y^{(i)}(t) &> 0 \quad \text{for } \ell \leq i \leq n. \end{aligned} \tag{1.1}$$

Such a $y(t)$ is said to be a (nonoscillatory) solution of degree ℓ and the totality of solutions of degree ℓ is denote by $\mathcal{N}_\ell$. If we denote the set of all nonoscillatory solutions of (E) by $\mathcal{N}$, then we have

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2.$$

It is known that in the case where $\tau(t) \equiv t$ Eq. (E) always has solutions of degree 0 and 2, that is, $\mathcal{N}_0 \neq \emptyset$ and $\mathcal{N}_2 \neq \emptyset$. The situation for (E) with $\tau(t) \not\equiv t$ is different. In fact, it may happen that $\mathcal{N}_0 = \emptyset$ or $\mathcal{N}_2 = \emptyset$ when the deviating argument $\tau(t)$ is retarded $(\tau(t) \leq t)$ or advanced $(\tau(t) \geq t)$ and the deviation $|t - \tau(t)|$ is large enough, see Ladas et al. [20] or Koplatadze and Chanturia [12] who formulated the following results:

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Theorem A. *If $\tau(t) \le t$ and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (s-\tau(t))p(s)\,\mathrm{d}s > 1, \tag{1.2}$$

then $\mathcal{N}_0 = \emptyset$ for (E).

Theorem B. *If $\tau(t) \ge t$ and*

$$\limsup_{t\to\infty} \int_{t}^{\tau(t)} (\tau(t)-s)p(s)\,\mathrm{d}s > 1, \tag{1.3}$$

then $\mathcal{N}_2 = \emptyset$ for (E).

The effort of mathematicians has been oriented to improve those results or extend them to more general differential equations. The present authors also contributed to the subject, see [2].

We are interested in the situation in which $\mathcal{N} = \emptyset$, that is, all proper solutions of (E) are oscillatory. The deviating argument $\tau(t)$ is said to be of mixed type if its delay part

$$\mathcal{D}_\tau = \{t \in (t_0,\infty) : \tau(t) < t\}$$

and its advanced part

$$\mathcal{A}_\tau = \{t \in (t_0,\infty) : \tau(t) > t\}$$

are both unbounded subset of (t_0,∞). In view of above-mentioned results it is natural to expect that the presence of deviating argument of mixed type will be sufficient to force all solutions of (E) to oscillate. Kusano [13] was the first who proved this conjecture. After introducing two sequences $\{t_k\}$, $\{s_k\}$ such that

$$t_k \in \mathcal{D}_\tau, \quad t_k \to \infty \quad \text{as} \quad k\to\infty \tag{1.4}$$

and

$$s_k \in \mathcal{A}_\tau, \quad s_k \to \infty \quad \text{as} \quad k\to\infty \tag{1.5}$$

Kusano formulated the following criterion:

Theorem C. *Assume that there exist two sequences $\{t_k\}$, $\{s_k\}$ satisfying* (1.4) *and* (1.5). *If*

$$\limsup_{k\to\infty} \int_{\tau(t_k)}^{t_k} \big(\tau(t_k)-\tau(s)\big)p(s)\,\mathrm{d}s > 1 \tag{1.6}$$

and

$$\limsup_{k\to\infty} \int_{s_k}^{\tau(s_k)} \big(\tau(t)-\tau(s_k)\big)p(t)\,\mathrm{d}t > 1, \tag{1.7}$$

then (E) is oscillatory.

Our aim in this work is to significantly improve the above mentioned results and the progress will be demonstrated via set of illustrative examples in which we shall compare our results with known ones.

2. MAIN RESULTS

It is easy to see that (H_2) guaranties the existence of the inverse function $\tau^{-1}(t)$ and therefore the auxiliary function $\xi(t) \in C^1([t_0,\infty))$ in this way

$$\xi(\xi(t)) = \tau^{-1}(t) \tag{2.8}$$

is well defined.

We are about to establish criterion for $\mathcal{N}_0 = \emptyset$ of (E). Suppose that there exists a sequence $\{t_k\}$ such that (1.4) holds. It is easy to see that $\xi(t_k) > t_k$. To simplify our notation we employ the following functions

$$\begin{aligned} P_1(t_k) &= \int_{t_k}^{\xi(t_k)} (s-t_k)p(s)\,\mathrm{d}s, \\ P_2(t_k) &= \int_{\xi(t_k)}^{\tau^{-1}(t_k)} (s-t_k)p(s)\,\mathrm{d}s, \\ P_3(t_k) &= \int_{\tau^{-1}(t_k)}^{\tau^{-1}(\xi(t_k))} (s-t_k)p(s)\,\mathrm{d}s. \end{aligned} \tag{2.9}$$

Theorem 2.1. *Assume that there exist a function $\xi(t) \in C^1([t_0,\infty))$ satisfying* (2.8) *and a sequence $\{t_k\}$ such that* (1.4) *holds. If*

$$\begin{aligned} \limsup_{k\to\infty} \Bigg[& \frac{P_1(t_k)P_1(\xi^{-1}(t_k)) + P_1(t_k)P_3(\xi^{-1}(t_k))}{(1-P_2(t_k))(1-P_2(\xi^{-1}(t_k)))} \\ & + \frac{P_3(t_k)P_1(\xi(t_k))}{(1-P_2(t_k))(1-P_2(\xi(t_k)))} \Bigg] > 1, \end{aligned} \tag{2.10}$$

then $\mathcal{N}_0 = \emptyset$ for (E).

Proof. Assume on the contrary that $y(t)$ is an eventually positive solution of (E) such that $y(t) \in \mathcal{N}_0$. Integrating twice of (E) from t to ∞ and changing order of integration we are led to

$$y(t) \geq \int_t^\infty (s-t)p(s)y(\tau(s))\,\mathrm{d}s. \tag{2.11}$$

Consequently

$$\begin{aligned} y(t) \geq & \int_t^{\xi(t)} (s-t)p(s)y(\tau(s))\,\mathrm{d}s + \int_{\xi(t)}^{\tau^{-1}(t)} (s-t)p(s)y(\tau(s))\,\mathrm{d}s \\ & + \int_{\tau^{-1}(t)}^{\tau^{-1}(\xi(t))} (s-t)p(s)y(\tau(s))\,\mathrm{d}s. \end{aligned} \tag{2.12}$$

It is useful to observe that for $t_k \in \mathcal{D}_\tau$ also $(\tau(t_k), \tau^{-1}(\xi(t_k))) \in \mathcal{D}_\tau$ Setting $t = t_k$ and using that $y(t)$ is decreasing function, we are led to

$$y(t_k) \geq y(\xi^{-1}(t_k))P_1(t_k) + y(t_k)P_2(t_k) + y(\xi(t_k))P_3(t_k),$$

which is

$$y(t_k) \geq \frac{1}{1-P_2(t_k)} \left[y(\xi^{-1}(t_k))P_1(t_k) + y(\xi(t_k))P_3(t_k) \right]. \tag{2.13}$$

We repeat this procedure and setting consequently $t = \xi^{-1}(t_k)$ and $t = \xi(t_k)$ into (2.12), we obtain

$$y(\xi^{-1}(t_k)) \geq \frac{1}{1-P_2(\xi^{-1}(t_k))} \left[y(\tau(t_k))P_1(\xi^{-1}(t_k)) + y(t_k)P_3(\xi^{-1}(t_k)) \right] \tag{2.14}$$

and

$$\text{(2.15)}\quad \begin{aligned} y(\xi(t_k)) &\geq \frac{1}{1-P_2(\xi(t_k))}\left[y(t_k)P_1(\xi(t_k)) + y(\tau^{-1}(t_k))P_3(\xi(t_k))\right] \\ &\geq \frac{P_1(\xi(t_k))}{1-P_2(\xi(t_k))}y(t_k), \end{aligned}$$

respectively. Combining (2.14), (2.15) and (2.13), one gets

$$\begin{aligned} y(t_k) \geq \frac{1}{1-P_2(t_k)}&\left[\frac{P_1(t_k)}{1-P_2(\xi^{-1}(t_k))}\left[y(\tau(t_k))P_1(\xi^{-1}(t_k)) + y(t_k)P_3(\xi^{-1}(t_k))\right]\right. \\ &\left.+ \frac{P_3(t_k)}{1-P_2(\xi(t_k))}y(t_k)P_1(\xi(t_k))\right] \end{aligned}$$

which in view of $y(\tau(t_k)) \geq y(t_k)$, leads to

$$\begin{aligned} y(t_k) \geq y(t_k)&\left[\frac{\left[P_1(t_k)P_1(\xi^{-1}(t_k)) + P_1(t_k)P_3(\xi^{-1}(t_k))\right]}{(1-P_2(t_k))(1-P_2(\xi^{-1}(t_k)))}\right. \\ &\left.+ \frac{P_3(t_k)P_1(\xi(t_k))}{(1-P_2(t_k))(1-P_2(\xi(t_k)))}\right]. \end{aligned}$$

This contradicts (2.10) and we conclude that $\mathcal{N}_0 = \emptyset$. □

Our next considerations are intended to derive sufficient conditions for $\mathcal{N}_2 = \emptyset$ of (E). Suppose that there exists a sequence $\{s_k\}$ such that (1.5) holds. It is easy to see that $\xi(s_k) < s_k$, where $\xi(t)$ is defined by (2.8). We shall use the notation

$$\text{(2.16)}\quad \begin{aligned} Q_1(s_k) &= \int_{\tau^{-1}(\xi(s_k))}^{\tau^{-1}(s_k)} (s_k - s)p(s)\,\mathrm{d}s, \\ Q_2(s_k) &= \int_{\tau^{-1}(s_k)}^{\xi(s_k)} (s_k - s)p(s)\,\mathrm{d}s, \\ Q_3(s_k) &= \int_{\xi(s_k)}^{s_k} (s_k - s)p(s)\,\mathrm{d}s. \end{aligned}$$

Theorem 2.2. *Assume that there exist a function $\xi(t) \in C^1([t_0,\infty))$ satisfying* (2.8) *and a sequence $\{s_k\}$ such that* (1.5) *holds. If*

$$\text{(2.17)}\quad \begin{aligned} \limsup_{k\to\infty}&\left[\frac{Q_3(s_k)Q_1(\xi^{-1}(s_k)) + Q_3(s_k)Q_3(\xi^{-1}(s_k))}{(1-Q_2(s_k))(1-Q_2(\xi^{-1}(s_k)))}\right. \\ &\left.+ \frac{Q_1(s_k)Q_3(\xi(s_k))}{(1-Q_2(s_k))(1-Q_2(\xi(s_k)))}\right] > 1, \end{aligned}$$

then $\mathcal{N}_2 = \emptyset$ for (E).

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t) \in \mathcal{N}_2$. Double integration of (E) from t_1 to t yields

$$y(t) \geq \int_{t_1}^{t} (t-s)p(s)y(\tau(s))\,\mathrm{d}s.$$

Therefore employing auxiliary function $\xi(t)$, we get

$$(2.18)\qquad \begin{aligned} y(t) \geq &\int_{\tau^{-1}(\xi(t))}^{\tau^{-1}(t)} (t-s)p(s)y(\tau(s))\,\mathrm{d}s + \int_{\tau^{-1}(t)}^{\xi(t)} (t-s)p(s)y(\tau(s))\,\mathrm{d}s \\ &+ \int_{\xi(t)}^{t} (t-s)p(s)y(\tau(s))\,\mathrm{d}s. \end{aligned}$$

It easy to see that for $s_k \in \mathcal{A}_\tau$ also $(\tau^{-1}(\xi(s_k)), s_k) \in \mathcal{A}_\tau$. Putting $t = s_k$ and taking into account that $y(t)$ is an increasing function, we have

$$y(s_k) \geq Q_1(s_k)y(\xi(s_k)) + Q_2(s_k)y(s_k) + Q_3(s_k)y(\xi^{-1}(s_k))$$

which is equivalent to

$$(2.19)\qquad y(s_k) \geq \frac{1}{1-Q_2(s_k)}\left[Q_1(s_k)y(\xi(s_k)) + Q_3(s_k)y(\xi^{-1}(s_k))\right].$$

Repeating this operation and setting consequently $t = \xi(s_k)$ and $t = \xi^{-1}(s_k)$ into (2.18), one gets

$$(2.20)\qquad \begin{aligned} y(\xi(s_k)) &\geq \frac{1}{1-Q_2(\xi(s_k))}\left[Q_1(\xi(s_k))y(\tau^{-1}(s_k)) + Q_3(\xi(s_k))y(s_k)\right] \\ &\geq \frac{Q_3(\xi(s_k))}{1-Q_2(\xi(s_k))}y(s_k) \end{aligned}$$

and

$$(2.21)\qquad y(\xi^{-1}(s_k)) \geq \frac{1}{1-Q_2(\xi^{-1}(s_k))}\left[Q_1(\xi^{-1}(s_k))y(s_k) + Q_3(\xi^{-1}(s_k))y(\tau(s_k))\right],$$

respectively. Setting (2.20) and (2.21) into (2.19) and using that $y(\tau(s_k)) \geq y(s_k)$, we get

$$\begin{aligned} y(s_k) \geq \frac{y(s_k)}{1-Q_2(s_k)}\Bigg[&\frac{Q_1(s_k)Q_3(\xi(s_k))}{1-Q_2(\xi(s_k))} \\ &+ \frac{Q_3(s_k)}{1-Q_2(\xi^{-1}(s_k))}\left[Q_1(\xi^{-1}(s_k)) + Q_3(\xi^{-1}(s_k))\right]\Bigg], \end{aligned}$$

which contradicts to (2.17) and we conclude that $\mathcal{N}_2 = \emptyset$. □

Combining Theorems 2.1 and 2.2 we obtain desired oscillation criterion.

Theorem 2.3. *Assume that there exist a function $\xi(t) \in C^1([t_0,\infty))$ satisfying* (2.8) *and two sequences $\{t_k\}$, $\{s_k\}$ satisfying* (1.4) *and* (1.5)*. If* (2.10) *and* (2.17) *hold, then every solution of* (E) *is oscillatory.*

In the following example we illustrate the quality of our criterion by comparing it with that of Kusano presented in Theorem C.

Example 2.1. Consider the equation

$$(E_1)\qquad y''(t) = p_0 y(t + \sin t),$$

where $p_0 > 0$, is a constant.

Clearly, the deviating argument $\tau(t) = t + \sin t$ is of mixed type. It is easy to see that if $t_k = (\pi/2) + 2k\pi$, $k = 1, 2, \ldots$, then $t_k \in \mathcal{D}_\tau$ moreover $\tau(t_k) = t_k - 1$, $\tau^{-1}(t_k) = t_k + 1$ and $\xi(t_k) = t_k + (1/2)$. It is easy to verify that

$$P_1(t_k) = \frac{p_0}{8}, \quad P_2(t_k) = \frac{3p_0}{8}, \quad \text{and} \quad P_3(t_k) = \frac{5p_0}{8}.$$

On the other hand, if $s_k = (-\pi/2) + 2k\pi$, $k = 1, 2, \dots$, then $s_k \in \mathcal{A}_\tau$ moreover $\tau(s_k) = s_k + 1$, $\tau^{-1}(s_k) = s_k - 1$ and $\xi(s_k) = s_k - (1/2)$. It is easy to verify that

$$Q_1(s_k) = \frac{5p_0}{8}, \quad Q_2(s_k) = \frac{3p_0}{8}, \quad \text{and} \quad Q_3(s_k) = \frac{p_0}{8}.$$

Therefore both conditions (2.10) and (2.17) reduce to

$$p_0 > \frac{8}{\sqrt{11}+3} \approx 1.2665$$

which according to Theorem 2.3 guarantees that (E_1) is oscillatory. we would like to point out that by Theorem C all solutions of (E_1) are oscillatory provided that

$$p_0 > \frac{1}{\sin 1 - 1/2} \approx 2.9285.$$

The progress is confessed.

Example 2.2. We consider the equation

$$(E_2) \qquad y''(t) = \frac{p_0}{t^2} y(t(1 + 0.5\sin(\ln t))), \quad p_0 > 0.$$

We set $t_k = \mathrm{e}^{-\pi/2+2k\pi}$, $k = 1, 2, \dots$. Then $t_k \in \mathcal{D}_\tau$ and $\tau(t_k) = t_k/2$, $\tau^{-1}(t_k) = 2t_k$ and $\xi(t_k) = \sqrt{2}\, t_k$. Thus simple computation yields

$$P_1(t_k) = p_0 \left(\ln\sqrt{2} + \frac{\sqrt{2}-2}{2} \right) = p_0 A_1, \quad P_2(t_k) = p_0 \left(\ln\sqrt{2} + \frac{1-\sqrt{2}}{2} \right) = p_0 A_2,$$

$$P_3(t_k) = p_0 \left(\ln\sqrt{2} + \frac{\sqrt{2}-2}{4} \right) = p_0 A_3.$$

Condition (2.10) takes the form

$$p_0 > \frac{\sqrt{A_1^2 + 2A_1A_3} - A_2}{A_1^2 + 2A_1A_3 - A_2^2} = 3.3833$$

which by Theorem 2.1 implies that $\mathcal{N}_0 = \emptyset$ for (E_2).

On the other hand, choosing $s_k = \mathrm{e}^{\pi/2+2k\pi}$, $k = 1, 2, \dots$ we see that $s_k \in \mathcal{A}_\tau$ and $\tau(s_k) = 3s_k/2$, $\tau^{-1}(s_k) = 2s_k/3$ and $\xi(s_k) = \sqrt{2/3}\, s_k$. Consequently

$$Q_1(t_k) = p_0 \left(\ln\sqrt{\frac{2}{3}} + \frac{3\sqrt{3}}{2\sqrt{2}} - \frac{3}{2} \right) = p_0 B_1, \quad Q_2(t_k) = p_0 \left(\ln\sqrt{\frac{2}{3}} + \frac{3}{2} - \sqrt{\frac{3}{2}} \right) = p_0 B_2,$$

$$Q_3(t_k) = p_0 \left(\ln\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}} - 1 \right) = p_0 B_3.$$

Condition (2.17) reduces to

$$p_0 > \frac{\sqrt{B_1^2 + 2B_1B_3} - B_2}{B_1^2 + 2B_1B_3 - B_2^2} = 4.3983$$

which by Theorem 2.2 implies that $\mathcal{N}_2 = \emptyset$ for (E_2) and moreover, by Theorem 2.3 all solutions of (E_2) are oscillatory.

Compared to Theorem C conditions (1.6) and (1.7) yield

$$p_0 \left(0.5 - \ln 2 - 0.5\cos\left(-\frac{\pi}{2} - \ln 2 \right) \right) > 1$$

and

$$p_0 \left(-0.5 + \ln 1.5 - 0.5\cos\left(\frac{\pi}{2} + \ln 1.5 \right) \right) > 1$$

respectively, which means that (E_2) is oscillatory provided that

$$p_0 > 9.7382.$$

Our progress is remarkable and it is caused by employing three integrals (see (2.9) and (2.16)) instead of only one (see (1.6) and (1.7)).

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