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Extreme solution for fractional differential equation with nonlinear boundary condition

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ABSTRACT. In this paper, we investigate a class of fractional equations with nonlinear boundary condition. We establish a new comparison principle related to linear fractional equation and show the existence of extreme solution by using monotone iterative method and lower and upper solutions method.

1. INTRODUCTION

In this paper, we consider the following fractional differential equation

$$\begin{cases} {}_0^{LC}D_t^{\alpha}u(t) + NI^{\beta}u(t) = f(t,u(t)), t \in (0,T], \\ g(u(0),u(1)) = 0, \end{cases} \tag{1.1}$$

where $0 < T < \infty$, $0 < \alpha < 1$, $\beta \geq \alpha$, $N \leq \frac{\alpha\Gamma(\beta)}{T^{\beta+\alpha}\Gamma(1-\alpha)}$, ${}_0^{LC}D_t^{\alpha}$ is Liouville-Caputo fractional derivative and I^{β} is Riemann-Liouville fractional integral. The nonlinear functions f and g are assumed to satisfy certain conditions, which will be specified later.

Fractional-order models have proven to be a valuable tool describing many phenomena in various fields of science and engineering. For example, in the study of a sphere subjected to gravity, Basset [3, 4] introduced a special hydraulic force which was interpreted by Mainardi [15] in terms of a fractional derivative of order $\frac{1}{2}$ of the velocity of the particle relative to the fluid. In 1994, Led S. Westerlund [11] used the equality with fractional order derivative to generalize Newton's second law. A comprehensive references on fractional order models, including an extensive list of applications, can be found in [6, 7, 12, 13, 18, 19] and references therein.

By their popular applications, fractional differential equations has attracted the attention of many researchers, see [1, 2, 5, 9, 10, 14, 16, 17, 20, 21, 22, 23, 24, 25] and the references therein. For boundary value problem with nonlinear boundary conditions, monotone iteration scheme is an interesting and powerful mechanism that offers theoretical existence results. In [26], Zhang considered the following fractional differential equation

$$\begin{cases} {}_0^{LC}D_t^{\alpha}u(t) = f(t,u(t)), t \in (0,T], \\ g\left(u(0),u(T)\right) = 0, \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$, ${}_0^{LC}D_t^{\alpha}$ is Liouville-Caputo fractional derivative. The author introduced the definition of coupled lower and upper solutions. A key condition in [26] is that (1.2) has coupled lower and upper solutions U and V respectively, that is,

$$U \leq V,\ \ {}_0^{LC}D_t^{\alpha}U(t) \leq f(t,U(t)),\ \ \ {}_0^{LC}D_t^{\alpha}V(t) \geq f(t,V(t)),$$
$$g(U(0),V(T)) \leq 0 \leq g(V(0),U(T))$$

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Combining with lipschitz-like condition for f and monotonicity condition about two variables of g, the author showed that (1.2) has a solution between U and V by using monotone iterative approach.

In [27], the fractional differential equation

$$\begin{cases} {}_0^{LC}D_t^{\alpha}u(t) - du(t) = h(t), t \in (0,T], \\ g(u(0)) = u(T), \end{cases} \tag{1.3}$$

was considered, where $d \geq 0, h \in C^1[0,T], 0 < \alpha < 1$. The author obtained the existence result for (1.3) by means of the upper and lower solutions method in reverse order

In [8], Fazli, Sun, Aghchi and Nieto studied the following fractional differential equation

$$\begin{cases} u^{(m)}(t) + M_0^{LC}D_t^{\alpha}u(t) = f(t,u(t)), m-1 < \alpha < m, t \in (0,T], \\ g\left(u^{(k)}(t_0), u^{(k)}(t_1), \cdots, u^{(k)}(t_r)\right) = 0, k = 0,1,\cdots,m-1,m \end{cases} \tag{1.4}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_r = T, m \in \mathbb{N}$ and ${}_0^{LC}D_t^{\alpha}$ is Liouville-Caputo fractional derivative of order α. The authors established the comparison theorem and applied the monotone iterative approach to show the existence of the extremal solutions.

Motivated by the work above, in this study, we established the existence result of extremal solutions for (1.1). Compared with (1.2), fractional integral term I^{β} is added to (1.1), which makes the system more complex and be difficult to handle. The comparison principle in [26] is not valid to (1.1). Moreover, we relax the restriction for f and g. Our result is new ever if $N = 0$.

The paper is organized as follows. In Section 2, we recall some theorems and derive some necessary lemmas. In Section 3, we show the existence of extremal solutions by utilizing the monotone iterative technique.

2. Preliminaries

Throughout the paper, $AC[0,T]$ denotes the set of absolutely continuous functions on $[0,T]$.

Definition 2.1. The Riemann-Liouville fractional integral of order $\gamma > 0$ for the function $u : [0,T] \to \mathbb{R}$ is defined as

$$I^{\gamma}u(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}u(s)ds,$$

provided that the right-hand side integral exists and is finite.

Definition 2.2. The Liouville-Caputo fractional derivative of order $\gamma > 0$ of a function $u : [0,T] \to \mathbb{R}$ is defined as

$${}_0^{LC}D_t^{\gamma}u(t) = (I^{n-\gamma}u^{(n)})(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1}u^{(n)}(s)ds,$$

where $n-1 < \gamma \leq n$ and $n \in \mathbb{N}$, provided that the right-hand side integral exists and is finite.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\gamma > 0$ of a function $u : [0,T] \to \mathbb{R}$ is defined as

$${}_0^{R}D_t^{\gamma}u(t) = \frac{d^n}{dt^n}(I^{n-\gamma}u)(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1}u(s)ds,$$

where $n-1 < \gamma \leq n$ and $n \in \mathbb{N}$, provided that the right-hand side integral exists and is finite.

Definition 2.4. For $a > 0$ and $b \in \mathbb{R}$, the two-parameter Mittag-Leffler function $E_{a,b}(z)$ is defined by

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}, \quad z \in \mathbb{R}$$

and

$$\frac{d^k}{dz^k} E_{a,b}(z) \equiv E_{a,b}^{(k)}(z) = \sum_{j=0}^{\infty} \frac{(k+j)! z^j}{j! \Gamma(a(k+j)+b)}, \quad z \in \mathbb{R}.$$

Lemma 2.1. *If* $a \in [0,1]$, $b \geq a$ *and* $x > 0$, *then*

$$E_{a,b}^{(k)}(-x) \geq 0, \quad k = 0, 1, 2, ...,$$

Proof. By [12], $E_{a,b}(-x)$ is completely monotone on $\mathbb{R}_+$. Hence, we have

$$(-1)^k \frac{d^k E_{a,b}(-x)}{dx^k} \geq 0, \quad x > 0, \quad k = 0, 1, \cdots,$$

that is,

$$(-1)^k \frac{d^k E_{a,b}(-x)}{dx^k} = (-1)^{2k} E_{a,b}^{(k)}(-x) \geq 0,$$

which implies that $E_{a,b}^{(k)}(-x) \geq 0$. □

Consider the linear problem

$$\begin{cases} {}_0^{LC}D_t^{\alpha} u(t) + Mu(t) + NI^{\beta} u(t) = h(t), t \in (0,T], \\ u(0) = x_0, \end{cases} \tag{2.5}$$

where M, N, x_0 are real constants and $h : [0,T] \to \mathbb{R}$ is continuous.

Lemma 2.2. *Let* $0 < \alpha < 1$ *and* $\beta \geq \alpha$, *then* (2.5) *has a unique solution* $u \in AC[0,T]$ *with*

$$\begin{aligned} u(t) = & \sum_{k=0}^{\infty} (-1)^k \frac{N^k}{k!} x_0 t^{k(\alpha+\beta)} E_{\alpha,k\beta+1}^{(k)}(-Mt^{\alpha}) \\ & + \int_0^t h(s) \sum_{k=0}^{\infty} (-1)^k \frac{N^k}{k!} (t-s)^{\alpha(k+1)+k\beta-1} E_{\alpha,\alpha+k\beta}^{(k)}(-M(t-s)^{\alpha}) ds, \end{aligned} \tag{2.6}$$

Proof. From Lemma 2.7 and Lemma 2.9 of [12], we have

$$L[I^{\beta} u](s) = s^{-\beta} L[u](s), \quad L[{}_0^{LC}D_t^{\alpha} u](s) = s^{\alpha} L[u](s) - s^{\alpha-1} u(0).$$

where L denotes the Laplace transform operator, $L[u]$ denotes the Laplace transform of u. We do Laplace transform to (2.5) and obtain that

$$s^{\alpha} U(s) - s^{\alpha-1} u(0) + MU(s) + Ns^{-\beta} U(s) = H(s),$$

where U, H denote the Laplace transform of u and h. Hence,

$$\begin{aligned} U(s) &= \frac{H(s)+s^{\alpha-1}u(0)}{s^\alpha+M+Ns^{-\beta}} = \frac{H(s)+s^{\alpha-1}u(0)}{s^\alpha+M}\frac{1}{1+\frac{Ns^{-\beta}}{s^\alpha+M}} \\ &= \frac{H(s)+s^{\alpha-1}u(0)}{s^\alpha+M}\sum_{k=0}^{\infty}(-1)^k\left(\frac{Ns^{-\beta}}{s^\alpha+M}\right)^k \\ &= (H(s)+s^{\alpha-1}u(0))\sum_{k=0}^{\infty}(-1)^k\frac{(Ns^{-\beta})^k}{(s^\alpha+M)^{k+1}} \\ &= \sum_{k=0}^{\infty}(-1)^kN^k\left(\frac{s^{\alpha-k\beta-1}u(0)}{(s^\alpha+M)^{k+1}}+\frac{s^{-k\beta}H(s)}{(s^\alpha+M)^{k+1}}\right). \end{aligned}$$

From the following equality in [6]

$$\int_0^\infty e^{-pt}t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^\alpha)dt = \frac{k!p^{\alpha-\beta}}{(p^\alpha\mp a)^{k+1}},$$

we have

$$L\left[t^{k(\alpha+\beta)}E^{(k)}_{\alpha,k\beta+1}(-Mt^\alpha)\right](s) = \frac{k!s^{\alpha-k\beta-1}}{(s^\alpha+M)^{k+1}},$$

and

$$L\left[t^{\alpha(k+1)+k\beta-1}E^{(k)}_{\alpha,\alpha+k\beta}(-Mt^\alpha)\right](s) = \frac{k!s^{-k\beta}}{(s^\alpha+M)^{k+1}}.$$

Therefore, by the inverse Laplace transform, we can obtain (2.6). Since h is continuous, u is continuous. From Proposition 4.6 of [12], one can obtain that $u \in AC[0,T]$. The proof is completed. □

Remark 2.1. If $u \in AC[0,T]$ satisfied equation (2.5), where $h \in L^1(0,T)$, then u satisfies (2.6).

Remark 2.2. Under the condition of Lemma 2.2, one can define an operator A in $C[0,1]$ by $Ah = u$, that is,

$$Ah(t) := \sum_{k=0}^{\infty}(-1)^k\frac{N^k}{k!}x_0t^{k(\alpha+\beta)}E^{(k)}_{\alpha,k\beta+1}(-Mt^\alpha) + \int_0^t h(s)K(t,s)ds,$$

where $K(t,s) = \sum_{k=0}^{\infty}(-1)^k\frac{N^k}{k!}(t-s)^{\alpha(k+1)+k\beta-1}E^{(k)}_{\alpha,\alpha+k\beta}(-M(t-s)^\alpha)$. Moreover $A : C[0,T] \to C[0,T]$ is compact.

Proof. Let $D \subset C[0,T]$ be a bounded set and $u \in D$, $t_1, t_2 \in [0,T], t_1 \geq t_2$, then

$$\begin{aligned} |(Au)(t_1)-(Au)(t_2)| \leq& \sum_{k=0}^{\infty}\frac{|N|^k}{k!}x_0\left|t_1^{k(\alpha+\beta)}E^{(k)}_{\alpha,k\beta+1}(-Mt_1^\alpha) - t_2^{k(\alpha+\beta)}E^{(k)}_{\alpha,k\beta+1}(-Mt_2^\alpha)\right| \\ &+ \int_{t_2}^{t_1}|u(s)||K(t_1,s)|ds + \int_0^{t_2}|u(s)||K(t_1,s)-K(t_2,s)|ds, \end{aligned}$$

Since functions $t^{k(\alpha+\beta)}E^{(k)}_{\alpha,k\beta+1}(-Mt^\alpha) : [0,T] \to \mathbb{R}$ and $K : [0,T]^2 \to \mathbb{R}$ are continuous, it follows that

$$|(Au)(t_1)-(Au)(t_2)| \to 0 \text{ as } |t_1-t_2| \to 0.$$

The proof is completed. □

Lemma 2.3. [12] *If* $0 < \alpha \le 1$ *and* $u \in AC[0,T]$, *then*

$$ {}_0^R D_t^\alpha u(t) = {}_0^{LC} D_t^\alpha u(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u(0^+). $$

Lemma 2.4. *(Comparison principle) Let* $0 < \alpha < 1, \beta \ge \alpha$, $M \ge 0$ *and* $N \le \frac{\alpha\Gamma(\beta)}{T^{\beta+\alpha}\Gamma(1-\alpha)}$. *If* $u \in AC[0,T]$ *satisfies*

$$ \left\{ \begin{array}{l} {}_0^{LC} D_t^\alpha u(t) + Mu(t) + NI^\beta u(t) \ge 0, t \in (0,T], \\ u(0) = x_0 \ge 0. \end{array} \right. \tag{2.7} $$

Then $u(t) \ge 0, \forall t \in [0,T]$.

Proof. Assume that the assertion is not true. From $u(0) = x_0 \ge 0$, there exist $t_0, t_1 \in [0,T]$ such that $u(t_0) = 0, u(t_1) < 0$ and $u(t) \ge 0$ for $t \in [0, t_0]$, $u(t) < 0$ for $t \in (t_0, t_1]$. We discussed the following two cases.

Case 1 $M \ge 0$ and $N > 0$. By Lemma 2.3, we have

$$ {}_0^R D_t^\alpha u(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x_0 + Mu(t) + NI^\beta u(t) \ge 0,\ t \in (0,T], $$

which implies that

$$ {}_0^R D_t^\alpha u(t) + Mu(t) + NI^\beta u(t) \ge 0,\ \ \forall t \in (t_0, t_1]. $$

Hence

$$ \int_{t_0}^t ({}_0^R D_s^\alpha u(s) + NI^\beta u(s)) ds \ge 0,\ \ \forall t \in (t_0, t_1]. $$

From Definition 2.3, we obtain that

$$ I^{1-\alpha} u(t) - I^{1-\alpha} u(t_0) + \int_{t_0}^t NI^\beta u(s) ds \ge 0, \forall t \in (t_0, t_1]. \tag{2.8} $$

If $t_0 = 0$, we have

$$ I^{1-\alpha} u(t) - I^{1-\alpha} u(0) + \int_0^t NI^\beta u(s) ds = \frac{\int_0^t (t-s)^{-\alpha} u(s) ds}{\Gamma(1-\alpha)} + N \int_0^t I^\beta u(s) ds < 0, \forall t \in (0, t_1] $$

since $u(t) < 0$ for $t \in (0, t_1]$, which contradicts (2.8).

If $t_0 > 0$, for $t \in (t_0, t_1]$, we have

$$ \begin{aligned} & I^{1-\alpha} u(t) - I^{1-\alpha} u(t_0) \\ = & \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds - \frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} (t_0 - s)^{-\alpha} u(s) ds \\ = & \frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} ((t-s)^{-\alpha} - (t_0 - s)^{-\alpha}) u(s) ds + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} u(s) ds \\ \le & \frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} ((t-s)^{-\alpha} - (t_0 - s)^{-\alpha}) u(s) ds \\ \le & \frac{1}{T^{2\alpha}\Gamma(1-\alpha)} \int_0^{t_0} ((t_0 - s)^\alpha - (t-s)^\alpha) u(s) ds, \end{aligned} $$

$$
\begin{aligned}
\int_{t_0}^{t} N I^{\beta} u(s)ds =& \frac{N}{\Gamma(\beta)} \int_{t_0}^{t}\int_0^{r} (r-s)^{\beta-1} u(s)dsdr \\
=& \frac{N}{\Gamma(\beta)} \left(\int_0^{t_0}\int_{t_0}^{t} (r-s)^{\beta-1} u(s)drds + \int_{t_0}^{t}\int_{s}^{t} (r-s)^{\beta-1} u(s)drds \right) \\
=& \frac{N}{\Gamma(\beta+1)} \left(\int_0^{t_0} ((t-s)^{\beta} - (t_0-s)^{\beta}) u(s)ds + \int_{t_0}^{t} (t-s)^{\beta} u(s)ds \right) \\
\le& \frac{N}{\Gamma(\beta+1)} \int_0^{t_0} ((t-s)^{\beta} - (t_0-s)^{\beta}) u(s)ds.
\end{aligned}
$$

From (2.8), we obtain that

$$
\begin{aligned}
0 \le \rho := & \int_0^{t_0} \left[\left(\frac{1}{\Gamma(1-\alpha)T^{2\alpha}} (t_0-s)^{\alpha} - \frac{N}{\Gamma(\beta+1)} (t_0-s)^{\beta} \right) \right. \\
& \left. - \left(\frac{1}{\Gamma(1-\alpha)T^{2\alpha}} (t-s)^{\alpha} - \frac{N}{\Gamma(\beta+1)} (t-s)^{\beta} \right) \right] u(s)ds, \quad t \in (t_0, t_1].
\end{aligned}
$$

Let

$$
y(x) = \frac{1}{\Gamma(1-\alpha)T^{2\alpha}} x^{\alpha} - \frac{N}{\Gamma(\beta+1)} x^{\beta}, x \in [0,T], \tag{2.9}
$$

then for $\forall x \in (0,T]$,

$$
y'(x) = \frac{\alpha}{\Gamma(1-\alpha)T^{2\alpha}} x^{\alpha-1} - \frac{N}{\Gamma(\beta)} x^{\beta-1} = \frac{\frac{\alpha}{\Gamma(1-\alpha)T^{2\alpha}} - \frac{N}{\Gamma(\beta)} x^{\beta-\alpha}}{x^{\alpha-1}} \ge 0.
$$

So y is strictly increasing and

$$
\left(\frac{1}{\Gamma(1-\alpha)T^{2\alpha}} (t_0-s)^{\alpha} - \frac{N}{\Gamma(\beta+1)} (t_0-s)^{\beta} \right) - \left(\frac{1}{\Gamma(1-\alpha)T^{2\alpha}} (t-s)^{\alpha} - \frac{N}{\Gamma(\beta+1)} (t-s)^{\beta} \right) < 0
$$

for $0 \le s \le t_0 < t \le t_1$, which implies that $\rho < 0$ since $u > 0$ for $t \in (0, t_0)$, which contradicts (2.8).

Case 2 $M \ge 0, N \le 0$. From Theorem 2.1 of [12], ${}_0^{LC}D_t^{\alpha} u(t) \in L^1(0,T)$ if $u \in AC[0,T]$. Let

$$
{}_0^{LC}D_t^{\alpha} u(t) + Mu(t) + N I^{\beta} u(t) = h(t), h \in L^1(0,T).
$$

From $N \le 0, x_0 \ge 0, h \ge 0$,(2.6), Lemma 2.1 and Remark 2.1, we can obtain that $u(t) \ge 0$. □

3. Main result

Definition 3.5. The function $u \in AC[0,T]$ is called a lower solution of (1.1) if

$$
\begin{cases} {}_0^{LC}D_t^{\alpha} u(t) + N I^{\beta} u(t) \le f(t, u(t)), t \in (0,T], \\ g(u(0), u(1)) \le 0 \end{cases} \tag{3.10}
$$

and it is an upper solution of (1.1) if the above inequalities are reverted.

We list the following assumptions for the convenience.

- (H_0) The constant $N \le \frac{\alpha\Gamma(\beta)}{T^{\beta+\alpha}\Gamma(1-\alpha)}$.
- (H_1) Problem (1.1) has lower and upper solutions u_0, v_0 respectively, and $u_0 \le v_0$ for $t \in [0,T]$.
- (H_2) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists constant $M > 0$ such that for $u_0 \le x \le y \le v_0$,

$$
f(t,x) + Mx \le f(t,y) + My.
$$

(H_3) $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exist constants $\lambda > 0, \mu \geq 0$ such that for $u_0 \leq \overline{t_i} \leq t_i \leq v_0, i = 1, 2,$

$$g(t_1, t_2) - g(\overline{t_1}, \overline{t_2}) \leq \lambda(t_1 - \overline{t_1}) - \mu(t_2 - \overline{t_2}).$$

Theorem 3.1. *Suppose that conditions $(H_0) - (H_3)$ hold. There exist sequences $\{u_n\}, \{v_n\} \subseteq AC[0, T]$ such that $u_n \to u^*, v_n \to v^*$ in $C[0, T]$ and u^*, v^* are minimal, maximal solutions of* (1.1) *in $[u_0, v_0] = \{u \in C[0, T] : u_0 \leq u \leq v_0\}$.*

Proof. The proof is divided into four steps. Step 1. Consider the linear problems

$$\begin{cases} {}_0^{LC}D_t^{\alpha} u_{n+1}(t) + Mu_{n+1}(t) + NI^{\beta}u_{n+1}(t) = f(t, u_n(t)) + Mu_n(t), t \in (0, T], \\ u_{n+1}(0) = \eta_n(0) := u_n(0) - \frac{1}{\lambda} g(u_n(0), u_n(1)), \end{cases} \tag{3.11}$$

$$\begin{cases} {}_0^{LC}D_t^{\alpha} v_{n+1}(t) + Mv_{n+1}(t) + NI^{\beta}v_{n+1}(t) = f(t, v_n(t)) + Mv_n(t), t \in (0, T], \\ v_{n+1}(0) = \sigma_n(0) := v_n(0) - \frac{1}{\lambda} g(v_n(0), v_n(1)). \end{cases} \tag{3.12}$$

From Lemma 2.2, (3.11) (or (3.12)) has a unique solution $u_n \in AC[0, T]$ (or $v_n \in AC[0, T]$) for $n = 1, 2, 3, ...$ and

$$\begin{aligned} u_{n+1}(t) = &\sum_{k=0}^{\infty} (-1)^k \frac{N^k}{k!} \eta_n(0) t^{k(\alpha+\beta)} E_{\alpha, k\beta+1}^{(k)}(-Mt^{\alpha}) \\ &+ \int_0^t (f(s, u_n(s)) + Mu_n(s)) \sum_{k=0}^{\infty} (-1)^k \frac{N^k}{k!} (t - s)^{\alpha(k+1)+k\beta-1} E_{\alpha, \alpha+k\beta}^{(k)}(-M(t - s)^{\alpha}) ds, \end{aligned}$$

$$\begin{aligned} v_{n+1}(t) = &\sum_{k=0}^{\infty} (-1)^k \frac{N^k}{k!} \sigma_n(0) t^{k(\alpha+\beta)} E_{\alpha, k\beta+1}^{(k)}(-Mt^{\alpha}) \\ &+ \int_0^t (f(s, v_n(s)) + Mv_n(s)) \sum_{k=0}^{\infty} (-1)^k \frac{N^k}{k!} (t - s)^{\alpha(k+1)+k\beta-1} E_{\alpha, \alpha+k\beta}^{(k)}(-M(t - s)^{\alpha}) ds. \end{aligned}$$

Step 2. We prove that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \leq v_{n+1} \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Note that

$$\begin{cases} {}_0^{LC}D_t^{\alpha} u_1(t) + Mu_1(t) + NI^{\beta}u_1(t) = f(t, u_0(t)) + Mu_0(t), \\ u_1(0) = u_0(0) - \frac{1}{\lambda} g(u_0(0), u_0(1)). \end{cases} \tag{3.13}$$

Let $p = u_1 - u_0$. From Definition 3.1, we have

$$\begin{cases} {}_0^{LC}D_t^{\alpha} p(t) + Mp(t) + NI^{\beta}p(t) \geq 0, \\ p(0) = -\frac{1}{\lambda} g(u_0(0), u_0(1)) \geq 0. \end{cases} \tag{3.14}$$

By Lemma 2.4, we obtain that $p \geq 0$ for $t \in [0, T]$, so $u_1 \geq u_0$. Now, from (3.13) and $(H_2), (H_3)$, we have

$$\begin{aligned} {}_0^{LC}D_t^{\alpha} u_1(t) + NI^{\beta}u_1(t) &= f(t, u_0(t)) + M(u_0(t) - u_1(t)) \leq f(t, u_1(t)), \\ g(u_1(0), u_1(1)) &\leq g(u_0(0), u_0(1)) + \lambda(u_1(0) - u_0(0)) - \mu(u_1(1) - u_0(1)) \\ &= -\mu(u_1(1) - u_0(1)) \leq 0. \end{aligned}$$

Therefore, u_1 is a lower solution of (1.1). We can repeat the argument above to deduce $u_2 \geq u_1, t \in [0, T]$ and then an induction verifies that $u_{n+1} \geq u_n, t \in [0, T]$. In the same way, we can prove that $v_n \geq v_{n+1}, t \in [0, T]$.

Let $w = v_1 - u_1$. We have

$$\begin{cases} {}_0^{LC}D_t^{\alpha} w(t) + Mw(t) + NI^{\beta}w(t) \geq 0, \\ w(0) \geq \frac{\mu}{\lambda}(v_0(1) - u_0(1)) \geq 0. \end{cases} \tag{3.15}$$

By Lemma 2.4, we obtain that $w \geq 0$ for $t \in [0,T]$, so that $v_1 \geq u_1$. Using mathematical induction, we obtain that $v_n \geq u_n$.

Step 3. The sequence of $\{u_n\}, \{v_n\}$ are uniformly bounded and equicontinuous in $[u_0, v_0]$. There exist $u^*, v^* \in C[0,T]$ such that

$$\lim_{n\to\infty} u_n(t) = u^*(t), \lim_{n\to\infty} v_n(t) = v^*(t),$$

uniformly on $[0,T]$, $u^*, v^* \in [u_0, v_0]$ and the limit functions u^*, v^* satisfy

$$\begin{aligned} x(t) = &\sum_{k=0}^{\infty}(-1)^k \frac{N^k}{k!}(x(0) - \lambda^{-1}g(x(0), x(1)))t^{k(\alpha+\beta)}E^{(k)}_{\alpha,k\beta+1}(-Mt^\alpha) \\ &+ \int_0^t (f(s,x(s)) + Mx(s))\sum_{k=0}^{\infty}(-1)^k\frac{N^k}{k!}(t-s)^{\alpha(k+1)+k\beta-1}E^{(k)}_{\alpha,\alpha+k\beta}(-M(t-s)^\alpha)ds. \end{aligned}$$

By Lemma 2.2, the above function x satisfies

$$\left\{ \begin{array}{l} {}^{LC}_0D^\alpha_t x(t) + Mx(t) + NI^\beta x(t) = f(t, x(t)) + Mx(t), t \in (0,T], \\ x(0) = x(0) - \lambda^{-1}g(x(0), x(1)). \end{array} \right. \tag{3.16}$$

Hence, u^*, v^* are solutions of (1.1).

Step 4. Finally, we prove that u^* and v^* are the extremal solutions of (1.1) in $[u_0, v_0]$. Let $x \in [u_0, v_0]$ be any solution of (1.1) and $u_n \leq x \leq v_n$ for some $n \in \mathbb{N}$. By (H_2), we have

$$\begin{aligned} {}^{LC}_0D^\alpha_t u_{n+1}(t) + Mu_{n+1}(t) + NI^\beta u_{n+1}(t) = f(t, u_n(t)) + Mu_n(t) \leq f(t, x(t)) + Mx(t) \\ =^C_0 D^\alpha_t x(t) + Mx(t) + NI^\beta x(t), \end{aligned}$$

$$\begin{aligned} u_{n+1}(0) - x(0) =& u_n(0) - \frac{1}{\lambda}g(u_n(0), u_n(1)) - x(0) + \frac{1}{\lambda}g(x(0), x(1)) \\ \leq& -\frac{\mu}{\lambda}(x(1) - u_n(1)) \leq 0. \end{aligned}$$

Hence $u_{n+1} \leq x$. Similarly, $x \leq v_{n+1}$. Hence, $u_n \leq x \leq v_n$ for $n = 0, 1, 2, \cdots$. Taking $n \to \infty$, we obtain that $u^* \leq x \leq v^*$. Thus u^* and v^* are the extremal solutions of (1.1) in $[u_0, v_0]$. □

Example 3.1. Consider the following fractional differential equation

$$\left\{ \begin{array}{l} {}^{LC}_0D^{\frac{1}{2}}_t u(t) - \frac{1}{4}I^{\frac{5}{4}}u(t) = t - u^2(t) - u(t), t \in (0,1], \\ 5u^2(0) - u(0) = 5u(1) - \frac{1}{2}u^2(1). \end{array} \right. \tag{3.17}$$

In fact,

$$\alpha = \frac{1}{2}, \ \beta = \frac{5}{4}, \ \ T = 1, \ N = -\frac{1}{4}, \ f(t,x) = t - x^2 - x, \ g(x,y) = 5x^2 - x - 5y + \frac{1}{2}y^2.$$

Clearly, $N \leq \alpha\Gamma(\beta)/T^{\alpha+\beta}\Gamma(1-\alpha) = \frac{\Gamma(\frac{5}{4})}{2\Gamma(\frac{1}{2})}$. Taking $u_0 = 0, v_0 = 4 - t, M = 10, \lambda = 40, \mu = \frac{1}{2}$. Then

$${}^{LC}_0D^{\frac{1}{2}}_t u_0(t) - \frac{1}{4}I^{\frac{5}{4}}u_0(t) = 0 \leq t = f(t, u_0),$$

$${}^{LC}_0D^{\frac{1}{2}}_t v_0(t) - \frac{1}{4}I^{\frac{5}{4}}v_0(t) = -\frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{1}{4\Gamma(\frac{5}{4})}(\frac{4}{9}t^{\frac{9}{4}} - \frac{4}{3}t^{\frac{7}{4}}) \geq -11 \geq f(t, v_0),$$

$$g(u_0(0), u_0(1)) = 0, \ \ g(v_0(0), v_0(1)) = 65.5.$$

Hence, u_0 is a lower solution and v_0 is an upper solution. For $0 \le x \le y \le 4-t$,

$$\begin{aligned} f(t,y)+My-f(t,x)-Mx &= t-y^2-y+10y-t+x^2+x-10x \\ &=-\left(y^2-x^2\right)-(y-x)+10(y-x) \\ &=(-(y+x)-1+10)(y-x) \ge 0. \end{aligned}$$

Let $0 \le x_1 \le x_2 \le 4-t, 0 \le y_1 \le y_2 \le 4-t$, then

$$\begin{aligned} & 40(x_2-x_1)-\frac{1}{2}(y_2-y_1)-g(x_2,y_2)+g(x_1,y_1) \\ =& 40(x_2-x_1)-\frac{1}{2}(y_2-y_1)-(5x_2^2-x_2-5y_2+\frac{1}{2}y_2^2)+(5x_1^2-x_1-5y_1+\frac{1}{2}y_1^2) \\ =& 40(x_2-x_1)-\frac{1}{2}(y_2-y_1)-5(x_2^2-x_1^2)+(x_2-x_1)+5(y_2-y_1)-\frac{1}{2}(y_2^2-y_1^2) \\ =& 41(x_2-x_1)+\frac{9}{2}(y_2-y_1)-5(x_2^2-x_1^2)-\frac{1}{2}(y_2^2-y_1^2) \\ =& \left(41-5(x_2+x_1)\right)(x_2-x_1)+\left(\frac{9}{2}-\frac{1}{2}(y_2+y_1)\right)(y_2-y_1) \ge 0. \end{aligned}$$

Hence, (H_2) and (H_3) are satisfied. Therefore, (3.17) has extremal solutions $u^*, v^* \in [u_0, v_0]$. Moreover, the result of [26] cannot be applied to (3.17) ever if $N=0$ because the function g does not satisfy the condition (2.8) of [26].

Example 3.2. Consider the following fractional differential equation

$$\begin{cases} {}_0^{LC}D_t^{\frac{1}{3}}u(t)+\frac{1}{10}I^{\frac{2}{3}}u(t)=\frac{t}{100}(1+u^2(t)), t\in(0,1], \\ u^2(0)\sin(u(0))=u(1). \end{cases} \tag{3.18}$$

In fact,

$$\alpha=\frac{1}{3},\ \ \beta=\frac{2}{3},\ \ T=1,\ \ N=\frac{1}{10},\ \ f(t,x)=\frac{t}{100}(1+x^2),\ \ g(x,y)=x^2\sin x-y.$$

Clearly, $N \le \alpha\Gamma(\beta)/T^{\alpha+\beta}\Gamma(1-\alpha)=\frac{1}{3}$. Taking $u_0^j(t)=2\pi j-2t, v_0^j(t)=2\pi j+1+2t, j=1,2, M=1, \lambda=300, \mu=\frac{1}{2}$. Then

$${}_0^{LC}D_t^{\frac{1}{3}}u_0^j(t)+\frac{1}{10}I^{\frac{2}{3}}u_0^j(t)=-\frac{3}{\Gamma(\frac{2}{3})}t^{\frac{2}{3}}+\frac{1}{10\Gamma(\frac{2}{3})}\left(3\pi j t^{\frac{2}{3}}-\frac{9}{5}t^{\frac{5}{3}}\right) \le 0 \le f(t,u_0^j),$$

$${}_0^{LC}D_t^{\frac{1}{3}}v_0^j(t)+\frac{1}{10}I^{\frac{2}{3}}v_0^j(t)=\frac{3}{\Gamma(\frac{2}{3})}t^{\frac{2}{3}}+\frac{1}{10\Gamma(\frac{2}{3})}\left(\frac{3(2\pi j+1)}{2}t^{\frac{2}{3}}+\frac{9}{5}t^{\frac{5}{3}}\right) \ge f(t,v_0^j),$$

$$g(u_0^j(0),u_0^j(1))=-2\pi j+2<0,\ \ g(v_0^j(0),v_0^j(1))=(2\pi j+1)^2\sin 1-2\pi j-3>0.$$

Hence, u_0^j and v_0^j are lower and upper solutions of (3.18). In addition, it is easy to verify that (H_2) and (H_3) are satisfied. Therefore, (3.18) has extremal solutions $u^{*j}, v^{*j} \in [u_0^j, v_0^j]$.

4. CONCLUSIONS

This paper focuses on the existence of extreme solution for the Liouville-Caputo fractional differential equation with nonlinear boundary condition. We obtain the specific expression of the solution for the corresponding linear problem using the laplace transform and establish a new comparison principle. We prove the existence of extreme solution by using monotone iterative method. Since the case that $0<\alpha<1$ is considered in present paper, we will discuss the existence of solutions for the Liouville-Caputo fractional differential equation when $n-1<\alpha<n$ in follow-up research.

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