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Generalized Split Feasibility Problem: Solution by Iteration

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ABSTRACT. In real Hilbert spaces, given a single-valued Lipschitz continuous and monotone operator, we study generalized split feasibility problem (GSFP) over solution set of monotone variational inclusion problem. An inertia iterative method is proposed to solve this problem, by showing that the sequence generated by the iteration converges strongly to solution of GSFP. As against previous methods, our step size is chosen to be simple and not depending on norm of associated bounded linear map as well as Lipschitz constant of the single-valued operator. The obtained result was applied to study split linear inverse problem, precisely, the LASSO problem. Lastly, with the aid of numerical examples, we exhibited efficiency of our algorithm and its dominance over other existing schemes.

1. INTRODUCTION

In 1994, Censor and Elfvin [12] were the first to formulate and study the Split Feasibility Problem (SFP). It is formulated as: Let $\mathcal{C} \subset \mathbb{R}^N$ and $\mathcal{Q} \subset \mathbb{R}^M$ be convex, nonempty, closed and $T \in \mathbb{R}^{M\times N}$ be a real matrix.

$$\text{Find } u^* \in \mathcal{C} \text{ that satisfies } z^* = Tu^* \in \mathcal{Q}. \tag{1.1}$$

The SFP hitherto has different applications in image and signal processing, phase retrieval, data compression and Intensity-Modulated Radiation Therapy (IMRT) treatment plans, etc. Consequently, many Researchers have investigated the problem under varying settings (see [20, 26, 52, 53, 54] and the references therein).

Some generalizations of the SFP have been investigated by other authors. For example, the following Split Variational Inequality problem (SVIP) was formulated by Censor *et al.* [13]:

Assume H_1 and H_2 are Hilbert spaces, $\mathcal{C} \subset H_1$ and $\mathcal{Q} \subset H_2$ are convex, closed and nonempty, the operator $T : H_1 \to H_2$ is linear and bounded. Given the operators $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$, for any $z \in \mathcal{C}$,

$$\text{find } u^* \in \mathcal{C} \text{ that satisfies } \langle A_1 u^*, z - u^*\rangle \geq 0, \text{ for any } z \in \mathcal{C}, \tag{1.2}$$

$$\text{and such that } z^* = Tu^* \in \mathcal{Q} \text{ and solves } \langle A_2 z^*, z - z^*\rangle \geq 0, \text{ for any } z \in \mathcal{Q}. \tag{1.3}$$

In fact, combining the SFP (1.3) and the classical variational inequality problem (VIP) yields SVIP. Another generalization of SFP is Split Monotone Variational Inclusion Problem (SMVIP) (see [33]), it is given as: Suppose H_1 and H_2 are Hilbert spaces, mappings

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$f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are single- valued, $G_1 : H_1 \to 2^{H_1}$ and $G_2 : H_2 \to 2^{H_2}$ are maximal monotone, operator $T : H_1 \to H_2$ is linear and bounded.

$$\text{(1.4)} \qquad \text{Find } u^* \in H_1 \quad \text{that satisfies } 0 \in f_1(u^*) + G_1(u^*) \text{ and } 0 \in f_2(Tu^*) + G_2(Tu^*).$$

Suppose we neglect f_2 and G_2, we arrive at a Monotone Inclusion problem (MIP), which is a particular case of (SMVIP) (see [32]). Mehra et al. [32] proposed the following algorithm to solve (MIP):

$$\text{(1.5)} \qquad \begin{cases} u_n = x_n + \epsilon(x_n - x_{n-1}), \\ v_n = (1-\alpha_n)u_n + \alpha_n J^{A,B}_{\lambda,M}(u_n), \\ \kappa_n = J^{A,B}_{\lambda,M}((1-\beta_n)v_n + \beta_n J^{A,B}_{\lambda,M}(v_n)), \\ x_{n+1} = \gamma_n h(x_n) + (1-\gamma_n-\delta_n)J^{A,B}_{\lambda,M}(\kappa_n) + \delta_n S_n \kappa_n, \end{cases}$$

where $J^{A,B}_{\lambda,M} = (I + \lambda M^{-1}B)^{-1}(I - \lambda M^{-1}A)$, M is a linear, self-adjoint, positive and bounded operator. The authors prove a strong convergence of the sequence $\{x_n\}$ generated by algorithm 1.5 to a point x^* belonging to solution set of (MIP) and intersection of Fix(S_i), with respect to an $M-$norm induced by the operator M.

Censor *et al.* [13] gave these algorithm to solve SVIP; Let T^* denote adjoint of T, γ the spectral radius of the operator T^*T, and $\eta \in (0, 1/\gamma)$. For any $u_1 \in H_1$, generate the sequence $\{u_n\}$ by

$$\text{(1.6)} \qquad u_{n+1} = P_{\mathcal{C}}(I - \lambda A_1)(u_n + \eta T^*(P_{\mathcal{Q}}(I - \lambda A_2) - I)Tu_n), \ n \geq 1,$$

A_1 and A_2 are α_1, α_2 - inverse strongly monotone operators, $\lambda \in (0, 2\alpha)$ (where $\alpha := \min\{\alpha_1, \alpha_2\}$) and for all u that solve (1.3), provided

$$\text{(1.7)} \qquad \langle A_1 z, P_{\mathcal{C}}(I - \lambda A_1)(z) - u\rangle \geq 0 \ \ \forall\, z \in H_1.$$

They proved that $\{u_n\}$ converges weakly to a solution of SVIP (1.2) and (1.3).

We highlight that assumption (1.7) is quite restrictive and constitute a drawback to the method. Recently, some authors have been able to do away with this assumption in solving SVIP and related problems (see [17, 25, 37]). Unfortunately, their methods still require that the operators A_1 and A_2 be inverse strongly monotone (again, a restrictive condition, for disadvantages of inverse strongly monotone assumption, see Remark 5.3 of [26]).

Inspired by the $\mathcal{CQ}-$algorithm of [9], the following weakly convergent algorithm was introduced by Moudafi [33] to approximate a solution of (1.4)

$$\text{(1.8)} \qquad u_{n+1} = J^{B_1}_{\lambda}(I - \eta f_1)(u_n - \lambda T^*(I - J^{B_2}_{\eta}(I - \eta f_2)))Tu_n, n \geq 1.$$

where $\eta \in (0, 1/\gamma)$, and γ is the spectral radius of T^*T. For more on CQ$-$algorithms see [4, 41].

Over the solution set of VIP, Tian and Jiang [44] formulated and studied a general class of SVIP called Generalized Split Feasibility Problem (GSFP). The problem is to

$$\text{(1.9)} \qquad \text{find } q^* \in \mathcal{C} \text{ such that } \langle Tq^*, q - q^*\rangle \geq 0, \ \forall\, q \in \mathcal{C} \ \text{ and } Tq^* \in F(S),$$

where S is nonexpansive and $F(S)$ is fixed points set of S. To solve GSFP (1.9), they introduced this algorithm: Pick arbitrary $z_1 \in \mathcal{C}$, define the sequence $\{z_n\}$ by

$$\text{(1.10)} \qquad \begin{cases} v_n = P_{\mathcal{C}}(q_n - \tau_n T^*(I - S)Tq_n), \\ r_n = P_{\mathcal{C}}(v_n - \lambda_n A(v_n)), \\ y_n = P_{\mathcal{C}}(v_n - \lambda_n A(r_n)), \\ q_{n+1} = \beta_n f(z_n) + (1-\beta_n)y_n, \ n \geq 1, \end{cases}$$

where $\{\beta_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$, $f : H_1 \to H_1$ is a contraction mapping, $S : H_2 \to H_2$ is a nonexpansive mapping and operator $A : \mathcal{C} \to H_1$ is monotone and $L-$Lipschitz continuous, $\{\tau_n\} \subset [a,b]$ for some $a,b \in (0, 1/||T||^2)$, $\{\lambda_n\} \subset [c,d]$ for some $c,d \in (0,1/L)$ and operator $T : H_1 \to H_2$ is linear and bounded. Strong convergence of sequence $\{z_n\}$ was proved.

Algorithm (1.10) has the following advantages; A is L-Lipschitz continuous and monotone, this assumption is weaker than inverse strong monotonicity assumed by many other authors (see [17, 25, 37] and the references therein). In establishing strong convergence of Algorithm (1.10), the restrictive assumption (1.7) of Censor *et al.* [13] was dispensed with. These not withstanding, the condition on the step size $\{\lambda_n\}$ is very restrictive and the method involves evaluation of many projections. The Lipschitz constant L not possible to compute in most real-world applications (see [26], Remark 5.3). Hence, iterative methods devoid of knowing the Lipschitz constant L is more desirable and would handle a larger class of problems. Some important results have been proved in which the methods do not require knowing the Lipschitz constant ahead of time (see [30, 50]] and the references therein). In the light of GSFP, Izuchukwu *et.al.* [26] recently studied the following GSFP over solution set of monontone variational inculsion problem (MVIP):

$$\text{Find } u^* \in H_1 \text{ satisfying } 0 \in (A+B)(u^*) \text{ and } z^* = Tu^* \in F(S), \tag{1.11}$$

where the operators $T : H_1 \to H_2$ is linear and bounded, $A : H_1 \to H_1$ is monotone and Lipschitz continuous, $B : H_1 \to 2^{H_1}$ is multivalued and maximal monotone, and $S : H_2 \to H_2$ is a nonexpansive map. We can note that a particular case of (1.11) is problem (1.9) if B is a normal cone. In addition, we can see (1.11) as an interesting generalization of the (SMVIP) in Moudafi [33] and the GSFP of Tian and Jiang [44]. It is important to note that the result of Moudafi [33] assumes the underlying single valued operators is inverse strongly monotone. Hence, the result in [26] is more encompassing and include a lot of interesting optimization problems such as; split minimization problems, split common null point problems, split feasibility problems, and so on (see [27, 39, 42, 43, 49]).

For solving problem (1.11), Izuchukwu *et al.* [26] constructed the algorithm below, they proved strong convergence of $\{u_n\}$ and that $u^* = \lim\limits_{n\to\infty} u_n$ solves problem (1.11).

$$\begin{cases} x_n = u_n + \alpha_n(u_n - u_{n-1}), \\ v_n = x_n - \tau_n T^*(I-S)Tx_n \\ z_n = J^B_{\lambda_n}(I - \lambda_n A)v_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)v_n, \\ \text{where } 0 < b \le \tau_n \le c < 1/||T||^2. \\ u_{n+1} = (1 - \theta_n - \beta_n)v_n + \theta_n q_n, \\ \text{where } q_n = z_n - \lambda_n(Az_n - Av_n) \text{ and} \end{cases} \tag{1.12}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|v_n - z_n\|}{\|Av_n - Az_n\|}, \lambda_n\}, Av_n \ne Az_n \\ \lambda_n, \qquad \text{otherwise.} \end{cases} \tag{1.13}$$

Very interesting and remarkable features of algorithm (1.12) studied by Izuchukwu are: the operator A, is monotone and Lipschitz continuous with Lipschitz constant L, the stepzize $\{\lambda_n\}$ is self adaptive and independent of L. In addition, we point out that the parameter $\{\tau_n\}$ in (1.12) is dependent on the norm $\|T\|$ of operator T. This constitutes a serious draw-back to the efficiency of the scheme (see Remark 5.3 of Izuchukwu *et al.* [26]). Furthermore, in the proof process, the authors introduced an auxillairy sequence, $\{y_n\}$ which depends on the restrictive condition imposed on the parameter $\{\tau_n\}$, this

played a central role in their convergence analysis. This auxilliary sequence was first introduced by Xu [51] and has been used by many authors (see for instance [24, 19, 51]). Although the auxilliary sequence method yields correct proof, we consider it labouriuos and restrictive. In order to extablish boundedness of the sequence (1.12), the authors, deployed Lemma 2.7 of their work. This is indeed superfulous as a simpler argument could have yielded the boundedness conclusion.
Construction of efficient and fast convergent algorithms has been of interest to many researchers in recent years. Considering discrete analogue of a dynamical system of second order, the inertial technique was developed, this improved and enhanced the rate of convergence for iterative methods. Polyak [35] first considered this method to solve smooth convex minimization problems. Nesterov's [34] went ahead to amplify this method by his accelerated gradient method [34]. Further development was made for structured convex minimization problems by Beck and Teboulle [8]. Reader may see [6, 11, 16, 38, 43] where this approach has helped to enhance rate of convergence for iterative methods.

Drawing motivations from Izuchukwu *et al.* [26], Tian and Jiang [44] and similar works, our concern herein is to answer positively, the following questions
Can an iterative scheme be constructed for solving problem (1.11) such that the underlisted features are preserved.

- none of the iterative parameters should depend on norm of the involved bounded linear map
- operator A is Lipschitz continuous and monotone
- the step size is independent of the Lipschitz constant
- convergence analysis of the scheme does not involve constructing an auxilliary sequence
- the scheme involves an inertia term.

Henceforth, we follow these outline; in Section 2 lies the needed definitions and Lemmas, Section 3, contains our main Theorem, the convergence analysis and some import corollaries. Section 4 is devoted to application while Section 5 contains numerical examples.

2. Preliminaries

H is a real Hilbert space henceforth. Let $S : H \to H$, by $F(S)$ we mean the fixed points set of S.

Definition 2.1. ([31]). Let $(\mathcal{U}, d)$ be a metric space, $f : \mathcal{U} \to \mathcal{U}$ is a Meir-Keeler contraction map if

$$\forall\, \varepsilon > 0,\ \exists\, \sigma > 0 \quad \text{s.t} \quad \varepsilon \leq d(w, z) < \varepsilon + \sigma \Rightarrow d(f(w), f(z)) < \varepsilon,\ \ \forall\, w, z \in \mathcal{U}.$$

Remark 2.1. Obviously, the collection of contraction mappings is contained in the class of Meir-Keeler contraction mappings.

Definition 2.2. A map $S : H \to H$ is called;

(i) nonexpansive if $||Sw - Sz|| \leq ||w - z||\ \forall\, w, z \in H$,

(ii) quasi-nonexpansive if $||Sw - p|| \leq ||w - p||\ \forall\, w \in H,\ p \in F(S)$,

(iii) $\kappa-$demimetric (see Takahashi [45]) if $F(S) \neq \emptyset$ and there is $\kappa \in (-\infty, 1)$ such that

$$\langle w - Sw, w - p\rangle \geq \frac{(1-k)}{2}||w - Sw||^2\ \forall,\ w \in H,\ p \in F(S),$$

(iv) demicontractive if $F(S) \neq \emptyset$ and there is $\tau \in (0,1)$ satisfying

$$||Sv - p||^2 \leq ||v - p||^2 + \tau||v - Sv||^2 \ \forall, \ v \in H, \ p \in F(S).$$

Remark 2.2. Notice that

$$\begin{aligned} ||v - Sv||^2 &= \langle v - Sv, v - Sv\rangle = \langle v - p + p - Sv, v - p + p - Sv\rangle \\ &= ||v - p||^2 + 2\langle v - p, p - Sv\rangle + ||Sv - p||^2 \end{aligned} \tag{2.14}$$

From Definition 2.2 (iii) and (2.14), we have

$$\langle v - Sv, v - p\rangle \geq \frac{(1-\kappa)}{2}||v - Sv||^2, \ \forall\, v \in H, \ p \in F(S).$$

So

$$\begin{aligned} 2\langle v - Sv, v - p\rangle &= ||v - Sv||^2 - \kappa||v - Sv||^2 \\ &= ||v - p||^2 + 2\langle v - p, p - Sv\rangle + ||Sv - p||^2 - \kappa||v - Sv||^2, \end{aligned}$$

that is,

$$2\langle v - Sv, v - p\rangle - 2\langle v - p, p - Sv\rangle \geq ||v - p||^2 + ||Sv - p||^2 - k||v - Sv||^2. \tag{2.15}$$

Rearranging (2.15), gives

$$||Sv - p||^2 \leq ||v - p||^2 + \kappa||v - Sv||^2, \ \forall\, v \in H, \ p \in F(S). \tag{2.16}$$

If $\kappa \leq 0$ in (2.16), then

$$||Sv - p||^2 \leq ||v - p||^2, \quad \forall\, v \in H, \ p \in F(S). \tag{2.17}$$

Hence, S is quasi-nonexpansive. Thus every demimetric map in the sense of Takahashi [45] is quasi-nonexpansive. For $\kappa \in (0,1)$, then every demimetric map in the sense of Takahashi [45] is demicontractive.

Lemma 2.1. ([29]) *Let H be a real Hilbert space. If the operators $A_1 : H \to H$ is monotone and Lipschitz continuous, and $A_2 : H \to 2^H$ is maximal monotone, then $(A_1 + A_2) : H \to 2^H$ is a maximal monotone operator.*

Lemma 2.2. ([15] *Suppose $T : H \to H$ is a $k-$ demicontractive mapping and $T_\mu := (1-\mu)I + \mu T$ for any $\mu \in (0, 1-k)$, then $||T_\mu v - v^*||^2 \leq ||v - v^*||^2 - (1-k-\mu)||(I-T)v||^2$, $\forall\, v \in H, v^* \in F(T)$.*

Remark 2.3. From Lemma 2.2, it is obvious that T_μ is quasi- nonexpansive with $v^* \in F(T) \Leftrightarrow v^* \in F(T_\mu)$.

Lemma 2.3. ([40, 55]) *Let X be a Banach space and $\mathcal{C} \subset X$ be closed and convex. Then, $f : \mathcal{C} \to \mathcal{C}$ is a Meir-Keeler contraction mapping if and only if for each $\varepsilon > 0$, we can find a number $\delta \in (0,1)$ such that*

$$||w - z|| \geq \epsilon \Rightarrow ||f(w) - f(z)|| \leq \delta||w - z|| \ \forall\, w, z \in \mathcal{C}.$$

Lemma 2.4. *The following properties hold, for every $w, z \in H$.*

(i) $||w + z||^2 = ||w||^2 + ||z||^2 + 2\langle w, z\rangle$,

(i) $||w + z||^2 \leq ||w||^2 + 2\langle z, w + z\rangle$,

(iii) $||\lambda w + (1-\lambda)z||^2 = \lambda||w||^2 + (1-\lambda)||z||^2 - \lambda(1-\lambda)||w - z||^2$.

Lemma 2.5. ([47]*) Given a nonexpansive map $S : H \to H$ with $F(S) \neq \emptyset$, if $\{u_n\} \subset H$ converges weakly to u^* and $||(I - S)u_n||$ strongly converges to z then $(I - S)u^* = z$.*

3. MAIN CONTRIBUTIONS

One of the major contributions of this work is presented here.

Assumptions

(A1) $H_1,\ H_2$ are real Hilbert spaces, $L : H_1 \to H_2$ is a bounded linear operator whose adjoint operator is $L^* : H_2 \to H_1$.

(A2) $A : H_1 \to H_1$ is monotone and Lipschitz continuous.

(A3) $B : H_1 \to 2^{H_1}$ is set-valued and maximal monotone.

(A4) $f : H_1 \to H_1$ is a Meir-Keeler contraction mapping,

(A5) $T : H_1 \to H_1$ is a $k-$ demicontractive map with $(I - T)$ demiclosed at 0, where $k \in [0, 1),\ F(T) \neq \emptyset$ and $S : H_2 \to H_2$ is $\zeta-$ demimetric mapping, with $\zeta \in (-\infty, 1),\ F(S) \neq \emptyset$.

Self-Adaptive Algorithm for GSFP

Algorithm 3.1. *Initialization:* Pick $\alpha_n \in [0, \alpha] \subset [0, 1), \delta_n \in [a, b] \subset (0, 1), \sigma_n \in (0, 1), \mu \in (0, 1 - k)$. Take $u_0, u_1 \in H_1$. Given the iterate u_n and u_{n-1}, compute

$$\begin{cases} x_n = u_n + \alpha_n(u_n - u_{n-1}), \\ v_n = x_n - \delta_n L^*(I - S)Lx_n, \\ z_n = J^B_{\lambda_n}(I - \lambda_n A)v_n, \\ q_n = z_n - \lambda_n(Av_n - Az_n), \\ u_{n+1} = \beta_n f(v_n) + (1 - \beta_n)(\mu I + (I - \mu)T)q_n, \end{cases} \tag{3.18}$$

where $\delta_n = \sigma_n \tau_n$, and

$$\tau_n = \begin{cases} \frac{(1-\zeta)\|(I-S)Lx_n\|^2}{\|L^*(I-S)Lx_n\|^2}, Lx_n \neq SLx_n, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{3.19}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|v_n - z_n\|}{\|Av_n - Az_n\|}, \lambda_n\}, Av_n \neq Az_n, \\ \lambda_n, \qquad \text{otherwise}. \end{cases} \tag{3.20}$$

The control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions:
C1 $\sum_{n=1}^{\infty} \beta_n = \infty,\ \ \lim_{n\to\infty} \beta_n = 0.$
C2 $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| = 0.$
C3 Denote by $\Gamma = \{u^* \in (A + B)^{-1}(0) \cap F(T) : Lu^* \in F(S)\} \neq \emptyset$ the solution set.

Remark 3.4. Demimetric mappings are crucial in optimization because they contain a lot of the commonly used operators in optimization. For example, it is known that the class of $k-$ demimetric mappings with $\zeta \in (-\infty, 1)$ includes the resolvents of maximal monotone operators and the metric projections (these are very useful tools in solving optimization problems) in Hilbert spaces (see e.g. [21, 48]). The class of $k-$ demicontractive mapping is quite general and contains the class of maps studied for instance in ([9, 19, 26, 33]). Furthermore, the problem studied in this manuscript, whose solution set is indicated in condition **C3** above is more general than the problem considered in [9, 26]. Hence, we recover the results of Tian and Jiang [9] and Izuchukwu *et al.* [26] as important corollaries. See remark 4.1 below.

4. Convergence Analysis

Observe that for $u^* \in \Gamma$,

$$\begin{aligned} ||L^*(I-S)Lx_n||||x_n-u^*|| &\geq \langle L^*(I-S)Lx_n, x_n-u^*\rangle \\ &= \langle (I-S)Lx_n, Lx_n-Lu^*\rangle \\ &\geq \frac{(1-\zeta)}{2}||(I-S)Lx_n||^2 \text{ since } S \text{ is } \zeta- \text{demimetric.} \end{aligned} \tag{4.21}$$

If $Lx_n \neq SLx_n$, Then $||L^*(I-S)Lx_n|| > 0$. Hence δ_n is well defined.

Lemma 4.6. *Suppose conditions* $\mathbf{C1}, \mathbf{C2}, \mathbf{C3}$ *hold, then the sequence* $\{u_n\}$ *given by Algorithm 3.18 is bounded.*

Proof. Let $u^* \in \Gamma$, if for any $\varepsilon > 0$, $||u_n - u^*|| \leq \varepsilon$ then the sequence $\{u_n\}$ is bounded. If on the contrary, $||u_n - u^*|| \geq \varepsilon$ then there exists a number $\rho \in (0,1)$ by Lemma 2.3 such that $||f(u_n) - f(u^*)|| \leq \rho||u_n - u^*||$. Using Remark 2.3, we have the following estimate:

$$\begin{aligned} \|u_{n+1}-u^*\| &\leq \beta_n\|f(v_n)-f(u^*)\| + \beta_n\|f(u^*)-u^*\| + (1-\beta_n)\|(1-\mu)I+\mu Tq_n-u^*\| \\ &\leq \beta_n\rho\|v_n-u^*\| + \beta_n\|f(u^*)-u^*\| + (1-\beta_n)\|q_n-u^*\|. \end{aligned} \tag{4.22}$$

From (3.20)), we have that

$$\|Av_n - Az_n\| \leq \frac{\mu}{\lambda_{n+1}}\|v_n - z_n\|.$$

Utilizing q_n in Algorithm 3.18 we get

$$\begin{aligned} \|q_n-u^*\|^2 &= \|z_n - \lambda_n(Av_n - Az_n) - u^*\|^2 \\ &= \|z_n-u^*\|^2 + \lambda_n^2\|Az_n-Av_n\|^2 - 2\lambda_n\langle z_n-u^*, Az_n-Av_n\rangle \\ &= \|z_n-v_n\|^2 + \|v_n-u^*\|^2 + 2\langle z_n-v_n, v_n-u^*\rangle + \lambda_n^2\|Az_n-Av_n\|^2 \\ &\quad - 2\lambda_n^2\langle z_n-u^*, Az_n-Av_n\rangle \\ &= \|v_n-u^*\|^2 + \|z_n-v_n\|^2 + \lambda_n^2\|Az_n-Av_n\|^2 + 2\langle z_n-u^*, z_n-v_n\rangle \\ &\quad - 2\langle z_n-v_n, z_n-v_n\rangle - 2\lambda_n\langle z_n-u^*, Az_n-Av_n\rangle \\ &= \|v_n-u^*\|^2 - \|z_n-v_n\|^2 + 2\langle z_n-u^*, z_n-v_n-\lambda_n(Az_n-Av_n)\rangle \\ &\quad + \lambda_n^2\|Az_n-Av_n\|^2 \\ &\leq \|v_n-u^*\|^2 - (1-\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2})\|z_n-v_n\|^2 + 2\langle z_n-u^*, z_n-v_n-\lambda_n(Az_n-Av_n)\rangle. \end{aligned} \tag{4.23}$$

Using the maximal monotonicity of B, we know from the definition of z_n that

$$\frac{1}{\lambda_n}(v_n - \lambda_n Av_n - z_n) \in Bz_n,$$

it follows from this fact that

$$Az_n + \frac{1}{\lambda_n}(v_n - \lambda_n Av_n - z_n) \in (A+B)z_n.$$

Since $0 \in (A+B)(u^*)$, we conclude from Lemma 2.1 that

$$\langle z_n - u^*, z_n - v_n - \lambda_n(Az_n - Av_n)\rangle \leq 0. \tag{4.24}$$

Substituting (4.24) into (4.23), we obtain

$$\|q_n - u^*\|^2 \leq \|v_n - u^*\|^2 - (1 - \lambda_n^2 \frac{\mu^2}{\lambda_{n+1}^2})\|z_n - v_n\|^2. \tag{4.25}$$

Clearly, $\lim\limits_{n\to\infty} \lambda_n$ exists since λ_n is a monotone nonincreasing. Therefore, without loss of generality we can assume that $\lim_{n\to\infty} \lambda_n = \lambda$. Using this idea, we get that

$$\lim_{n\to\infty}(1 - \lambda_n^2 \frac{\mu^2}{\lambda_{n+1}^2}) = 1 - \mu^2 > 0. \tag{4.26}$$

Therefore, using (4.26) in (4.25), we get

$$\|q_n - u^*\|^2 \leq \|v_n - u^*\|^2. \tag{4.27}$$

Consequently,

$$\|q_n - u^*\| \leq \|v_n - u^*\|. \tag{4.28}$$

Observe also from the condition (**C2**) that

$$\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \to 0. \tag{4.29}$$

So, there is a number $K_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \leq K_1, \forall n \in N. \tag{4.30}$$

Thus, using (4.30) and the definition of $\{u_n\}$, we obtain

$$\begin{aligned} \|x_n - u^*\| &= \|u_n + \alpha_n(u_n - u_{n-1}) - u^*\| \\ &\leq \|u_n - u^*\| + \alpha_n\|u_n - u_{n-1}\| \\ &= \|u_n - u^*\| + \beta_n \frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \\ &\leq \|u_n - u^*\| + \beta_n M_1, \forall n \in N. \end{aligned} \tag{4.31}$$

Recall that

$$2\delta_n\langle Lx_n - Lu^*, (I-S)Lx_n\rangle \geq \delta_n(1-\zeta)\|(I-S)Lx_n\|^2, \text{ since } S \text{ is } \zeta-\text{demimetric}. \tag{4.32}$$

Using the definition of z_n, (4.32) for all $u^* \in \Gamma$, then from Algorithm 3.1, we get

$$\begin{aligned} \|v_n - u^*\|^2 &= \|x_n - \delta_n L^*(I-S)Lx_n - u^*\| \\ &= \|x_n - u^*\|^2 + \delta_n^2\|L^*(I-S)Lx_n\|^2 - 2\delta_n\langle x_n - u^*, L^*(I-S)Lx_n\rangle \\ &= \|x_n - u^*\|^2 + \delta_n^2\|L^*(I-S)Lx_n\|^2 - 2\delta_n\langle Lx_n - Lu^*, (I-S)Lx_n\rangle \\ &\leq \|x_n - u^*\|^2 + \delta_n^2\|L^*(I-S)Lx_n\|^2 - \delta_n(1-\zeta)\|(I-S)Lx_n\|^2 \\ &= \|x_n - u^*\|^2 + \delta_n^2\|L^*(I-S)Lx_n\|^2 - \delta_n\tau_n\|L^*(I-S)Lx_n\|^2 \\ &\leq \|x_n - u^*\|^2 - \delta_n(\tau_n - \delta_n)\|L^*(I-S)Lx_n\|^2 \\ &= \|x_n - u^*\|^2 - (1-\sigma_n)(1-\zeta)\delta_n\|(I-S)Lx_n\|^2 \\ &\leq \|x_n - u^*\|^2. \end{aligned} \tag{4.33}$$

Using (4.22), (4.28), (4.31) and (4.32), we get

$$
\begin{aligned}
\|u_{n+1}-u^*\| &\leq (1-\beta_n(1-\rho))\|u_n-u^*\|+\beta_n(1-\rho)[\frac{\|f(u^*)-u^*\|}{(1-\rho)}+\frac{\alpha_n\|u_n-u_{n-1}\|}{\beta_n(1-\rho)}] \\
&\leq (1-\beta_n(1-\rho))\|u_n-u^*\|+\beta_n(1-\rho)K \\
&\leq max\{\|u_n-u^*\|, K\} \\
&\vdots \\
&\leq max\{\|u_1-u^*\|, K\} \qquad (4.34)
\end{aligned}
$$

where $K = \frac{\|f(u^*)-u^*\|}{(1-\rho)} + K_2$ with $K_2 = \sup \frac{\alpha_n\|u_n-u_{n-1}\|}{\beta_n(1-\rho)} > 0$.
Therefore, $\{u_n\}$ is bounded. Consequently, $\{v_n\}, \{z_n\}$ and $\{x_n\}$ are all bounded sequences. □

Theorem 4.1. *The sequence $\{u_n\}$ given by Algorithm 3.18 is convergent in norm to a point $u^* = P_\Gamma f(u^*)$ i.e.,*

$$\langle f(u^*)-u^*, u-u^*\rangle \leq 0 \,\forall\, u \,\in\, \Gamma.$$

Proof. From the definition of Γ, we know Γ is closed and convex. Furthermore, assuming $\Gamma \neq \emptyset$ we have that P_Γ is well defined. Utilizing (4.33) gives

$$\|x_n-u^*\|^2 \quad\leq\quad \|x_n-u^*\|^2-\delta_n(1-\sigma_n)(1-\zeta)\|(I-S)Lx_n\|^2. \qquad (4.35)$$

Set $Q_n = \delta_n(1-\sigma_n)(1-\zeta)\|(I-S)Lx_n\|^2$. Notice from Lemma 2.4 (ii) that

$$
\begin{aligned}
\|x_n-u^*\|^2 &\leq \|u_n-u^*\|^2+2\alpha_n\langle x_n-u^*, u_n-u_{n-1}\rangle \\
&\leq \|u_n-u^*\|^2+2\alpha_n\|x_n-u^*\|.\|u_n-u_{n-1}\|. \qquad (4.36)
\end{aligned}
$$

By convexity of $||\cdot||^2$, we obtain;

$$
\begin{aligned}
&\|\beta_n(f(v_n)-f(u^*))+(1-\beta_n)[((1-\mu)I+\mu T)q_n-u^*]\|^2 \qquad (4.37)\\
&\leq \beta_n\|f(v_n)-f(u^*)\|^2+(1-\beta_n)\|((1-\mu)I+\mu T)q_n-u^*\|^2 \\
&\leq \beta_n\|f(v_n)-f(u^*)\|+(1-\beta_n)\|q_n-u^*\|^2-\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|^2 \\
&\leq \beta_n\rho^2\|v_n-u^*\|^2+(1-\beta_n)\|v_n-u^*\|^2-\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|^2 \\
&= (1-\beta_n(1-\rho^2))\|v_n-u^*\|^2-\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|^2 \\
&\leq (1-\beta_n(1-\rho^2))\|x_n-u^*\|^2-(1-\beta_n(1-\rho^2))\delta_n(1-\sigma_n)(1-\zeta)\|(I-S)Lx_n\|^2 \\
&\quad -\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|^2 \\
&\leq (1-\beta_n(1-\rho))\|u_n-u^*\|^2+2(1-\beta_n(1-\rho))\alpha_n\|x_n-u^*\|\|u_n-u_{n-1}\| \\
&\quad -(1-\beta_n(1-\rho))Q_n-\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|^2. \qquad (4.38)
\end{aligned}
$$

Also, from Lemma 2.4 (ii) and (4.37), we get that

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &\leq \|\beta_n(f(v_n)-f(u^*))+(1-\beta_n)[((1-\mu)I+\mu T)q_n-u^*]\|^2\\ &\quad +2\beta_n\langle f(u^*)-u^*,u_{n+1}-u^*\rangle\\ &\leq (1-\beta_n(1-\rho))\|u_n-u^*\|^2+2(1-\beta_n(1-\rho))\alpha_n\|x_n-u^*\|\|u_n-u_{n-1}\|\\ &\quad +2\beta_n\langle f(u^*)-u^*,u_{n+1}-u^*\rangle-(1-\beta_n(1-\rho))Q_n\\ &\quad -\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|. \end{aligned} \tag{4.39}$$

Set $R_n=\|u_n-u^*\|^2, \theta_n=\beta_n(1-\rho), C_n=(1-\theta_n)Q_n+\mu(1-\beta_n)(1-k-\mu)\|(I-T)q_n\|^2$, $d_n=2(1-\theta)\alpha_n\|x_n-u^*\|.\|u_n-u_{n-1}\|+2\beta_n\langle f(u^*)-u^*,u_{n+1}-u^*\rangle$, and $b_n=\frac{d_n}{\theta_n}$. Then,

$$R_{n+1}\leq(1-\theta_n)D_n+b_n\beta_n$$

and

$$R_{n+1}\leq D_n-C_n+d_n,\forall n\geq 1.$$

By condition **C1**, we have $\{\theta_n\}\subset(0,1),\sum_{n=1}^{\infty}\theta_n=\infty$ and $\lim_{n\to\infty}d_n=0$.
Assume that $\lim_{k\to\infty}C_{n_k}=0$ for any subsequence $\{C_{n_k}\}$ of $\{C_n\}$.
Then,

$$\lim_{k\to\infty}Q_{n_k}=0=\lim_{k\to\infty}||(I-T)q_{n_k}||^2. \tag{4.40}$$

It follows that

$$\lim_{k\to\infty}\|L^*(I-S)Lx_{n_k}\|^2=0 \text{ and } \lim_{k\to\infty}\|I-S)Lx_{n_k}\|^2=0, \text{ from (4.21)}. \tag{4.41}$$

Furthermore,

$$\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\|^2=0. \tag{4.42}$$

To conclude the proof, we need the following fact which we state and prove as a proposition.

Proposition 4.1. *Let the sequence $\{u_n\}$ given by Algorithm 3.1 satisfy conditions* **C1**, **C2** *and* **C3**. *Suppose there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to a point $q\in H_1$ and*

$$\lim_{k\to\infty}\|v_{n_k}-z_{n_k}\|=\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=0.$$

Then, $q\in\Gamma$.

Proof. Recalling that $\{u_n\}$ is bounded, we can extract a subsequence $\{u_{n_{k_j}}\}$ of $\{u_n\}$ converging weakly to a point $q\in H_1$ and $\limsup_{k\to\infty}\langle f(u^*)-u^*,u_{n_k}-u^*\rangle=\lim_{j\to\infty}\langle f(u^*)-u^*,u_{n_{k_j}}-u^*\rangle$. Consequently, using (4.42), $\{u_{n_{k_j}}\}$ converges weakly to $q\in H_1$. Furthermore, from the hypothesis we have that $\{z_{n_{k_j}}\}$ and $\{v_{n_{k_j}}\}$ converge weakly to q. By linearity of L, we conclude that $\{Lx_{n_{k_j}}\}$converges weakly to Lq. By (4.41) and Lemma 2.5, we have that

$$Lq\in F(S). \tag{4.43}$$

Suppose that $(\tilde{v},v)\in G(A+B)$, then $v-A\tilde{v}\in B(\tilde{v})$. Furthermore, from the Algorithm (3.18) we get

$$\frac{1}{\lambda_{n_{k_j}}}(v_{n_{k_j}}-\lambda_{n_{k_j}}Av_{n_{k_j}}-z_{n_{k_j}})\in B(z_{n_{k_j}}). \tag{4.44}$$

Using monotonicity of B we obtain

$$\langle \tilde{v} - v_{n_{k_j}}, v - A\tilde{v} - \frac{1}{\lambda_{n_{k_j}}}(v_{n_{k_j}} - \lambda_{n_{k_j}} A v_{n_{k_j}} - z_{n_{k_j}})\rangle \geq 0. \tag{4.45}$$

We then obtain from (4.45) above and from the monotonicity of A that

$$\begin{aligned}
\langle \tilde{v} - z_{n_{k_j}}, v\rangle &\geq \langle \tilde{v} - z_{n_{k_j}}, A\tilde{v} + \frac{1}{\lambda_{n_{k_j}}}(v_{n_{k_j}} - z_{n_{k_j}}) - Av_{n_{k_j}}\rangle \\
&= \langle \tilde{v} - z_{n_{k_j}}, A\tilde{v} - A(z_{n_{k_j}})\rangle + \langle \tilde{v} - z_{n_{k_j}}, A(z_{n_{k_j}}) - A(v_{n_{k_j}})\rangle \\
&\quad + \langle \tilde{v} - z_{n_{k_j}}, \frac{1}{\lambda_{n_{k_j}}}(v_{n_{k_j}} - z_{n_{k_j}})\rangle \\
&\geq \langle \tilde{v} - z_{n_{k_j}}, A(z_{n_{k_j}}) - A(v_{n_{k_j}})\rangle + \langle \tilde{v} - z_{n_{k_j}}, \frac{1}{\lambda_{n_{k_j}}}(w_{n_{k_j}} - y_{n_{k_j}})\rangle. \qquad (4.46)
\end{aligned}$$

From (4.26), we know that $\lim_{k\to\infty} \lambda_{n_{k_j}} > 0$. Furthermore, from hypothesis, $\lim_{k\to\infty} \|v_{n_{k_j}} - z_{n_{k_j}}\| = 0$. Using the Lipschitz continuity of A, we get that $\lim_{k\to\infty} \|Av_{n_{k_j}} - Az_{n_{k_j}}\| = 0$. Thus, we obtain from (4.46) that

$$\langle \tilde{v} - q, v\rangle \geq 0. \tag{4.47}$$

Clearly, $A + B$ is maximal monotone by Lemma 2.1, hence

$$0 \in (A + B)q. \tag{4.48}$$

Next, we show that $\lim_{n\to\infty} \|u_{n_{k_j}+1} - u_{n_{k_j}}\| = 0$, indeed we have

$$\begin{aligned}
\|u_{n_{k_j}+1} - q_{n_{k_j}}\| &\leq \beta_{n_{k_j}} \|f(v_{n_{k_j}}) - q_{n_{k_j}}\| + (1 - \beta_{n_{k_j}})\|((1-\mu)I + \mu T)q_{n_{k_j}} - q_{n_{k_j}}\| \\
&\leq \beta_{n_{k_j}} \|f(v_{n_{k_j}}) - q_{n_{k_j}}\| + (1 - \beta_{n_{k_j}})\|(I - T)q_{n_{k_j}}\| \to 0. \qquad (4.49)
\end{aligned}$$

Therefore,

$$\|u_{n_{k_j}+1} - u_{n_{k_j}}\| \leq \|u_{n_{k_j}+1} - q_{n_{k_j}}\| + \|q_{n_{k_j}} - x_{n_{k_j}}\| + \|x_{n_{k_j}} - u_{n_{k_j}}\| \to 0. \tag{4.50}$$

It follows that

$$\|u_{n_{k_j}} - q_{n_{k_j}}\| \leq \|u_{n_{k_j}} - u_{n_{k+1}}\| + \|u_{n_{k_j}+1} - q_{n_{k_j}}\| \to 0. \tag{4.51}$$

Since $\{u_{n_{k_j}}\}$ converges weakly to $q \in H_1$, then (4.51) implies that $\{q_{n_{k_j}}\}$ also converges weakly to $q \in H_1$. Utilizing conclusion (4.40) and Lemma 2.5, we get

$$q \in F(T). \tag{4.52}$$

Combining (4.43), (4.48) and (4.52), we immediately get that $q \in \Gamma$. This conclude s proof of the proposition. □

Now, we can conclude the proof of Theorem 4.1. From (4.39) we get $\limsup_{k\to\infty}\langle f(u^*) - u^*, u_{n_k} - u^*\rangle \leq 0$. Also, $\alpha_{n_{k_j}} \frac{\|x_{n_k} - u^*\|\|u_{n_k} - u_{n_k - 1}\|}{\beta_{n_{k_j}}(1-\rho)} \to 0$.
This gives $\limsup_{k\to\infty} b_{n_k} \leq 0$. Using Lemma 2.5, we get $\|u_n - u^*\| \to 0$, which means, $u_n \to u^* = P_\Gamma f(u^*)$. The proof of Theorem 4.1 is completed. □

Remark 4.5. According to Remark 2.1, the class of Meir-Keeler contraction mappings studied in Theorem 4.1 properly contains the class of contraction mappings. So, if we take the mapping f in Algorithm 3.18 to be a contraction map, Theorem 4.1 still holds. Furthermore, if we take the mapping S in Algorithm 3.18 to be nonexpansive, Theorem 4.1 also holds. Hence, Theorem 4.1 includes the main result of Izuchukwu *et al.* [26].

Many authors (see e.g Xu [51]) have studied a very interesting problem of finding a vector of minimum norm in different applications. In the next result, we solve minimum norm problem by applying Theorem 4.1.
Observe that if $f \equiv 0$, then Algorithm 3.18 reduces to:

$$(4.53)\qquad \begin{cases} x_n = u_n + \alpha_n(u_n - u_{n-1}), \\ v_n = x_n - \delta_n L^*(I-S)Lx_n, \\ z_n = J^B_{\lambda_n}(I - \lambda_n A)v_n, \\ q_n = z_n - \lambda_n(Av_n - Az_n), \\ u_{n+1} = (1-\beta_n)(\mu I + (I-\mu)T)q_n, \end{cases}$$

where $\delta_n = \sigma_n \tau_n$, and

$$(4.54)\qquad \tau_n = \begin{cases} \frac{\|(I-S)Lx_n\|^2}{\|L^*(I-S)Lx_n\|^2}, Lx_n \neq SLx_n, \\ 0, \qquad \text{otherwise}, \end{cases}$$

$$(4.55)\qquad \lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|v_n - z_n\|}{\|Av_n - Az_n\|}, \lambda_n\}, Av_n \neq Az_n, \\ \lambda_n, \qquad \text{otherwise}. \end{cases}$$

Theorem 4.2. *Assume that* $(A1) - (A3)$ *and* $(A5)$ *of Algorithm 3.18 hold and that conditions* $(\mathbf{C1}) - (\mathbf{C3})$ *of Algorithm 3.18 are also satisfied. Then,* $\{u_n\}$ *the sequence generated by Algorithm 4.53 converges in norm to* $u^* = P_\Gamma(0)$. *This means the minimum-norm element of* Γ *is* u^*.

Proof. It is directly derived from the proof of Theorem 4.1. □

Remark 4.6. (1) If T is $k-$strictly pseudo-contractive with $F(T) \neq \emptyset$, then it is $k-$demicontractive and the complement $(I - T)$ is demiclosed at 0.
(2). Quasi-nonexpansive mappings are $0-$ demicontractive mappings and include the nonexpansive mappings having nonempty fixed point sets.
If we set the map T to be either $k-$strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ or Quasi-nonexpansive mapping, in Algorithm (3.18), Therorem 4.1 still holds. Therefore Theorem 4.1 of this article is an important generalization and improvement on many existing results in the literature (see [10, 28, 33, 37]).

Remark 4.7. Note that if we set $S = J^M_\lambda(I - \lambda f)$, with the operators $M : H_2 \to 2^{H_2}$ maximal monotone and $f : H_2 \to H_2$ being $\alpha-$ inverse strongly monotone, and $\lambda \in (0, 2\alpha)$. Then S is nonexpansive, since it is an averaged-nonexpansive mapping.

5. Applications

The split linear inverse problem

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert paces. Let $\mathcal{F} : \mathcal{H}_1 \to \mathbb{R}$ be a continuously differentiable and convex function, and $\mathcal{G} : \mathcal{H}_1 \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and $S : \mathcal{H}_2 \to \mathcal{H}_2$ a non-linear single-valued mapping. We consider the problem

$$(5.56)\ \text{Find}\ \ u^* \in \mathcal{H}_1\ \ \text{such that}\ \ \mathcal{F}(u^*) + \mathcal{G}(u^*) = \min[\mathcal{F}(u) + \mathcal{G}(u)], \text{and}\ \ Tu^* \in F(S).$$

The problem (5.56) is a finite dimensional Split Linear Inverse Problem (SLIP) (see [15]).

Assuming $u^* \in \Psi$ the solution set of (5.56). It is known that whenever $\mathcal{F}$ is continuously differentiable and convex, then the gradient of $\mathcal{F}$ i.e., $\nabla\mathcal{F}$ is continuous and monotone.

Also, the subdifferential $\partial\mathcal{G}$ of $\mathcal{G}$ is always maximal monotone whenever $\mathcal{G}$ is lower semi-continuous and convex (see [15]). Obviously,

$$\mathcal{F}(u^*) + \mathcal{G}(u^*) = \min[\mathcal{F}(u) + \mathcal{G}(u)] \Leftrightarrow 0 \in \nabla\mathcal{F}(u^*) + \partial\mathcal{G}(u^*). \tag{5.57}$$

Setting $A = \nabla\mathcal{F}$, $B = \partial\mathcal{G}$ and S to be nonexpansive in Algorithm 3.18, we deduce the following method for SLIP (5.56).

Algorithm 5.1: Adaptation of Algorithm (3.18) **to SLIP 5.56**

Initialization: Take $\alpha_n \in [0, \alpha] \subset [0, 1), \delta_n \in [a, b] \subset (0, 1), \theta_n \in (0, 1), \mu \in (0, 1-k)$. Take $u_0, u_1 \in \mathcal{H}_1$. Given the iterate u_n and u_{n-1},
Compute

$$\begin{cases} x_n = u_n + \alpha_n(u_n - u_{n-1}), \\ v_n = x_n - \delta_n T^*(I - S)Tx_n, \\ z_n = J^B_{\lambda_n}(I - \lambda_n A)v_n = (I + \lambda_n\partial\mathcal{G})^{-1}(I - \lambda_n\nabla\mathcal{F})v_n, \\ q_n = z_n - \lambda_n(\nabla\mathcal{F}v_n - \nabla\mathcal{F}z_n), \\ u_{n+1} = \beta_n f(v_n) + (1 - \beta_n)((\mu I) + (I - \mu)T)q_n, \end{cases} \tag{5.58}$$

where $\delta_n = \sigma_n\tau_n$, and

$$\delta_n = \begin{cases} \frac{\|(I-S)Tx_n\|}{\|T^*(I-S)Tx_n}, Tx_n \neq STx_n \\ 0, \qquad \text{otherwise}, \end{cases} \tag{5.59}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|v_n - z_n\|}{\|\nabla\mathcal{F}v_n - \nabla\mathcal{F}z_n\|}, \lambda_n + \rho_n\}, \nabla\mathcal{F}v_n \neq \nabla\mathcal{F}z_n, \\ \lambda_n, \qquad \text{otherwise}. \end{cases} \tag{5.60}$$

Remark 5.8. We remark that the Least Absolute Selection and Shrinking Operator (LASSO) problem is indeed a particular case of the SLIP (5.56) in that, for $\mathcal{F}(u) = \frac{1}{2}\|Du - z\|_2^2,$ and gradient $\nabla\mathcal{F}(u) = D^*(Du - z)$ is monotone and $||D||^2-$ Lipschitz continuous. Thus, the LASSO problem becomes the minimization problem:

$$\min_{u \in \mathbb{R}^N} \frac{1}{2}\|Du - z\|_2^2 + \lambda\|u\|_1, \tag{5.61}$$

where $\lambda > 0, z \in \mathbb{R}^N$ and $D \in \mathbb{R}^{M\times N}$ is a matrix (see [8, 52] for detials).

We note that the minimization problem (5.61) can be recast into a problem of second-order cone programming, which is central in the development of two classical algorithms: the Iteration Shrinking Thresholding Algorithm (ISTA) and the Fast Iteration Shrinking Thresholding Algorithm (FISTA). Both are known to efficiently solve SLIP (5.56) (see [8, 14, 52]).

Remark 5.9. The following example indicates that approximation or computation of Lipschitz constants of Lipschitz continuous operators or norm of operators is generally difficult. Consequently, the applicability of the results in [33, 40, 44] to the LASSO problem may be affected negatively.

Theorem 5.3 ([23] Theorem 2.3)**.** *Let a rational number $p \in [1, \infty)$, $p \neq 1, 2$ be given. Except for $p = np$, no algorithm exists to compute the $p-$norm of a matrix having its entries in $\{-1, 0, 1\}$ to relative error with running time polynomial in the dimensions.*

Therefore, our algorithm (3.18) which is independent of the Lipschitz constant and norm of the associated bounded linear map is better for application purposes than the results in [26, 33, 44].

6. Numerical illustrations

In infinite dimensional real Hilbert spaces, we consider an example to demonstrate that our algorithm is implementable. Furthermore, efficiency of our scheme is compared with those of Izuchukwu *et al* [26]. All our codes were written in MATLAB R2021a. The specification of the personal computer used in our computations is; 11th Gen Intel(R) Core(TM) i7-1165G7, 2.80 GHz with 16.00 Gb-RAM and 64-bit-OS.
The vectors $u_0, u_1 \in H_1 = L^2([0,1])$. In Algorithm (3.18) of our work and Algorithm 1.12 of Izuchukwu *et al.* [26], we generate the step size λ_n at each iteration. In Algorithm (1.12) taking $\alpha = 4, 8, 10$, we compute α_n at each iteration step by choosing $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n := \begin{cases} \min\left\{\dfrac{n-1}{n-1+\alpha}, \dfrac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\}, & \text{if } u_n \neq u_{n-1}, \\[2ex] \dfrac{n-1}{n-1+\alpha}, & \text{otherwise.} \end{cases}$$

Meanwhile, for our Algorithm (3.18), $\alpha_n \in (0,1)$ is easily chosen and is fixed. Our stopping criteria is set as $\|e_n\| = \|u_{n+1} - u_n\|_{L^2([0,1])} \leq 10^{-4}$.

Example 6.1. Let $H_1 = H_2 = L_2([0,1])$. We define an inner product and norm on H_1, H_2 as;

$$\langle u, z\rangle := \int_0^1 u(t)z(t)dt \ \ \forall u, z \in L_2([0,1])$$

and

$$\|u\|_{L_2([0,1])} := \left(\int_0^1 |u(t)|^2 dt\right)^{1/2} \quad \forall u \in L_2([0,1]).$$

Let the operators $A, B : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Au(t) = \frac{1}{2}u(t), \ \text{ and } \ Bu(t) = \max\{0, u(t)\}, \ t \in [0,1], \ \ u \in L_2([0,1]).$$

Clearly, A is monotone and Lipschitz while B is maximal monotone, on $L_2([0,1])$ (see [22]), moreover, we have

$$J^B_{\lambda_n} u(t) = (I + \lambda_n B)^{-1} u(t) := \begin{cases} \dfrac{1}{1+\lambda_n} u(t), & \text{if } u(t) > 0, \\[2ex] u(t), & \text{if } u(t) \leq 0. \end{cases}$$

We define $L : L_2([0,1]) \to L_2([0,1])$ by

$$Lu(s) = \int_0^1 K(s,t)u(t)dt \quad \forall u \in L_2([0,1]),$$

where $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Thus, L is a bounded linear operator whose adjoint is

$$L^* u(s) = \int_0^1 K(t,s)u(t)dt \quad \forall u \in L_2([0,1]).$$

We define $g : L_2([0,1]) \to [0,\infty)$ by $g(u) = \|u\|_{L_2([0,1])}$. We take $S = J^{\partial g}_{\gamma}$ the resolvent of ∂g (subdifferential of g) with respect to $\gamma > 0$. Obviously, g is continuous, convex and maximal monotone. We deduce that the resolvent of ∂g given as $J_\gamma^{\partial g} = (I+\gamma\partial g)^{-1}$ is $(-1)-$demimetric (see [46]). In addition, S is nonexpansive and we can explicitly compute S as (see [7] Example 24.20):

$$Su = J_\gamma^{\partial g}u = \text{Prox}_{\gamma\|\cdot\|_{L_2([0,1])}}u = \left(1 - \frac{\gamma}{\max\{\|u\|_{L_2([0,1])}, \gamma\}}\right)u.$$

Next, we take the operator $T \equiv I$, the identity map on $L^2([0,1])$, consequently, T is 0-demicontractive. Furthermore, let's define $f : L_2([0,1]) \to L_2([0,1])$ as

$$fu(t) = \int_0^1 \frac{t}{2}u(s)ds, \qquad t \in [0,1].$$

Therefore, f is a Mier-Keeler contraction map, because it is clearly a contraction map.
The following scenarios are studied for our numerical results appearing in the figures and tables below.
Case 1: Take $u_0(t) = e^t,\ u_1(t) = t^2+1$;
Case 2: Take $u_0(t) = t,\ u_1(t) = te^t$;
Cases 3: Take $u_0(t) = t^2+1,\ u_1(t) = \cos t + 2t$;
Cases 4: Take $u_0(t) = e^t,\ u_1(t) = t\sin(t^2)$.

TABLE 1. Comparison of our proposed Algorithm (3.18) and Algorithm (1.12), with $S = J_\gamma^{\partial g}$.

λ_1	CASES	$\alpha_n = 10^{-2}$	Time	Iter.	$\alpha = 4$	Time	Iter.
		Alg. (3.18)			Alg. (1.12)		
0.01	I		3.9755	15		6.9229	24
	II		3.9304	15		6.3071	22
	III		6.6694	20		31.5942	31
	IV		1.1627	5		1.8860	7
λ_1	CASES	$\alpha_n = 10^{-3}$	Time	Iter.	$\alpha = 8$	Time	Iter.
		Alg. (3.18)			Alg. (1.12)		
0.5	I		3.3784	13		5.9493	21
	II		3.0593	12		5.7421	20
	III		5.5292	17		8.2208	28
	IV		0.8742	4		1.5470	6
λ_1	CASES	$\alpha_n = 10^{-4}$	Time	Iter.	$\alpha = 10$	Time	Iter.
		Alg. (3.18)			Alg. (1.12)		
1	I		3.0800	12		5.6116	20
	II		2.7826	11		5.4660	19
	III		5.3319	16		7.8963	27
	IV		0.8648	4		1.6077	6

Remark 6.10. The following parameters were used for generating Table 1. $S = J_\gamma^{\partial g}$ for both Algorithm (3.18) and Algorithm (1.12). $\sigma_n = \dfrac{1}{2}$ in Algorithm (3.18) and Algorithm

(1.12), respectively. $\theta_n = \frac{1}{2} - \beta_n$, $\varepsilon_n = \frac{\beta_n}{n^{0.01}}$, and $\gamma = 0.01$, $\mu_n = 0.01$ and $\beta_n = \frac{1}{5n+2}$ in both Algorithms . It is inferred from Table 1 that Algorithm (3.18) has better performance than Algorithm (1.12), based on number of iterations and speed of convergence. Also, as seen from Table 1 the best performance recorded by our proposed Algorithm (3.18) is when $\lambda_1 = 1$, $\alpha_n = 10^{-4}$.

Now, for the purpose of further numerical comparison, we recall the operator $S : L_2([0,1]) \to L_2([0,1])$ given by Izuchukwu *et al* [26] and define by

$$Su(t) = \int_0^1 tu(s)ds, \quad t \in [0,1].$$

S is nonexpansive, since

$$\|S(u) - S(z)\|^2_{L_2([0,1])} = \int_0^1 |Su(t) - Sz(t)|^2 \le \|u - z\|^2_{L_2([0,1])}.$$

Therefore, for any $0 \le \zeta < 1$, S is a $\zeta-$strict pseudo-contraction, thus S is $\zeta-$demimetric for any $0 \le \zeta < 1$.

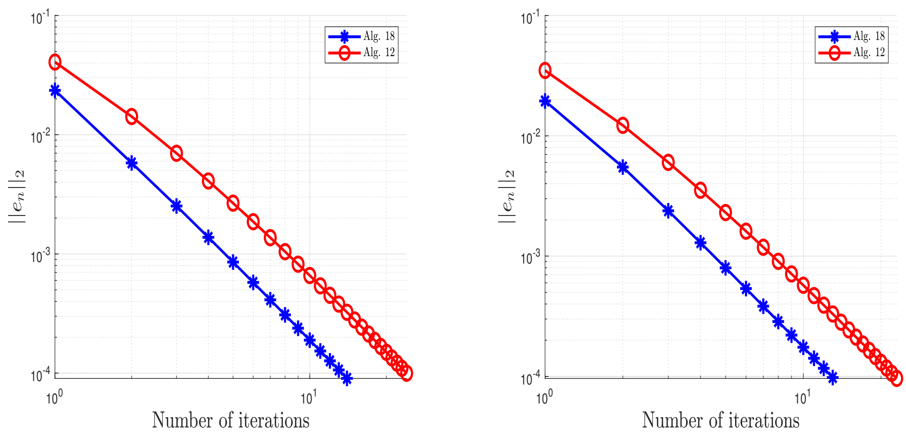
TABLE 2. Comparison of our proposed Algorithm (3.18) with Algorithm (1.12), when $Su(t) = \int_0^1 tu(s)ds$.

λ_1	CASES	$\alpha_n = 10^{-2}$	Time	Iter.	$\alpha = 4$	Time	Iter.
		Alg. (3.18)			Alg. (1.12)		
0.01	I		5.5851	17		8.0885	24
	II		5.4368	17		6.7190	23
	III		120.1772	25		172.2655	33
	IV		1.2314	5		1.9669	7
λ_1	CASES	$\alpha_n = 10^{-3}$	Time	Iter.	$\alpha = 8$	Time	Iter.
		Alg. (3.18)			Alg. (1.12)		
0.5	I		3.9468	14		7.9246	23
	II		3.4981	13		6.3024	21
	III		6.8531	20		164.7988	31
	IV		0.8939	4		1.5869	6
λ_1	CASES	$\alpha_n = 10^{-4}$	Time	Iter.	$\alpha = 10$	Time	Iter.
		Alg. (3.18)			Alg. (1.12)		
1	I		3.6095	13		6.8195	22
	II		3.2789	12		6.2676	20
	III		6.6391	19		225.1881	30
	IV		0.9347	4		1.3055	5

Remark 6.11. For Table 2 we considered the operator $Sx(t) = \int_0^1 tx(s)ds$. We take $\sigma_n = \frac{1}{10}$ in Algorithm (3.18) and in Algorithm (1.12) we take $\theta_n = \frac{1}{2} - \beta_n$, $\varepsilon_n = \frac{\beta_n}{n^{0.01}}$,

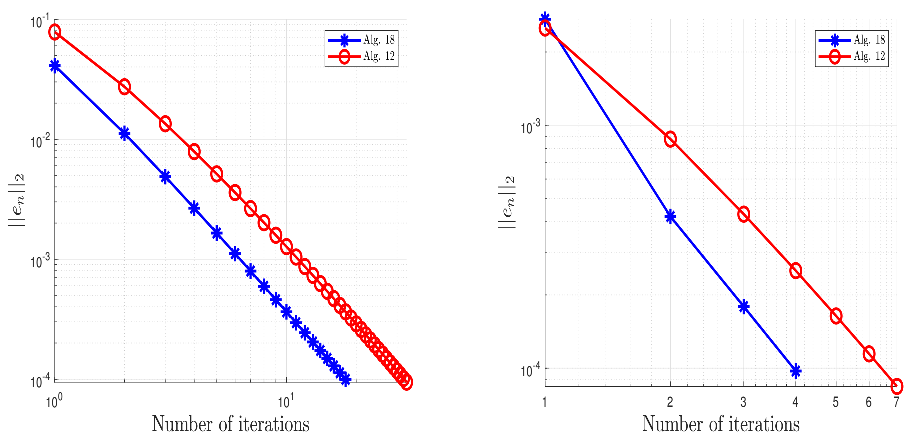
then in both Algorithms we take $\gamma = 0.01$, $\mu_n = 0.01$ and $\beta_n = \dfrac{1}{5n+2}$. Again, the results shows that for the operator $Su(t) = \int_0^1 tu(s)ds$ given in Izuchukwu *et al.*[21], our proposed Algorithm (3.18) performed better than Algorithm (1.12), based on the speed of convergence and number of iterations.

The figures below; Figure 1 and Figure 2 graphically compares the performance of Algorithm (3.18) and Algorithm (1.12) of Izuchukwu *et al.*[21]. By plotting the error $\|e_n\|_{L^2([0,1])} = \|x_{n+1} - x_n\|_{L^2([0,1])}$ against the number of iterations, our proposed Algorithm (3.18) performs better in terms of speed of convergence and the number of iterations. The parameters in Remark 6.10 were used here.



(A) Case I; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$ (B) Case II; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$

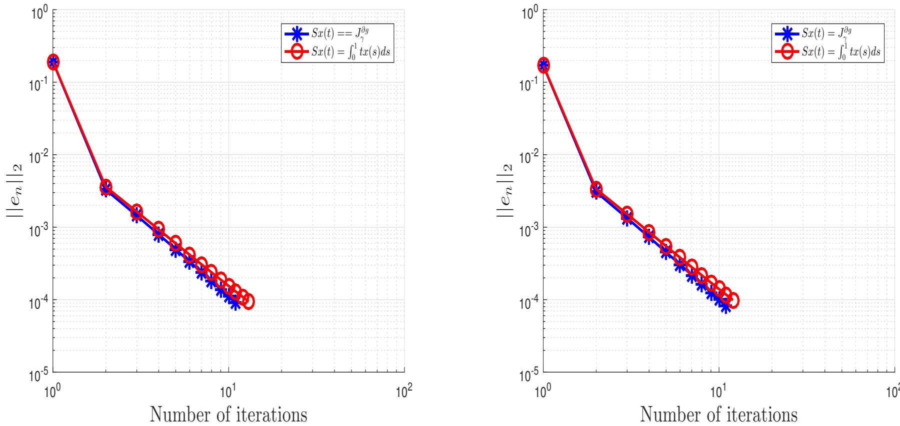
FIGURE 1. Error comparisons of the Proposed **Alg.** (3.18) and **Alg.** (1.12)



(A) Case III; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$ (B) Case IV; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$

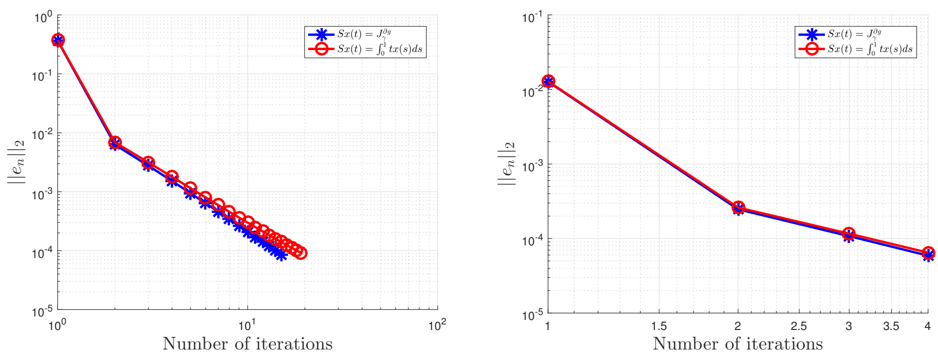
FIGURE 2. Error comparisons of the Proposed **Alg.** (3.18) and **Alg.** (1.12)

In the next figures, that is Figures 3 and 4, we present graphical comparison for performance of our proposed Algorithm (3.18) with respect to the two operators $Su(t) = J_\gamma^{\partial g}u(t)$ and $Su(t) = \int_0^1 tu(s)ds$. It is evident that Algorithm (3.18) performs better with $Su(t) = J_\gamma^{\partial g}u(t)$, both in terms of speed of convergence and number of iteration. This conclusion is also evident from Table 1 and Table 2.



(A) Case I; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$ (B) Case II; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$

FIGURE 3. Comparisons of the Proposed **Alg.** (3.18), for $Su(t) = J_\gamma^{\partial g}u(t)$ and $Su(t) = \int_0^1 tu(s)ds$



(A) Case III; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$ (B) Case IV; $\lambda_1 = 0.01$, $\alpha = 4$, $\alpha_n = 10^{-2}$

FIGURE 4. Comparisons of the Proposed **Alg.** (3.18) with $Su(t) = J_\gamma^{\partial g}u(t)$ and **Alg.** (1.12) with $Su(t) = \int_0^1 tu(s)ds$

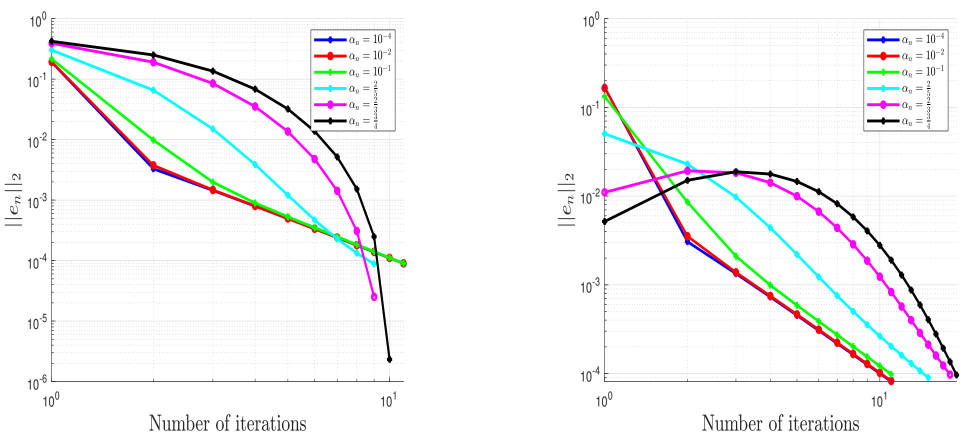
.

The table below i.e., Table 3 comperes the performance of our proposed Algorithm (3.18) for different values of α_n and different initial points in $L^2([0,1])$. We take the operator $S = J_\gamma^{\partial g}$. The following parameters are considered, in addition to those found on Table 3, $\sigma_n = \frac{1}{10}$, $\gamma = 0.01$, $\lambda_1 = 0.01$, $\mu_n = 0.01$ and $\beta_n = \frac{1}{5n+2}$.

TABLE 3. Comparison of the effects of different values of α_n on our proposed Algorithm (3.18), with $S = J_{\gamma}^{\partial g}$.

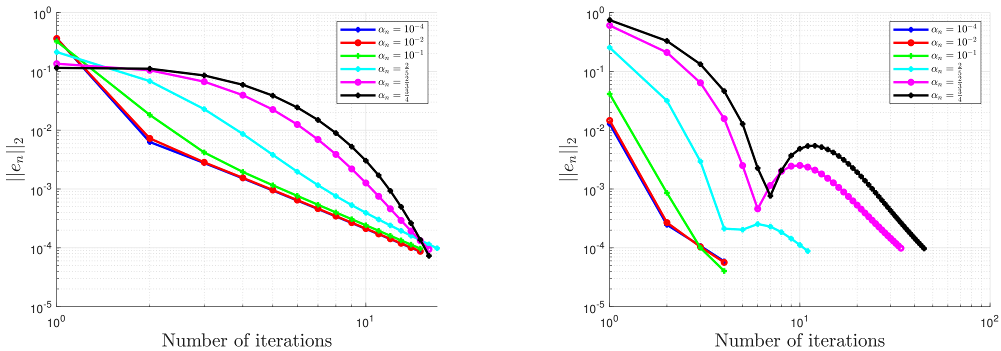
λ_1	CASES	Time	Iter.	Time	Iter.	Time	Iter.	Time	Iter.
		$\alpha_n = 10^{-4}$		$\alpha_n = 10^{-2}$		$\alpha_n = \frac{2}{3}$		$\alpha_n = \frac{3}{4}$	
0.5	I	3.1112	12	3.0568	12	2.2834	9	2.6477	10
	II	2.8471	11	2.7049	11	8.9506	22	8.0656	23
	III	5.1134	16	5.0964	16	7.5916	20	6.4028	21
	IV	0.8373	4	0.9292	4	2.0889e+03	36	2.2514e+03	46
λ_1	CASES	Time	Iter.	Time	Iter.	Time	Iter.	Time	Iter.
		$\alpha_n = 10^{-4}$		$\alpha_n = 10^{-2}$		$\alpha_n = \frac{2}{3}$		$\alpha_n = \frac{3}{4}$	
1	I	3.0800	12	2.8611	11	2.2992	9	2.5142	10
	II	2.7826	11	2.8485	11	5.1826	18	5.9806	19
	III	4.3214	13	4.7041	15	5.5497	16	6.1214	16
	IV	0.8648	4	0.8836	4	2.3362e+03	34	2.6567e+03	45

In Figure 5—Figure 6 below, with different initial points in $L^2([0,1])$ the graphs compares responses of our Algorithm (3.18) to different values of α_n with respect to the operator $S = J_{\gamma}^{\partial g}$. We took $\lambda_n = 1$, $\mu_n = 0.01$. The results in the graphs are also corroborated by the numerical data in Table 3.



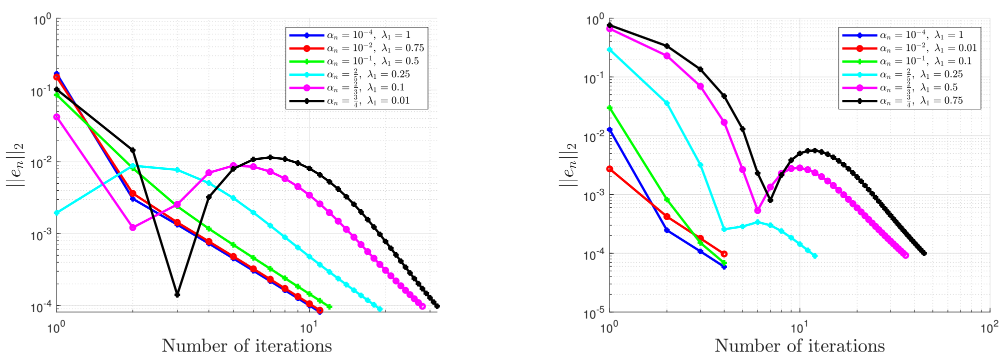
(A) Case I (B) Case II

FIGURE 5. Comparisons for different α_n values for **Alg.** (3.18) with $Su(t) = J_{\gamma}^{\partial g}u(t)$



(A) Case III (B) Case IV

FIGURE 6. Comparisons for different α_n values for **Alg.** (3.18) with $Su(t) = J_\gamma^{\partial g} u(t)$



(A) Case II (B) Case IV

FIGURE 7. Comparisons for different α_n values for **Alg.** (3.18) with $Su(t) = J_\gamma^{\partial g} u(t)$

Example 6.2.

For the purpose of further validating the performance of our proposed algorithm, we make use of the following preconditioning forward-backward splitting algorithm introduced by Altiparmak and Karahan [5], they proved that the scheme converges strongly to solution set of (MIP):

$$\begin{cases} u_n = x_n + \epsilon_n(x_n - x_{n-1}), \\ v_n = J_{\lambda,M}^{A,B}((1-\beta_n)u_n + \beta_n J_{\lambda,M}^{A,B}(u_n)), \\ x_{n+1} = (1-\gamma_n)J_{\lambda,M}^{A,B} + \gamma_n h(x_n), \end{cases} \tag{6.62}$$

where $\epsilon_n \in [0, \theta]$ with $\theta \in [0, 1)$, $\beta_n\gamma_n \in (0, 1)$, while $h : H \to H$ is a $k-$contraction mapping with respect to $M-$norm.
In the current example, we compare the performances of Algorithm 1.5, Algorithm 3.18 and Algorithm 6.62, by solving a Monotone Inclusion Problem (MIP). In line with Example 6.1, we set $L \equiv 0$ and $S \equiv 0$ which implies $\delta_n = 0$ in Algorithm 3.18. We also take

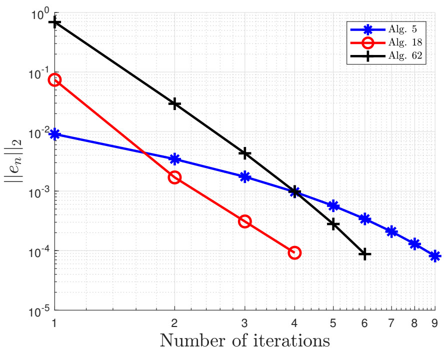
$M \equiv I$ in Algorithm 1.5 and Algorithm 6.62, where I is the identity operator.
The following scenarios are studied for our numerical results appearing in the figures and tables below.
Case 1: Take $u_0(t) = e^t,\ u_1(t) = t^2 + 1$;
Case 2: Take $u_0(t) = e^t,\ u_1(t) = e^t \sin t$;
Cases 3: Take $u_0(t) = t^2 + 1,\ u_1(t) = \cos t + 2t$;
Cases 4: Take $u_0(t) = t^3,\ u_1(t) = t \sin(t^2)$.

TABLE 4. Comparison of our proposed Algorithm 3.18, Algorithm 1.5 (with $\lambda = 0.5$) and Algorithm 6.62 (with $\lambda = 0.5$).

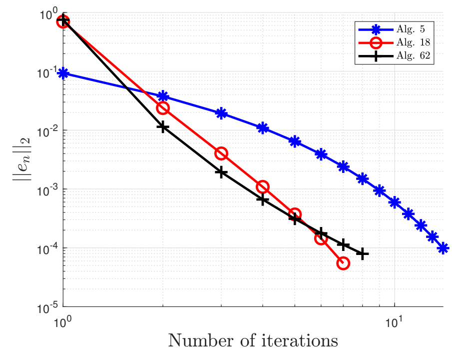
λ_1	CASES	Time	Iter.	Time	Iter.	Time	Iter.
		Alg. (3.18), $\alpha_n = 10^{-1}$		Alg. (1.5), $\epsilon = 10^{-1}$		Alg. (6.62), $\epsilon = 10^{-1}$	
0.5	I	0.8726	4	1.7342	7	1.5012	6
	II	1.8811	6	3.1398	13	2.9701	10
	III	2.7124	8	4.1015	16	3.7832	12
	IV	1.0258	4	2.1323	9	1.7945	7
λ_1	CASES	Time	Iter.	Time	Iter.	Time	Iter.
		Alg. (3.18), $\alpha_n = 10^{-2}$		Alg. (1.5), $\epsilon = 10^{-2}$		Alg. (6.62), $\epsilon = 10^{-2}$	
0.01	I	1.0795	4	1.1040	5	1.0907	5
	II	2.2125	7	4.0865	16	3.7209	13
	III	2.4709	8	4.3520	29	4.2845	22
	IV	0.9884	4	2.3351	5	1.8724	5

TABLE 5. Comparison of our proposed Algorithm 3.18, Algorithm 1.5 (with $\lambda = 1$) and Algorithm 6.62 (with $\lambda = 1$).

λ_1	CASES	Time	Iter.	Time	Iter.	Time	Iter.
		Alg. (3.18), $\alpha_n = 10^{-3}$		Alg. (1.5), $\epsilon = 10^{-3}$		Alg. (6.62), $\epsilon = 10^{-3}$	
1	I	0.9166	4	2.0922	8	1.7213	6
	II	1.6105	6	3.6934	12	3.201	10
	III	2.6667	7	4.3122	14	3.9213	11
	IV	0.8590	4	1.8212	8	1.4371	7
λ_1	CASES	Time	Iter.	Time	Iter.	Time	Iter.
		Alg. (3.18), $\alpha_n = 10^{-4}$		Alg. (1.5), $\epsilon = 10^{-4}$		Alg. (6.62), $\epsilon = 10^{-4}$	
1	I	0.8236	4	1.6342	8	1.3210	6
	II	1.5425	6	2.4302	12	2.014	11
	III	1.5874	7	2.6677	14	2.1942	12
	IV	0.8383	4	1.4689	8	1.3584	6

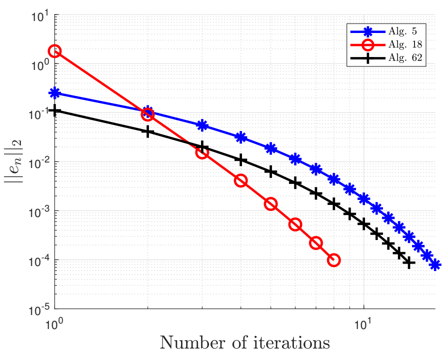


(A) Case I; $\lambda_1 = \lambda = 0.5$, $\epsilon = \alpha_n = 10^{-3}$

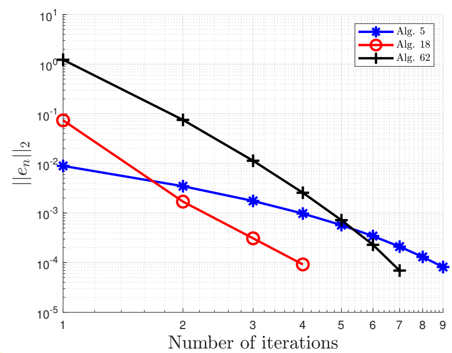


(B) Case II; $\lambda_1 = \lambda = 0.5$, $\epsilon = \alpha_n = 10^{-3}$

FIGURE 8. Error comparisons of the Proposed **Alg.** (3.18), **Alg.** (1.5) and **Alg.** (6.62)



(A) Case III; $\lambda_1 = \lambda = 0.5$, $\epsilon = \alpha_n = 10^{-3}$



(B) Case IV; $\lambda_1 = \lambda = 0.5$, $\epsilon = \alpha_n = 10^{-3}$

FIGURE 9. Error comparisons of the Proposed **Alg.** (3.18), **Alg.** (1.5) and **Alg.** (6.62)

By plotting the error $\|e_n\|_{L^2([0,1])} = \|x_{n+1} - x_n\|_{L^2([0,1])}$ against the number of iterations, Table 4 and Figure 8–Figure 9 show that our proposed Algorithm (3.18) performs better than Algorithm (1.5) in speed of convergence and number of iterations.

Remark 6.12. From the examples given above and from the Tables and Figures, most glaringly from Table 3–Table 5 and Figure 5–Figure 7 our proposed algorithm has the most optimal performance when $\lambda_1 = 1$ and $\alpha_n = 10^{-4}$.

7. CONCLUSION

We proposed and studied inertia-based iterative scheme to solve generalised split feasibility problem over the solution set of monotone variational inclusion problem. We established strong convergence of the scheme under a mild assumption that the stepsize is independent of any knowledge of Lipschitz constant of the involved single-valued operator and the norm of the bounded linear operator. The associated nonlinear maps are quite general and contains for instance nonexpansive and the projection maps, they are highly used in solving optimization problems in real Hilbert space as has been explained in Remark 3.4 above. Important Corollaries of our result were given; Remark 4.5, Theorem 4.2 and Remark 4.7. As application, we study split linear inverse problem, precisely, the LASSO problem. Furthermore, with the aid of numerical examples, we compared our method with the methods studied in [5, 26, 32]. In our comparison, we saw that our method performs better than the methods in [5, 26, 32]. Hence, our method is more general and improves many important results in the literature, for instance [5, 9, 26, 32, 33, 44].

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