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Some remarks on expansive mappings in metric spaces

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ABSTRACT. The aim of this paper is to generalize the results on expansive mappings of Yeşilkaya and Aydin from *Fixed Point Results of Expansive Mappings in Metric Spaces* [see Yeşilkaya, S. S.; Aydin, C. Fixed Point Results of Expansive Mappings in Metric Spaces. *Mathematics* **8** (2020), 1800]. In the present paper we show that the conditions imposed on the function θ can be relaxed. Thus, we present more general fixed point results for q-expansive mappings in metric spaces and prove some fixed point theorems for this class of mappings, via a different approach. Finally, we present some examples to support the new results.

1. INTRODUCTION AND PRELIMINARIES

In 1984, Wang et. al. [19] started the study of expansive mappings and proved some fixed point theorems for such mappings, which correspond to some contractive mappings in metric spaces. Thereafter, several authors generalised and extended the results of Wang, see [3], [5], [7], [8], [11], [16], [18]. Recently, Yeşilkaya and Aydin (see [20]) introduced the concept of θ-expansive mappings in ordered metric spaces and extended the main results for expansive mappings from the current literature. For example, they obtained a common fixed point theorem of two weekly compatible mappings in metric spaces.

In 1982, Sessa [17] defined the concept of weak commutativity for two mappings and proved a common fixed point theorem for such mappings. In 1986, Jungck [10] introduced the concept of weakly compatible mappings.

Definition 1.1. [17] Let U and V be self mappings of a set M. A point $x \in M$ is called a coincidence point of U and V if and only if $Uz = Vz$. In this case, $w = Uz = Vz$ is called a coincidence of U and V.

Definition 1.2. [10] Two self mappings U and V of a metric space (M, d) are said to be weakly compatible if and only if at every point $z \in M$ which is a coincidence point of U and V, the mappings commute, that is $UVz = VUz$.

Remark 1.1. It is worth noting that the condition $UVz = VUz$ for weakly compatible mappings in Definition 1.2 is equivalent to the condition $U^2z = V^2z$. In other words, if u is the a coincidence point of U and V, then it is also a coincidence point of U^2 and V^2.

The following fixed point result for expansive mappings that was proved by Wang in [19] is essential for the current paper.

Theorem 1.1. [19] *Let (M, d) be a complete metric space and U a self mapping of M. If U is surjective and satisfies*

$$d(Ux, Uz) \geq qd(x, z),$$

for all $x, z \in M$, with $q > 1$, then U has a unique fixed point in M.

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In 2004, Ran and Reurings [15] proved a fixed point theorem in a partially ordered metric space.

Theorem 1.2. [15] *Let $(M, \leq)$ be an ordered set and d be a metric on M such that (M, d) is a complete metric space. Let $U : M \to M$ be a nondecreasing mapping, i.e. $Ux \leq Uy$, for every $x, y \in M$ with $x \leq y$. Suppose that there exists $x_0 \in M$ with $x_0 \leq Ax_0$ and $L \in [0, 1)$ such that*

$$d(Ux, Uy) \leq Ld(x, y),$$

for every $x, y \in M$ with $x \leq y$. If U is continuous, then it has a fixed point in M.

Thereafter, many authors considered the problem of the existence of a fixed point for contraction type mappings on partially ordered set, see [1], [2], [4], [12], [13]. In 2014, Jleli and Samet introduced in [9] the class of θ-contractions. They considered Θ, the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(θ_1) θ is non-decreasing;
(θ_2) for each sequence $\{t_n\} \in (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;
(θ_3) there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l.$$

The following lemma, proved by Górnicki in [6] is an important tool in this theory.

Lemma 1.1. [6] *Let (M, d) be a metric space and $U : M \to M$ a surjective mapping. Then, U has a right inverse mapping, i.e., there exists a mapping $U^* : M \to M$ such that $U \circ U^* = I_M$, where I_M is the identity mapping on M.*

If $(M, \leq)$ is an ordered set and d is a metric on M, we say that $(M, \leq, d)$ is an ordered metric space. If for every increasing sequence $\{x_n\} \subseteq M$ with $x_n \to x^* \in M$ we have $x_n \leq x^*$ for all $n \in \mathbb{N}$, then we say that M is a regular ordered metric space.

Very recently, Yeşilkaya and Aydin introduced in [20] the notion of θ-expansive mapping in ordered metric spaces.

Definition 1.3. [20] Let (M, d) be an ordered metric space. A mapping $U : M \to M$ is said to be θ-expansive if there exists $\theta \in \Theta$ and $\eta > 1$ such that

$$\theta(d(Ux, Uz)) \geq [\theta(d(x, z))]^\eta,$$

for all $(x, z) \in M_0$, where

$$M_0 = \{(x, z) \in M \times M \ : \ x \leq z, \ d(Ux, Uz) > 0\}.$$

They proved the following theorems:

Theorem 1.3. [20] *Let $(M, \leq, d)$ be an ordered complete metric space, $U : M \to M$ a surjective θ-expansive mapping and U^* a right inverse of U such that U^* is $\leq$ increasing. Suppose that there exists $x_0 \in M$ such that $x_0 \leq U^* x_0$. If U is continuous or M is regular, then U has a fixed point.*

Theorem 1.4. [20] *Let (M, d) be a complete metric space and $U : M \to M$ a continuous surjective θ-expansive mapping. If there exists $\eta > 1$ such that*

$$\theta(d(Ux, Uz)) \geq [\theta(\min\{d(x, z), d(x, Ux), d(z, Uz)\})]^\eta,$$

for all $x, z \in M$, then U has a fixed point.

We show that in these theorems it is not necessary that $\theta \in \Theta$. We can prove that the results are available even if θ has only the property that it is a non-increasing function. Thus, our results are much more less restrictive and consequently, more general than the results existing in literature.

In [20], there is provided the following common fixed point theorem for weakly compatible mappings (see Theorem 5).

Theorem 1.5. [20] *Let (M, d) be a complete metric space. Let U and V be weakly compatible self mappings of M and $V(M) \subseteq U(M)$. Suppose that $\theta \in \Theta$ and there exists a constant $\eta > 1$ such that*

$$\theta(d(Ux, Uz)) \geq [\theta(d(Vx, Vz))]^{\eta},$$

for all $x, z \in M$. If one of the subspaces $U(M)$ or $V(M)$ is complete, then U and V have a unique common fixed point in M.

In this paper we give a more general equivalent of Theorem 1.5 and thus, we provide a new significant common fixed point result for weakly compatible mappings.

An essential tool in the proofs of our results is the following Lemma proved by Popescu in [14]:

Lemma 1.2. [14] *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X which is not Cauchy and $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then there exists $\varepsilon > 0$ and two sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon^{+}.$$

2. Main results

First, let us start with the definition of φ-expansive mappings in ordered metric spaces.

Definition 2.4. Let $(M, \leq, d)$ be an ordered metric space. A mapping $U : M \to M$ is said to be φ-expansive if there exist a non-decreasing function $\varphi : (0, \infty) \to (1, \infty)$ and $\eta > 1$ such that

$$\varphi(d(Ux, Uz)) \geq [\varphi(d(x, z))]^{\eta},$$

for all $(x, z) \in M_0$, where

$$M_0 = \{(x, z) \in M \times M \, : \, x \leq z, \, d(Ux, Uz) > 0\}.$$

The first result is a generalization of Theorem 1.3.

Theorem 2.6. *Let $(M, \leq, d)$ be an ordered complete metric space, $U : M \to M$ be a surjective φ-expansive mapping and U^* a right inverse of U such that U^* is $\leq$ increasing. Suppose that there exists $x_0 \in M$ such that $x_0 \leq U^* x_0$. If U is continuous or M is regular, then U has a fixed point.*

Proof. Let $x_0 \in M$ with $x_0 \leq U^* x_0$. We define the sequence $\{x_n\}$ by $x_{n+1} = U^* x_n$. Then, we have

$$Ux_{n+1} = UU^* x_n = x_n,$$

for all $n = 0, 1, 2, \ldots$

Since $x_0 \leq U^* x_0 = x_1$ and U^* is increasing, we get

$$U^* x_0 \leq U^* x_1,$$

i.e., $x_1 \leq x_2$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_{n+1} is a fixed point of U.

Now assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Inductively, by $x_n \leq x_{n+1}$ we obtain $U^*x_n \leq U^*x_{n+1}$, i.e., $x_{n+1} \leq x_{n+2}$, so

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \ldots$$

Let $s = \dfrac{1}{\eta}$. Since $\eta > 1$, we have $s < 1$. Since $d(Ux_n, Ux_{n+1}) = d(x_{n-1}, x_n) > 0$ and $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, then $(x_n, x_{n+1}) \in M_0$. So, we have for all $n \in \mathbb{N}$

$$\varphi(d(x_{n-1}, x_n)) = \varphi(d(Ux_n, Ux_{n+1})) \geq [\varphi(d(x_n, x_{n+1}))]^\eta$$

by where

$$\varphi(d(x_n, x_{n+1})) \leq [\varphi(d(x_{n-1}, x_n))]^s < \varphi(d(x_{n-1}, x_n)).$$

Since φ is a non-decreasing function, we get

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$, hence $\{d(x_n, x_{n+1})\}_{n \geq 0}$ is a decreasing sequence of positive numbers. Therefore, $\{d(x_n, x_{n+1})\}_{n \geq 0}$ converges to some $d \geq 0$.

Suppose $d > 0$. Then, we have

$$\varphi(d^+) \leq \varphi(d(x_n, x_{n+1}))$$

and

$$\varphi(d^+) \leq [\varphi(d(x_{n-1}, x_n))]^s,$$

for all $n \in \mathbb{N}$.

Since φ is non-decreasing, letting n tend to ∞, we obtain

$$1 < \varphi(d) \leq \varphi(d^+) \leq \varphi(d^+)^s,$$

which is a contradiction. Therefore, $d = 0$.

Now, we suppose that $\{x_n\}$ is not a Cauchy sequence. Then, by Lemma 1.2, there exist $\varepsilon > 0$ and two subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) \to \varepsilon^+, \quad d(x_{n(k)+1}, x_{m(k)+1}) \to \varepsilon^+.$$

Since $(x_{m(k)+1}, x_{n(k)+1}) \in M_0$, we have

$$\begin{aligned} \varphi(d(x_{n(k)}, x_{m(k)})) &= \varphi(d(Ux_{n(k)+1}, Ux_{m(k)+1})) \\ &\geq [\varphi(d(x_{n(k)+1}, x_{m(k)+1}))]^\eta, \end{aligned}$$

for every $k \geq 1$.

Letting $k \to \infty$, we obtain

$$\varphi(\varepsilon^+) \geq \varphi(\varepsilon^+)^\eta \geq \varphi(\varepsilon) > 1,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since (M, d) is complete, we get that there exists $x^* \in M$ such that $\lim_{n\to\infty} x_n = x^*$.

Now we shall show that x^* is a fixed point of U. If U is continuous, then we have

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ux_{n+1} = U(\lim_{n\to\infty} x_{n+1}) = Ux^*,$$

i.e., x^* is a fixed point of U. If M is regular, than $x_n \leq x^*$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x^*$, than we have

$$U^*x^* = U^*x_{n_0} = x_{n_0+1} \leq x^*$$

and

$$x^* = x_{n_0} \leq x_{n_0+1} = U^*x_{n_0} = U^*x^*.$$

Hence $U^*x^* = x^*$. Otherwise, we have $x_n \neq x^*$ for every $n \in \mathbb{N}$. If $U^*x^* \neq x^*$, we have

$$\varphi(d(x_n, x^*)) = \varphi(d(Ux_{n+1}, UU^*x^*))$$

$$\geq [\varphi(d(x_{n+1}, U^*x^*))]^\eta \geq \varphi(d(x_{n+1}, U^*x^*)).$$

Then, we get

$$d(x_n, x^*) \geq d(x_{n+1}, U^*x^*).$$

Letting n tend to ∞, we obtain $d(x^*, U^*x^*) \leq 0$, which is a contradiction.

Thus, we conclude that $d(x^*, U^*x^*) = 0$, that is $U^*x^* = x^*$ and $UU^*x^* = Ux^*$. Therefore, $x^* = Ux^*$. □

Corollary 2.1. *Let (M, d) be a complete metric space and $U : M \to M$ be a continuous surjective φ-expansive mapping. If there exists $\eta > 1$ such that*

$$\varphi(d(Ux, Uz)) \geq [\varphi(d(x, z))]^\eta$$

for all $x, z \in M$, then U has a unique fixed point in M.

Proof. By Theorem 2.6 we have that U has a fixed point $x^* \in M$. Suppose that $y^* \in M$ is another fixed point of U. Then

$$\varphi(d(x^*, y^*)) = \varphi(d(Ux^*, Uy^*)) \geq [\varphi(d(x^*, y^*))]^\eta.$$

Since $\varphi(d(x^*, y^*)) > 1$ and $\eta > 1$, this is a contradiction. Thus, U has a unique fixed point. □

We provide an example to illustrate our results.

Example 2.1. Let $Y = \left\{ \frac{1}{r+1}, \, r \in \mathbb{N} \cup \{0\} \right\} \cup \{0\}$ endowed with the metric

$$d\left(\frac{1}{r}, \frac{1}{r+p}\right) = \frac{1}{r} \quad \text{and} \quad d\left(0, \frac{1}{r}\right) = \frac{1}{r},$$

for every $r, p \in \mathbb{N}$. Let us consider the order relation $\preccurlyeq$ on Y defined as

$$x \preccurlyeq z \iff x = z \text{ or } x < z < 1.$$

where $<$ is the usual order.

Then $(Y, \preccurlyeq, d)$ is an ordered complete metric space.

Let $U : Y \to Y$ be defined as

$$Ux = \begin{cases} \frac{1}{r}, & x = \frac{1}{r+1}, \; r \in \mathbb{N} \\ 0, & x = 0 \\ 1, & x = 1 \end{cases}$$

Taking

$$U^*x = \begin{cases} \frac{1}{r+1}, & x = \frac{1}{r}, \; r \in \mathbb{N} \\ 0, & x = 0, \end{cases}$$

clearly U^* is $\preccurlyeq$-increasing.

Let $\varphi(t) = e^{e^{-\frac{1}{t}}}$ for every $t > 0$ and let $1 < \eta < e$ Then, it is true that

$$e^{e^{-\frac{1}{d(Ux,Uz)}}} \geq e^{\eta e^{-\frac{1}{d(x,z)}}},$$

for every $x, z \in Y$ with $x \preccurlyeq z$, since $e^{-r} \geq \eta e^{-(r+1)}$ for every $r \in \mathbb{N}$.

Thus, U satisfies the hypothesis of Theorem 2.6 which implies that it has a unique fixed point in Y.

The mapping U is not an expansive mapping in metric spaces since

$$\lim_{r \to \infty} \frac{d(Ax, Az)}{d(x, z)} = \frac{r+1}{r} = 1.$$

Remark 2.2. Let us note that the function φ provided does not belong to the class Θ as it does not verify θ_3. Thus, our results are more general than those presented in [20].

The following Theorem is a generalization of Theorem 1.4.

Theorem 2.7. *Let (M, d) be a complete metric space and $U : M \to M$ a continuous surjective φ-expansive mapping. If there exists $\eta > 1$ such that*

$$\varphi(d(Ux, Uz)) \geq [\varphi(\min\{d(x, z), d(x, Ux), d(z, Uz)\})]^{\eta} \tag{2.1}$$

for all $x, z \in M \setminus \{t \in M : Ut = t\}$ with $Ux \neq Uz$ then U has a fixed point.

Proof. Let x_0 be an arbitrary point in M. Since U is surjective, there exists $x_1 \in M$ such that $x_0 = Ux_1$. In general, if $x_n \in M$, we can choose $x_{n+1} \in M$ such that $x_n = Ux_{n+1}$, for all $n = 0, 1, 2, \ldots$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_n is a fixed point of U. Otherwise, we have $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from equation (2.1), for $x = x_n$ and $z = x_{n+1}$, we have

$$\begin{aligned}\varphi(d(x_{n-1}, x_n)) &= \varphi(d(Ux_n, Ux_{n+1})) \\ &\geq [\varphi(\min\{d(x_n, x_{n+1}), d(x_n, Ux_n), d(x_{n+1}, Ux_{n+1})\})]^{\eta},\end{aligned}$$

where $\min\{d(x_n, x_{n+1}), d(x_n, Ux_n), d(x_{n+1}, Ux_{n+1})\} = \min\{d(x_n, x_{n+1}), d(x_n, x_{n-1})\}$.

If $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then we get

$$\varphi(d(x_{n-1}, x_n)) \geq [\varphi(d(x_{n-1}, x_n))]^{\eta},$$

which is a contradiction. Therefore $d(x_{n-1}, x_n) > d(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Then, we obtain

$$\varphi(d(x_{n-1}, x_n)) \geq [\varphi(d(x_n, x_{n+1}))]^{\eta}.$$

Since $\{d(x_n, x_{n+1})\}_{n \geq 0}$ is a decreasing sequence of positive numbers we get that there exists $d \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = d$.

Suppose $d > 0$. Then, letting n tend to ∞ in the above equation we obtain

$$\varphi(d^+) \geq [\varphi(d^+)]^{\eta},$$

which is a contradiction. Therefore, $d = 0$.

Now, we suppose that $\{x_n\}$ is not a Cauchy sequence. Then, by Lemma 1.2, there exist $\varepsilon > 0$ and two subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) \to \varepsilon^+, \quad d(x_{n(k)+1}, x_{m(k)+1}) \to \varepsilon^+.$$

Taking $x = x_{n(k)+1}$ and $z = x_{m(k)+1}$ in equation (2.1) we obtain

$$\begin{aligned}&\varphi(d(x_{n(k)}, x_{m(k)})) = \varphi(d(Ux_{n(k)+1}, Ux_{m(k)+1})) \geq \\ &\geq [\varphi(\min\{d(x_{n(k)+1}, x_{m(k)+1}), d(x_{n(k)+1}, Ux_{n(k)+1}), d(x_{m(k)+1}, Ux_{m(k)+1})\})]^{\eta},\end{aligned}$$

where

$$\begin{aligned}&\min\{d(x_{n(k)+1}, x_{m(k)+1}), d(x_{n(k)+1}, Ux_{n(k)+1}), d(x_{m(k)+1}, Ux_{m(k)+1})\} = \\ &= \min\{d(x_{n(k)+1}, x_{m(k)+1}), d(x_{n(k)+1}, x_{n(k)}), d(x_{m(k)+1}, x_{m(k)})\}.\end{aligned}$$

Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, for k large enough we have that $d(x_{n(k)+1}, x_{n(k)}) < \varepsilon$ and $d(x_{m(k)+1}, x_{m(k)}) < \varepsilon$, hence

$$\begin{aligned}\min\{d(x_{n(k)+1}, x_{m(k)+1}), d(x_{n(k)+1}, Ux_{n(k)+1}), d(x_{m(k)+1}, Ux_{m(k)+1})\} = \\ = \min\{d(x_{n(k)+1}, x_{m(k)+1})\}.\end{aligned}$$

This implies that

$$\varphi(d(x_{n(k)}, x_{m(k)})) \geq [\varphi(d(x_{n(k)+1}, x_{m(k)+1}))]^{\eta},$$

so letting $k \to \infty$, we get

$$\varphi(\varepsilon^+) \geq \varphi(\varepsilon^+)^\eta,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence and there exists $x^* \in M$ such that $\lim_{n\to\infty} x_n = x^*$.

Since U is continuous, we have

$$x^* = \lim_{n\to\infty} x_n = \lim Ux_{n+1} = U(\lim_{n\to\infty} x_{n+1}) = Ux^*,$$

hence x^* is a fixed point of U. □

Theorem 2.8. *Let (M, d) be a complete metric space. Let U and V be weakly compatible self mappings of M and $V(M) \subseteq U(M)$. Suppose that φ is a non-decreasing function $\varphi : (0, \infty) \to (1, \infty)$ and there exists a constant $\eta > 1$ such that*

$$\varphi(d(Ux, Uz)) \geq [\varphi(d(Vx, Vz))]^\eta \tag{2.2}$$

for all $x, z \in M$ with $Vx \neq Vz$. If one of the subspaces $U(M)$ or $V(M)$ is complete, then U and V have a unique common fixed point in M.

Proof. Let x_0 be an arbitrary point in M. Since $V(M) \subseteq U(M)$, choose $x_1 \in M$ such that $y_1 = Ux_1 = Vx_0$. In general, for $x_n \in M$ we can choose $x_{n+1} \in M$ such that $y_{n+1} = Ux_{n+1} = Vx_n$.

If $y_n = y_{n+1}$, then we have

$$y_n = Ux_n = Vx_{n-1} = Ux_{n+1} = Vx_n = y_{n+1}.$$

Since $Ux_n = Vx_n$, x_n is a coincidence of U and V, so the weak compatibility of U and V ensures that

$$UVx_n = VUx_n = UUx_n = VVx_n.$$

Then, we have two possibilities. If $Vx_n \neq VVx_n$, then from (2.2) we get

$$\varphi(d(Ux_n, UVx_n)) \geq [\varphi(d(Vx_n, VVx_n))]^\eta,$$

but since $Ux_n = Vx_n$ and $UVx_n = VVx_n$, the above inequality becomes

$$\varphi(d(Vx_n, VVx_n)) \geq [\varphi(d(Vx_n, VVx_n))]^\eta,$$

which is a contradiction.

On the other hand, if $Vx_n = VVx_n$, then we obtain

$$Vx_n = VVx_n = UVx_n,$$

and thus Vx_n is a common fixed point of U and V.

Now, suppose that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$. Let $s = \frac{1}{\eta}$. Since $\eta > 1$, we get $s < 1$. Then, from (2.2) for $x = y_{n+1}$, $z = y_{n+2}$, we obtain

$$\begin{aligned}\varphi(d(y_{n+1}, y_{n+2})) = \varphi(d(Vx_n, Vx_{n+1})) &\leq [\varphi(d(Ux_n, Ux_{n+1}))]^s \\ &= [\varphi(d(Vx_{n-1}, Vx_n))]^s = [\varphi(d(y_n, y_{n+1}))]^s.\end{aligned}$$

Like in the proof of Theorem 2.6, we obtain that $\{y_n\}$ is a Cauchy sequence. Since $V(M) \subseteq U(M)$ and $V(M)$ or $U(M)$ is a complete subspace of M, we get that there exists $w \in U(M)$ such that $\lim_{n\to\infty} d(y_n, w) = 0$. So, we can find $u \in M$ such that $Uu = w$. We shall show that $Vu = w$.

Let us suppose that $Vu \neq w$. Since $y_n \neq y_{n+1}$ for every $n \in \mathbb{N}$, there exists a subsequence $\{x_{k(n)}\}$ such that $Vx_{k(n)} \neq Vu$. Thus, from (2.2) we have

$$[\varphi(d(Vx_{k(n)}, Vu))]^\eta \leq \varphi(d(Ux_{k(n)}, Uu)) \tag{2.3}$$

Since $\lim_{n\to\infty} d(y_{n(k)}, w) = \lim_{n\to\infty} d(Ux_{k(n)}, w) = 0$ and $d(Uu, w) = 0$, we have

$$\lim_{n\to\infty} d(Ux_{k(n)}, Uu) = 0$$

so there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$d(Ux_{k(n)}, Uu) \leq \frac{d(Vu, w)}{2}$$

so

$$\varphi(d(Ux_{k(n)}, Uu)) \leq \varphi\left(\frac{d(Vu, w)}{2}\right). \tag{2.4}$$

On the other hand we have $\lim_{n\to\infty} d(y_{n(k)+1}, w) = \lim_{n\to\infty} d(Vx_{k(n)}, w) = 0$, so

$$\lim_{n\to\infty} d(Vx_{k(n)}, Vu) = \lim_{n\to\infty} d(w, Vu) > 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ we have $d(Vx_{n(k)}, Vu) \geq \dfrac{d(w, Vu)}{2}$ and

$$\varphi(d(Vx_{n(k)}, Vu)) \geq \varphi\left(\frac{d(w, Vu)}{2}\right). \tag{2.5}$$

Combining relations (2.3), (2.4) and (2.5), we obtain

$$[\varphi(d(Vx_{n(k)}, Vu))]^{\eta} \leq d(Vx_{n(k)}, Vu),$$

which is a contradiction. Thus, $Vu = w$, and w is a coincidence of U and V and thus we have

$$VUu = UVu.$$

Moreover, $VVu = UVu$, which means that Vu is a coincidence of U and V.

To prove that u is a common fixed point of U and V, let us assume that $Vu \neq u$. Then, we can apply (2.2) and we have

$$\varphi(d(UVu, Uu)) \geq [\varphi(d(VVu, Vu))]^{\eta},$$

which is a contradiction since $UVu = VVu$ and $Uu = Vu$.

Thus, we have $Vu = Uu = u$, so u is a common fixed point of U and V.

Let us suppose that u is not the unique common fixed point of U and V. Then, there exists $t \in M$, $t \neq u$ such that $Ut = Vt = t$. Then, we have

$$\varphi(d(t, u)) = \varphi(d(Ut, Uu)) \geq [\varphi(d(Vt, Vu))]^{\eta} = [\varphi(d(t, u))]^{\eta},$$

which is a contradiction. □

We provide an example to illustrate our results. The example is similar to the one provided in [20] to illustrate Theorem 1.5. However, the function $\theta : (0, \infty) \to (1, \infty)$ given by $\theta(t) = e^t$ for every $t > 0$, does not belong to the class Θ (as it does not verify θ_3), but it is indeed a non-decreasing function as is required in the context of the previous theorem.

Example 2.2. The space $Y = [0, 1]$ endowed with the usual metric $d(x, z) = |x - z|$, for every $x, z \in Y$ is a complete metric space. Let $U : Y \to Y$, $Ux = \dfrac{x}{4}$, for every $x \in Y$ and $V : Y \to Y$, $Vx = \dfrac{x}{12}$, for every $x \in Y$. We have $V(Y) \subseteq U(Y)$ and $U(Y)$ is complete. Let $\theta : (0, \infty) \to (1, \infty)$, given by $\theta(t) = e^t$, for every $t > 0$, which is a non-decreasing function. Then, for every $x, z \in Y$, $x \neq z$ we have

$$e^{\frac{1}{4}}|x - z| \geq e^{\frac{\eta}{12}}|x - z|,$$

for $1 < \eta < 3$. U and V are weekly compatible mappings and 0 is the unique common fixed point.

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