CARPATHIAN J. MATH.
Volume **40** (2024), No. 3,
Pages 753 - 768

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2024.03.14

Convergence of self-adaptive Tseng-type algorithms for split variational inequalities and fixed point problems

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ABSTRACT. In this paper, we survey iterative algorithms for solving split variational inequalities and fixed point problems in Hilbert spaces. The investigated split problem is involved in two pseudomonotone operators and two pseudocontractive operators. We propose a self-adaptive Tseng-type algorithm for finding a solution of the split problem. Strong convergence of the suggested algorithm is shown under weaker conditions than sequential weak-to-weak continuity imposed on two pseudomonotone operators.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H. Let $\phi : H \to H$ be a nonlinear operator. Consider the variational inequality (shortly VI) of finding a point $x^\dagger \in C$ such that

$$\langle \phi(x^\dagger), x - x^\dagger \rangle, \ \forall x \in C. \tag{1.1}$$

Denote by $\mathrm{Sol}(C, \phi)$ the solution set of VI (1.1).

Definition 1.1. Recall that an operator ϕ is said to be

- strongly monotone if there exists a positive constant γ such that
  $$\langle \phi(x) - \phi(\hat{x}), x - \hat{x} \rangle \geq \gamma \|x - \hat{x}\|^2, \ \forall x, \hat{x} \in H.$$
  In this case, we call ϕ γ-strongly monotone.
- monotone if
  $$\langle \phi(x) - \phi(\hat{x}), x - \hat{x} \rangle \geq 0, \ \forall x, \hat{x} \in H.$$
- pseudomonotone if
  $$\langle \phi(\hat{x}), x - \hat{x} \rangle \geq 0 \Rightarrow \langle \phi(x), x - \hat{x} \rangle \geq 0, \ \forall x, \hat{x} \in H.$$
- Lipschitz continuous if there exists a positive constant L such that
  $$\|\phi(x) - \phi(\hat{x})\| \leq L\|x - \hat{x}\|, \ \forall x, \hat{x} \in H.$$
  In this case, we call ϕ L-Lipschitz.

As a powerful means, VI has been investigated and applied extensively to obstacle problems, optimization and control problems, traffic network problems, equilibrium problems, fixed point problems to name just a few, see [1, 3, 9, 12, 13, 22, 25, 29]. Now, we briefly recall several representative iterative algorithms for solving VI. Selecting $\phi = \nabla\psi(x)$ where $\psi : C \to C$ is a convex function, solving VI (1.1) is equivalent to $\min_C \psi(x)$. This implies that one can use the following projection gradient algorithm ([11, 14]) for solving VI (1.1):

$$x^{k+1} = \mathrm{proj}_C[x^k - \tau\phi(x^k)], k \geq 0, \tag{1.2}$$

Received: 30.06.2022. In revised form: 06.01.2024. Accepted: 13.01.2024

2010 *Mathematics Subject Classification.* 47J25, 65K10, 90C99.

Key words and phrases. *variational inequality, fixed point, split problem, pseudomonotone operator, pseudocontractive operator.*

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where $\tau > 0$ is stepsize and proj_C is the orthogonal projection onto C.

To ensure the convergence of (1.2), strong monotonicity and Lipschitz continuity of ϕ are indispensable (see [20]). To weaken the strong monotonicity imposed on ϕ, Korpelevich ([21]) proposed a well known extragradient method by using a double projection technique. Extragradient method provides an available approach for solving a classical monotone variational inequality. Consequently, extragradient method was exploited and developed in a variety of ways, see, e.g., [2, 7, 18, 26, 33]. Ceng, Teboulle and Yao [4] investigated extragradient method for solving pseudomonotone variational inequality and fixed point problem under the hypothesis that the pseudomonotone operator ϕ is sequently weak-to-strong continuous. Vuong [28] weaken this hypothesis to the sequentially weak-to-weak continuity. An inevitable drawback of extragradient algorithm is that we have to calculate two projections onto the closed convex set C in each iteration ([23]). This is very time-consuming and will seriously affect the execution of the algorithm. For avoiding this obstacle, as a transformation of extragradient algorithm is the following remarkable algorithm introduced by Tseng [27]

$$\begin{cases} y^k = \mathrm{proj}_C[x^k - \tau\phi(x^k)], \\ x^{k+1} = y^k + \tau[\phi(x^k) - \phi(y^k)], k \geq 0. \end{cases} \tag{1.3}$$

On the other hand, in projection gradient algorithm, extragradient algorithm and Tseng algorithm, the stepsize τ depends upon the Lipschitz-type constant of ϕ. The prior information of such constant imposes some restrictions on implementing these methods because these Lipschitz-type constants are normally not known or hard to compute. To overcome this flaw, Iusem [19] used a self-adaptive technique without prior knowledge of Lipschitz constant of ϕ for solving VI (1.1). Some related works on self-adaptive methods for solving (1.1), please refer to [15, 16, 32, 33].

In this paper, we investigate the following split problem of finding a point $\hat{x} \in C$ such that

$$\hat{x} \in \mathrm{Fix}(f) \cap \mathrm{Sol}(C, \phi) \text{ and } A\hat{x} \in \mathrm{Fix}(g) \cap \mathrm{Sol}(Q, \varphi), \tag{1.4}$$

where C and Q are two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively, $\mathrm{Fix}(f)$ and $\mathrm{Fix}(T)$ are the fixed point sets of two pseudocontractive operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, respectively, $\phi : H_1 \to H_1$ and $\varphi : H_2 \to H_2$ are two pseudomonotone operators and $A : H_1 \to H_2$ is a bounded linear operator.

The solution set of (1.4) is denoted by Γ, i.e.,

$$\Gamma = \{\hat{x} \in \mathrm{Fix}(f) \cap \mathrm{Sol}(C, \phi), A\hat{x} \in \mathrm{Fix}(g) \cap \mathrm{Sol}(Q, \varphi)\}.$$

It is clear that the split problem (1.4) include the split fixed point problem ([8]) of finding a point $\hat{x} \in C$ with the property

$$\hat{x} \in \mathrm{Fix}(f) \text{ and } A\hat{x} \in \mathrm{Fix}(g) \tag{1.5}$$

and the split variational inequality problem ([6]) of finding a point $\hat{x} \in C$ satisfying

$$\hat{x} \in \mathrm{Sol}(C, \phi) \text{ and } A\hat{x} \in \mathrm{Sol}(Q, \varphi) \tag{1.6}$$

as special cases.

The solution sets of (1.5) and (1.6) are denoted by Γ_1 and Γ_2, respectively, i.e., $\Gamma_1 = \{\hat{x} \in \mathrm{Fix}(f), A\hat{x} \in \mathrm{Fix}(g)\}$ and $\Gamma_2 = \{\hat{x} \in \mathrm{Sol}(C, \phi), A\hat{x} \in \mathrm{Sol}(Q, \varphi)\}$.

The split problems have emerged their powerful applications in image recovery and signal processing, control theory, biomedical engineering and geophysics. Some iterative algorithms for solving the split problems have been studied and extended by many scholars, see [5, 17, 24, 31].

Motivated and inspired by the above works, in this paper, we further survey the split problem (1.4). This split problem is involved in two pseudomonotone operators and two pseudocontractive operators. We propose a self-adaptive Tseng-type algorithm for finding a solution of the split problem (1.4). Strong convergence of the suggested algorithm is shown under weaker conditions than sequential weak-to-weak continuity imposed on two pseudomonotone operators ϕ and φ.

2. PRELIMINARIES

Let H be a real Hilbert space. Then, we have following equality

$$\|\alpha x + (1-\alpha)x^\dagger\|^2 = \alpha\|x\|^2 + (1-\alpha)\|x^\dagger\|^2 - \alpha(1-\alpha)\|x - x^\dagger\|^2, \tag{2.7}$$

for any $x, x^\dagger \in H$ and $\alpha \in \mathbb{R}$.

For a given $u^\dagger \in H$ and a closed convex set $C \subset H$, recall that the orthogonal projection of $u^\dagger$ onto C, denoted by $\mathrm{proj}_C[u^\dagger]$, is the unique point in C such that

$$\|u^\dagger - \mathrm{proj}_C[u^\dagger]\| = \inf_{x\in C} \|x - u^\dagger\|.$$

Moreover, one has

$$\langle \hat{x} - \mathrm{proj}_C[\hat{x}], x^\dagger - \mathrm{proj}_C[\hat{x}]\rangle \le 0,\ \forall \hat{x} \in H, x^\dagger \in C. \tag{2.8}$$

It is easy to check that proj_C satisfies

$$\|\mathrm{proj}_C[\hat{x}] - \mathrm{proj}_C[x^\dagger]\|^2 \le \langle \mathrm{proj}_C[\hat{x}] - \mathrm{proj}_C[x^\dagger], \hat{x} - x^\dagger\rangle,$$

and

$$\|\mathrm{proj}_C[\hat{x}] - \mathrm{proj}_C[x^\dagger]\| \le \|\hat{x} - x^\dagger\|$$

for all $\hat{x}, x^\dagger \in H$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $F\colon C \to C$ is said to be α-contractive, if there exists a constant $\alpha \in [0,1)$ such that $\|F(x) - F(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$. F is said to be pseudocontractive if

$$\|F(x) - F(x^\dagger)\|^2 \le \|x - x^\dagger\|^2 + \|(I-F)x - (I-F)x^\dagger\|^2, \forall x, x^\dagger \in C.$$

Lemma 2.1 ([35]). *Let C be a nonempty, convex and closed subset of a Hilbert space H. Let $F\colon C \to C$ be a κ-Lipschitz pseudocontractive operator. For all $\hat{u} \in C$ and $u^\dagger \in \mathrm{Fix}(F)$, we have*

$$\|F((1-\beta)\hat{u} + \beta F(\hat{u})) - u^\dagger\|^2 \le \|\hat{u} - u^\dagger\|^2 + (1-\beta)\|\hat{u} - F((1-\beta)\hat{u} + \beta F(\hat{u}))\|^2,$$

where β is a constant in $(0, \frac{1}{\sqrt{1+\kappa^2}+1})$.

Lemma 2.2 ([10]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \to H$ be a continuous and pseudomonotone operator. Then $x^\dagger \in \mathrm{Sol}(C, \phi)$ iff $x^\dagger$ solves the following variational inequality*

$$\langle \phi(x), x - x^\dagger\rangle \ge 0,\ \forall x \in C.$$

In what follows, the symbol “ $\rightharpoonup$ ” denotes the weak convergence and the symbol “ $\to$ ” denotes the strong convergence.

Lemma 2.3 ([34]). *Let C be a nonempty, convex and closed subset of a Hilbert space H. Let $F\colon C \to C$ be a continuous pseudocontractive operator. Then, F is demi-closedness, i.e., $u^k \rightharpoonup \tilde{u}$ and $F(u^k) \to u^\dagger$ as $k \to \infty$ imply that $F(\tilde{u}) = u^\dagger$.*

Lemma 2.4 ([30]). *Let $\{a_k\} \subset (0,\infty)$, $\{b_k\} \subset (0,1)$ and $\{c_k\}$ be three real number sequences. If $a_{k+1} \le (1-b_k)a_k + c_k, \forall k \ge 0$, $\sum_{k=1}^\infty b_k = \infty$ and $\limsup_{k\to\infty} c_k/b_k \le 0$ or $\sum_{k=1}^\infty |c_k| < \infty$, then $\lim_{k\to\infty} a_k = 0$.*

3. MAIN RESULTS

In this section, we first describe our proposed algorithm to solve the split problem (1.4) and then prove its convergence.

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex subsets of H_1 and H_2, respectively. Let $A\colon H_1 \to H_2$ be a nonzero bounded linear operator and A^* be the adjoint of A. Let $f : H_1 \to H_1$ be an L_1-Lipschitz pseudocontractive operator and $g : H_2 \to H_2$ be an L_2-Lipschitz pseudocontractive operator with $L_1 > 1$ and $L_2 > 1$. Let the operator ϕ be pseudomonotone on H_1 and κ_1-Lipschitz continuous on C and the operator φ be pseudomonotone on H_2 and κ_2-Lipschitz continuous on Q. Let $F : C \to C$ be an α-contractive operator.

Let $\{\alpha_k\}$, $\{\mu_k\}$, $\{\tau_k\}$, $\{\sigma_k\}$ and $\{\beta_k\}$ be five real number sequences in $(0, 1)$. Let δ, λ, ω and μ be four positive constants in $(0, 1)$ and $\hat{\varepsilon}$ be a positive constant in $(0, 1/\|A\|^2)$.

The self-adaptive Tseng-type algorithm to solve the split problem (1.4) is defined as follows.

Algorithm 3.1. Choose an initial guess $x^0 \in C$ arbitrarily. Select two initial constants $\eta_0 > 0$ and $\zeta_0 > 0$. Set $k = 0$.

Step 1. Let x^k, η_k and ζ_k be given. Calculate

$$\begin{cases} \hat{v}^k = (1-\alpha_k)x^k + \alpha_k f(x^k) \text{ and } v^k = (1-\mu_k)x^k + \mu_k f(\hat{v}^k), & (3.9) \\ y^k = \mathrm{proj}_C[v^k - \eta_k \phi(v^k)], & (3.10) \\ u^k = (1-\delta)v^k + \delta y^k + \delta\eta_k[\phi(v^k) - \phi(y^k)], & (3.11) \\ w^k = \mathrm{proj}_Q[Au^k - \zeta_k \varphi(Au^k)], & (3.12) \\ t^k = (1-\lambda)Au^k + \lambda w^k + \lambda\zeta_k[\varphi(Au^k) - \varphi(w^k)], & (3.13) \\ \hat{q}^k = (1-\sigma_k)t^k + \sigma_k g(t^k) \text{ and } q^k = (1-\tau_k)t^k + \tau_k g(\hat{q}^k). & (3.14) \end{cases}$$

Step 2. Calculate x^{k+1} via the following form

$$x^{k+1} = \beta_k F(x^k) + (1-\beta_k)\mathrm{proj}_C[u^k + \hat{\varepsilon}A^*(q^k - Au^k)]. \tag{3.15}$$

Step 3. Set $k := k + 1$ and update

$$\eta_{k+1} = \begin{cases} \min\left\{\eta_k, \frac{\omega\|y^k - v^k\|}{\|\phi(y^k)-\phi(v^k)\|}\right\}, & \phi(y^k) \neq \phi(v^k), \\ \eta_k, & else. \end{cases} \tag{3.16}$$

and

$$\zeta_{k+1} = \begin{cases} \min\left\{\zeta_k, \frac{\mu\|w^k - Au^k\|}{\|\varphi(w^k)-\varphi(Au^k)\|}\right\}, & \varphi(w^k) \neq \varphi(Au^k), \\ \zeta_k, & else. \end{cases} \tag{3.17}$$

Then go back to Step 1.

Suppose that five real number sequences $\{\alpha_k\}$, $\{\mu_k\}$, $\{\tau_k\}$, $\{\sigma_k\}$ and $\{\beta_k\}$ satisfy the following conditions

(C1): $\lim_{k\to\infty} \beta_k = 0$ and $\sum_{k=0}^{\infty} \beta_k = \infty$;
(C2): $0 < \hat{\mu} < \mu_k < \overline{\mu} < \alpha_k < \overline{\alpha} < \frac{1}{\sqrt{1+L_1^2}+1}$ for all $k \geq 0$;
(C3): $0 < \hat{\tau} < \tau_k < \overline{\tau} < \sigma_k < \overline{\sigma} < \frac{1}{\sqrt{1+L_2^2}+1}$ for all $k \geq 0$.

Remark 3.1. We have the following observations:

(i) By (3.16) and (3.17), the sequences $\{\eta_k\}$ and $\{\zeta_k\}$ are all monotonically decreasing.

(ii) Since ϕ and φ are κ_1-Lipschitz and κ_2-Lipschitz, respectively, we have $\frac{\omega\|y^k-v^k\|}{\|\phi(y^k)-\phi(v^k)\|} \geq \frac{\omega}{\kappa_1}$ and $\frac{\mu\|w^k-Au^k\|}{\|\varphi(w^k)-\varphi(Au^k)\|} \geq \frac{\mu}{\kappa_2}$. Thus, $\eta_k \geq \min\{\eta_0, \frac{\omega}{\kappa_1}\}$ and $\zeta_k \geq \min\{\zeta_0, \frac{\mu}{\kappa_2}\}$ for all $k \geq 0$.

According to (i) and (ii), we know that $\lim_{k\to\infty}\eta_k$ and $\lim_{k\to\infty}\zeta_k$ exist. Therefore,

$$\lim_{k\to\infty}\big[2-\delta-\delta\omega^2\frac{\eta_k^2}{\eta_{k+1}^2}-2(1-\delta)\omega\frac{\eta_k}{\eta_{k+1}}\big]=2-\delta-\delta\omega^2-2(1-\delta)\omega>0$$

and

$$\lim_{k\to\infty}\big[2-\lambda-\lambda\mu^2\frac{\zeta_k^2}{\zeta_{k+1}^2}-2(1-\lambda)\mu\frac{\zeta_k}{\zeta_{k+1}}\big]=2-\lambda-\lambda\mu^2-2(1-\lambda)\mu>0.$$

Thus, there exist a common constant $\sigma>0$ and a positive integer N such that

$$2-\delta-\delta\omega^2\frac{\eta_k^2}{\eta_{k+1}^2}-2(1-\delta)\omega\frac{\eta_k}{\eta_{k+1}}\geq\sigma>0$$

and

$$2-\lambda-\lambda\mu^2\frac{\zeta_k^2}{\zeta_{k+1}^2}-2(1-\lambda)\mu\frac{\zeta_k}{\zeta_{k+1}}\geq\sigma>0,$$

when $k\geq N$.

Next, we give some conditions which are weaker than "the sequential weak-to-weak continuity" imposed on ϕ and φ.

Suppose that ϕ and φ satisfy the following conditions, respectively,

$$(\mathrm{adc1}):\qquad \left.\begin{array}{r}\text{for any given sequence } \{a^k\}\subset H_1\\ a^k\rightharpoonup a^\dagger\in H_1 \ as\ k\to+\infty\\ \liminf\limits_{k\to+\infty}\|\phi(a^k)\|=0\end{array}\right\}\Rightarrow\phi(a^\dagger)=0,$$

and

$$(\mathrm{adc2}):\qquad \left.\begin{array}{r}\text{for any given sequence } \{b^k\}\subset H_2\\ b^k\rightharpoonup b^\dagger\in H_2 \ as\ k\to+\infty\\ \liminf\limits_{k\to+\infty}\|\varphi(b^k)\|=0\end{array}\right\}\Rightarrow\varphi(b^\dagger)=0.$$

Remark 3.2. Recall that an operator $h:H\to H$ is said to be sequentially weak-to-weak continuous, if $H\ni u^k\rightharpoonup\tilde{u}$ implies that $h(u^k)\rightharpoonup h(\tilde{u})$. We can prove that if ϕ and φ are sequentially weak-to-weak continuous, then ϕ and φ satisfy the above conditions (adc1) and (adc2), respectively.

In order to show our main theorem, we first prove several important lemmas. In what follows, suppose that $\Gamma\neq\emptyset$. Set $\hat{x}=\mathrm{proj}_\Gamma F(\hat{x})$. Then, $\hat{x}\in\mathrm{Fix}(f)\cap\mathrm{Sol}(C,\phi)$ and $A\hat{x}\in\mathrm{Fix}(g)\cap\mathrm{Sol}(Q,\varphi)$.

Lemma 3.5. *The sequences $\{x^k\}$, $\{y^k\}$, $\{v^k\}$, $\{u^k\}$, $\{q^k\}$, $\{t^k\}$ and $\{w^k\}$ generated by Algorithm 3.1 are all bounded.*

Proof. By virtue of (2.7) and (3.9), we have

$$\begin{aligned}\|v^k-\hat{x}\|^2&=\|(1-\mu_k)(x^k-\hat{x})+\mu_k(f(\hat{v}^k)-\hat{x})\|^2\\&=(1-\mu_k)\|x^k-\hat{x}\|^2+\mu_k\|f(\hat{v}^k)-\hat{x}\|^2-(1-\mu_k)\mu_k\|f(\hat{v}^k)-x^k\|^2.\end{aligned}\tag{3.18}$$

Applying Lemma 2.1, we get

$$\|f(\hat{v}^k)-\hat{x}\|^2\leq\|x^k-\hat{x}\|^2+(1-\alpha_k)\|f(\hat{v}^k)-x^k\|^2.\tag{3.19}$$

Substituting (3.19) into (3.18), we obtain

$$\begin{aligned}\|v^k-\hat{x}\|^2 &\le (1-\mu_k)\|x^k-\hat{x}\|^2+\mu_k(1-\alpha_k)\|f(\hat{v}^k)-x^k\|^2+\mu_k\|x^k-\hat{x}\|^2\\ &\quad -(1-\mu_k)\mu_k\|f(\hat{v}^k)-x^k\|^2\\ &= \|x^k-\hat{x}\|^2-\mu_k(\alpha_k-\mu_k)\|f(\hat{v}^k)-x^k\|^2\\ &\le \|x^k-\hat{x}\|^2.\end{aligned} \tag{3.20}$$

Similarly, according to (2.7), Lemma 2.1 and (3.14), we have the following estimate

$$\begin{aligned}\|q^k-A\hat{x}\|^2 &\le \|t^k-A\hat{x}\|^2-(\sigma_k-\tau_k)\tau_k\|g(\hat{q}^k)-t^k\|^2\\ &\le \|t^k-A\hat{x}\|^2.\end{aligned} \tag{3.21}$$

In the light of (2.8) and (3.10), we obtain

$$\langle y^k-v^k+\eta_k\phi(v^k),y^k-\hat{x}\rangle\le 0. \tag{3.22}$$

Owing to $\hat{x}\in\mathrm{Sol}(C,\phi)$ and $y^k\in C$, we have $\langle\phi(\hat{x}),y^k-\hat{x}\rangle\ge 0$. Using the pseudomonotonicity of ϕ, we obtain

$$\langle\phi(y^k),y^k-\hat{x}\rangle\ge 0. \tag{3.23}$$

Thanks to (3.22) and (3.23), we get

$$\langle y^k-v^k,y^k-\hat{x}\rangle+\eta_k\langle\phi(v^k)-\phi(y^k),y^k-\hat{x}\rangle\le 0,$$

it leads to

$$\frac{1}{2}(\|y^k-v^k\|^2+\|y^k-\hat{x}\|^2-\|v^k-\hat{x}\|^2)+\eta_k\langle\phi(v^k)-\phi(y^k),y^k-\hat{x}\rangle\le 0,$$

which implies that

$$\|y^k-\hat{x}\|^2\le\|v^k-\hat{x}\|^2-2\eta_k\langle\phi(v^k)-\phi(y^k),y^k-\hat{x}\rangle-\|y^k-v^k\|^2. \tag{3.24}$$

By (3.11), we have

$$\begin{aligned}\|u^k-\hat{x}\|^2 &= \|(1-\delta)(v^k-\hat{x})+\delta(y^k-\hat{x})+\delta\eta_k[\phi(v^k)-\phi(y^k)]\|^2\\ &= \|(1-\delta)(v^k-\hat{x})+\delta(y^k-\hat{x})\|^2+\delta^2\eta_k^2\|\phi(v^k)-\phi(y^k)\|^2\\ &\quad +2\delta(1-\delta)\eta_k\langle v^k-\hat{x},\phi(v^k)-\phi(y^k)\rangle\\ &\quad +2\delta^2\eta_k\langle y^k-\hat{x},\phi(v^k)-\phi(y^k)\rangle.\end{aligned} \tag{3.25}$$

From (2.7), we obtain

$$\begin{aligned}\|(1-\delta)(v^k-\hat{x})+\delta(y^k-\hat{x})\|^2 &= (1-\delta)\|v^k-\hat{x}\|^2+\delta\|y^k-\hat{x}\|^2\\ &\quad -(1-\delta)\delta\|v^k-y^k\|^2.\end{aligned} \tag{3.26}$$

Substituting (3.24) and (3.26) into (3.25), we deduce

$$\begin{aligned}\|u^k-\hat{x}\|^2 &\le \|v^k-\hat{x}\|^2-(2-\delta)\delta\|v^k-y^k\|^2+\delta^2\eta_k^2\|\phi(v^k)-\phi(y^k)\|^2\\ &\quad -2\delta(1-\delta)\eta_k\langle\phi(v^k)-\phi(y^k),y^k-v^k\rangle\\ &\le \|v^k-\hat{x}\|^2-(2-\delta)\delta\|v^k-y^k\|^2+\delta^2\eta_k^2\|\phi(v^k)-\phi(y^k)\|^2\\ &\quad +2\delta(1-\delta)\eta_k\|\phi(v^k)-\phi(y^k)\|\|y^k-v^k\|.\end{aligned} \tag{3.27}$$

From (3.16), We have $\|\phi(v^k)-\phi(y^k)\| \le \frac{\omega\|y^k-v^k\|}{\eta_{k+1}}$. It follows from (3.27) that

$$\begin{aligned}\|u^k-\hat{x}\|^2 &\le \|v^k-\hat{x}\| - (2-\delta)\delta\|v^k-y^k\|^2 + \delta^2\omega^2\frac{\eta_k^2}{\eta_{k+1}^2}\|y^k-v^k\| \\ &\quad + 2\delta(1-\delta)\omega\frac{\eta_k}{\eta_{k+1}}\|y^k-v^k\|^2 \\ &= \|v^k-\hat{x}\| - \delta\left[2-\delta-\delta\omega^2\frac{\eta_k^2}{\eta_{k+1}^2} - 2(1-\delta)\omega\frac{\eta_k}{\eta_{k+1}}\right]\|y^k-v^k\|^2.\end{aligned}\tag{3.28}$$

By Remark 3.1 and (3.28), we get

$$\|u^k-\hat{x}\|^2 \le \|v^k-\hat{x}\| - \sigma\delta\|y^k-v^k\|^2.$$

It follows from (3.20) that

$$\|u^k-\hat{x}\|^2 \le \|x^k-\hat{x}\|^2 - \mu_k(\alpha_k-\mu_k)\|f(\hat{v}^k)-x^k\|^2 - \sigma\delta\|y^k-v^k\|^2.\tag{3.29}$$

Using the property (2.8) of proj_Q and from (3.12), we have

$$\langle w^k - Au^k + \zeta_k\varphi(Au^k), w^k - A\hat{x}\rangle \le 0.\tag{3.30}$$

Owing to $A\hat{x} \in \operatorname{Sol}(Q,\varphi)$ and $w^k \in Q$, we get $\langle\varphi(A\hat{x}), w^k - A\hat{x}\rangle \ge 0$. Using the pseudomonotonicity of φ, we derive

$$\langle\varphi(w^k), w^k - A\hat{x}\rangle \ge 0.\tag{3.31}$$

Taking into account (3.30) and (3.31), we obtain

$$\langle w^k - Au^k, w^k - A\hat{x}\rangle + \zeta_k\langle\varphi(Au^k)-\varphi(w^k), w^k - A\hat{x}\rangle \le 0,$$

which yields

$$\frac{1}{2}(\|w^k-Au^k\|^2 + \|w^k-A\hat{x}\|^2 - \|Au^k-A\hat{x}\|^2) + \zeta_k\langle\varphi(Au^k)-\varphi(w^k), w^k-A\hat{x}\rangle \le 0.$$

It follows that

$$\|w^k-A\hat{x}\|^2 \le \|Au^k-A\hat{x}\|^2 - 2\zeta_k\langle\varphi(Au^k)-\varphi(w^k), w^k-A\hat{x}\rangle - \|w^k-Au^k\|^2.\tag{3.32}$$

From (3.11), we receive

$$\begin{aligned}\|t^k-A\hat{x}\|^2 &= \|(1-\lambda)(Au^k-A\hat{x}) + \lambda(w^k-A\hat{x}) + \lambda\zeta_k[\varphi(Au^k)-\varphi(w^k)]\|^2 \\ &= \|(1-\lambda)(Au^k-A\hat{x}) + \lambda(w^k-A\hat{x})\|^2 + \lambda^2\zeta_k^2\|\varphi(Au^k)-\varphi(w^k)\|^2 \\ &\quad + 2\lambda(1-\lambda)\zeta_k\langle Au^k-A\hat{x}, \varphi(Au^k)-\varphi(w^k)\rangle \\ &\quad + 2\lambda^2\zeta_k\langle w^k-A\hat{x}, \varphi(Au^k)-\varphi(w^k)\rangle.\end{aligned}\tag{3.33}$$

According to (2.7), we achieve

$$\begin{aligned}\|(1-\lambda)(Au^k-A\hat{x}) + \lambda(w^k-A\hat{x})\|^2 &= (1-\lambda)\|Au^k-A\hat{x}\|^2 + \lambda\|w^k-A\hat{x}\|^2 \\ &\quad - (1-\lambda)\lambda\|Au^k-w^k\|^2.\end{aligned}\tag{3.34}$$

Substituting (3.32) and (3.34) into (3.33), we obtain

$$\begin{aligned}\|t^k-A\hat{x}\|^2 &\le \|Au^k-A\hat{x}\|^2 - (2-\lambda)\lambda\|Au^k-w^k\|^2 + \lambda^2\zeta_k^2\|\varphi(Au^k)-\varphi(w^k)\|^2 \\ &\quad - 2\lambda(1-\lambda)\zeta_k\langle w^k-Au^k, \varphi(Au^k)-\varphi(w^k)\rangle \\ &\le \|Au^k-A\hat{x}\|^2 - (2-\lambda)\lambda\|Au^k-w^k\|^2 + \lambda^2\zeta_k^2\|\varphi(Au^k)-\varphi(w^k)\|^2 \\ &\quad + 2\lambda(1-\lambda)\zeta_k\|w^k-Au^k\|\|\varphi(Au^k)-\varphi(w^k)\|.\end{aligned}\tag{3.35}$$

By (3.17), we have

$$\|\varphi(Au^k) - \varphi(w^k)\| \leq \frac{\mu\|Au^k - w^k\|}{\zeta_{k+1}}.$$

This together with (3.35) implies that

$$\begin{aligned}
\|t^k - A\hat{x}\|^2 &\leq \|Au^k - A\hat{x}\| - (2-\lambda)\lambda\|Au^k - w^k\|^2 + \lambda^2\mu^2\frac{\zeta_k^2}{\zeta_{k+1}^2}\|w^k - Au^k\| \\
&\quad + 2\lambda(1-\lambda)\mu\frac{\zeta_k}{\zeta_{k+1}}\|Au^k - w^k\|^2 \\
&= \|Au^k - A\hat{x}\| - \lambda\left[2 - \lambda - \lambda\mu^2\frac{\zeta_k^2}{\zeta_{k+1}^2} - 2(1-\lambda)\mu\frac{\zeta_k}{\zeta_{k+1}}\right]\|Au^k - w^k\|^2.
\end{aligned} \tag{3.36}$$

By Remark 3.1 and (3.36), we have

$$\|t^k - A\hat{x}\|^2 \leq \|Au^k - A\hat{x}\| - \sigma\lambda\|w^k - Au^k\|^2. \tag{3.37}$$

Owing to (3.21) and (3.37), we get

$$\|q^k - A\hat{x}\|^2 \leq \|Au^k - A\hat{x}\| - (\sigma_k - \tau_k)\tau_k\|g(\hat{t}^k) - t^k\|^2 - \sigma\lambda\|w^k - Au^k\|^2. \tag{3.38}$$

Note that

$$\begin{aligned}
\langle u^k - \hat{x}, A^*(q^k - Au^k)\rangle &= \langle Au^k - A\hat{x}, q^k - Au^k\rangle \\
&= \frac{1}{2}[\|q^k - A\hat{x}\|^2 - \|Au^k - A\hat{x}\|^2] - \frac{1}{2}\|q^k - Au^k\|^2.
\end{aligned} \tag{3.39}$$

Combining (3.38) and (3.39), we acquire

$$\begin{aligned}
\langle u^k - \hat{x}, A^*(q^k - Au^k)\rangle \leq &-\frac{1}{2}\sigma\lambda\|w^k - Au^k\|^2 - \frac{1}{2}\|q^k - Au^k\|^2 \\
&- \frac{1}{2}(\sigma_k - \tau_k)\tau_k\|g(\hat{t}^k) - t^k\|^2.
\end{aligned} \tag{3.40}$$

Set $z^k = \mathrm{proj}_C[u^k + \hat{\varepsilon}A^*(q^k - Au^k)]$ for all $k \geq 0$. It follows that

$$\begin{aligned}
\|z^k - \hat{x}\|^2 &= \|\mathrm{proj}_C[u^k + \hat{\varepsilon}A^*(q^k - Au^k)] - \mathrm{proj}_C[\hat{x}]\|^2 \\
&\leq \|u^k - \hat{x} + \hat{\varepsilon}A^*(q^k - Au^k)\|^2 \\
&= \|u^k - \hat{x}\|^2 + \|\hat{\varepsilon}A^*(q^k - Au^k)\|^2 + 2\hat{\varepsilon}\langle A^*(q^k - Au^k), u^k - \hat{x}\rangle.
\end{aligned}$$

By (3.29) and (3.40), we have

$$\begin{aligned}
\|z^k - \hat{x}\|^2 &\leq \|u^k - \hat{x}\|^2 + \hat{\varepsilon}^2\|A\|^2\|q^k - Au^k\|^2 - \hat{\varepsilon}\sigma\lambda\|w^k - Au^k\|^2 \\
&\quad - \hat{\varepsilon}\|q^k - Au^k\|^2 - \hat{\varepsilon}(\sigma_k - \tau_k)\tau_k\|g(\hat{t}^k) - t^k\|^2 \\
&= \|u^k - \hat{x}\|^2 - \hat{\varepsilon}(1 - \hat{\varepsilon}\|A\|^2)\|q^k - Au^k\|^2 - \hat{\varepsilon}\sigma\lambda\|w^k - Au^k\|^2 \\
&\quad - \hat{\varepsilon}(\sigma_k - \tau_k)\tau_k\|g(\hat{t}^k) - t^k\|^2 \\
&\leq \|x^k - \hat{x}\|^2 - \hat{\varepsilon}\sigma\lambda\|w^k - Au^k\|^2 - \hat{\varepsilon}(1 - \hat{\varepsilon}\|A\|^2)\|q^k - Au^k\|^2 \\
&\quad - \mu_k(\alpha_k - \mu_k)\|f(\hat{v}^k) - x^k\|^2 - \sigma\delta\|y^k - v^k\|^2 \\
&\quad - \hat{\varepsilon}(\sigma_k - \tau_k)\tau_k\|g(\hat{t}^k) - t^k\|^2.
\end{aligned} \tag{3.41}$$

By (3.15) and (3.41), we obtain

$$\begin{aligned}
\|x^{k+1}-\hat{x}\| &= \|\beta_k(F(x^k)-\hat{x})+(1-\beta_k)(z^k-\hat{x})\| \\
&\le \beta_k\|F(x^k)-\hat{x}\|+(1-\beta_k)\|z^k-\hat{x}\| \\
&\le \beta_k\|F(x^k)-F(\hat{x})\|+\beta_k\|F(\hat{x})-\hat{x}\|+(1-\beta_k)\|x^k-\hat{x}\| \\
&\le [1-(1-\alpha)\beta_k]\|x^k-\hat{x}\|+(1-\alpha)\beta_k\frac{\|F(\hat{x})-\hat{x}\|}{1-\alpha} \\
&\le \max\{\|x^k-\hat{x}\|,\frac{\|F(\hat{x})-\hat{x}\|}{1-\alpha}\} \\
&\le \cdots \\
&\le \max\{\|x^0-\hat{x}\|,\frac{\|F(\hat{x})-\hat{x}\|}{1-\alpha}\}.
\end{aligned}$$

Hence, the sequence $\{x^k\}$ is bounded. According to the above discussion, we can deduce that the sequences $\{q^k\}$, $\{v^k\}$, $\{t^k\}$, $\{u^k\}$, $\{w^k\}$ and $\{y^k\}$ are bounded. □

Lemma 3.6. *$\omega_w(x^k)\subset\Gamma$, where $\omega_w(x^k)$ denotes the set of the weak cluster points of the sequence $\{x^k\}$, i.e., $\omega_w(x^k):=\{z\in C:\exists\{x^{k_i}\}\subset\{x^k\}$ such that $x^{k_i}\rightharpoonup z(i\to\infty)\}$.*

Proof. Take into consideration of (3.15), we have

$$\begin{aligned}
\|x^{k+1}-\hat{x}\|^2 &= \|\beta_k(F(x^k)-\hat{x})+(1-\beta_k)(z^k-\hat{x})\|^2 \\
&= \beta_k\langle F(x^k)-\hat{x},x^{k+1}-\hat{x}\rangle+(1-\beta_k)\langle z^k-\hat{x},x^{k+1}-\hat{x}\rangle \\
&\le \beta_k\alpha\frac{1}{2}(\|x^k-\hat{x}\|^2+\|x^{k+1}-\hat{x}\|^2)+\beta_k\langle F(\hat{x})-\hat{x},x^{k+1}-\hat{x}\rangle \\
&\quad +(1-\beta_k)\frac{1}{2}(\|z^k-\hat{x}\|^2+\|x^{k+1}-\hat{x}\|^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x^{k+1}-\hat{x}\|^2 \le{}& \frac{\alpha\beta_k}{1+(1-\alpha)\beta_k}\|x^k-\hat{x}\|^2+\frac{1-\beta_k}{1+(1-\alpha)\beta_k}\|z^k-\hat{x}\|^2 \\
&+\frac{2\beta_k}{1+(1-\alpha)\beta_k}\langle F(\hat{x})-\hat{x},x^{k+1}-\hat{x}\rangle.
\end{aligned}\tag{3.42}$$

In view of (3.41) and (3.42), we receive

$$\begin{aligned}
\|x^{k+1}-\hat{x}\|^2 \le{}& \left[1-\frac{2(1-\alpha)\beta_k}{1+(1-\alpha)\beta_k}\right]\|x^k-\hat{x}\|^2+\frac{2(1-\alpha)\beta_k}{1+(1-\alpha)\beta_k}\Bigg(-\frac{(1-\beta_k)\sigma\delta}{2(1-\alpha)}\frac{\|y^k-v^k\|^2}{\beta_k} \\
&-\frac{(1-\beta_k)\hat{\varepsilon}(1-\hat{\varepsilon}\|A\|^2)}{2(1-\alpha)}\frac{\|q^k-Au^k\|^2}{\beta_k}-\frac{(1-\beta_k)\hat{\varepsilon}\sigma\lambda}{2(1-\alpha)}\frac{\|w^k-Au^k\|^2}{\beta_k} \\
&-\frac{(1-\beta_k)\mu_k(\alpha_k-\mu_k)}{2(1-\alpha)}\frac{\|f(\hat{v}^k)-x^k\|^2}{\beta_k}+\frac{1}{1-\alpha}\langle F(\hat{x})-\hat{x},x^{k+1}-\hat{x}\rangle \\
&-\frac{(1-\beta_k)\hat{\varepsilon}(\sigma_k-\tau_k)\tau_k}{2(1-\alpha)}\frac{\|g(\hat{t}^k)-t^k\|^2}{\beta_k}\Bigg).
\end{aligned}\tag{3.43}$$

For all $k \geq 0$, set $a_k = \frac{2(1-\alpha)\beta_k}{1+(1-\alpha)\beta_k}$ and

$$\begin{aligned} b_k = {} & \frac{1}{1-\alpha}\langle F(\hat{x}) - \hat{x}, x^{k+1} - \hat{x}\rangle - \frac{(1-\beta_k)\hat{\varepsilon}\sigma\lambda}{2(1-\alpha)} \frac{\|w^k - Au^k\|^2}{\beta_k} \\ & - \frac{(1-\beta_k)\hat{\varepsilon}(1-\hat{\varepsilon}\|A\|^2)}{2(1-\alpha)} \frac{\|q^k - Au^k\|^2}{\beta_k} - \frac{(1-\beta_k)\sigma\delta}{2(1-\alpha)} \frac{\|y^k - v^k\|^2}{\beta_k} \\ & - \frac{(1-\beta_k)\mu_k(\alpha_k - \mu_k)}{2(1-\alpha)} \frac{\|f(\hat{v}^k) - x^k\|^2}{\beta_k} - \frac{(1-\beta_k)\hat{\varepsilon}(\sigma_k - \tau_k)\tau_k}{2(1-\alpha)} \frac{\|g(\hat{t}^k) - t^k\|^2}{\beta_k}. \end{aligned} \tag{3.44}$$

It is clear that $b_k \leq \frac{1}{1-\alpha}\|F(\hat{x}) - \hat{x}\|\|x^{k+1} - \hat{x}\|$ and $\limsup_{k\to\infty} b_k$ exists.

According to Lemma 3.5, the sequence $\{x^k\}$ is bounded. Selecting any $p^\dagger \in \omega(x^k)$, there is a subsequence $\{k_i\}$ of $\{k\}$ such that $x^{k_i+1} \rightharpoonup p^\dagger \in C$ and $\limsup_{k\to\infty} b_k = \lim_{i\to\infty} b_{k_i}$. Moreover, from (3.44), we have

$$\begin{aligned} \lim_{i\to\infty} \Bigg[& -\frac{(1-\beta_{k_i})\hat{\varepsilon}\sigma\lambda}{2(1-\alpha)} \frac{\|w^{k_i} - Au^{k_i}\|^2}{\beta_{k_i}} - \frac{(1-\beta_{k_i})\hat{\varepsilon}(1-\hat{\varepsilon}\|A\|^2)}{2(1-\alpha)} \frac{\|q^{k_i} - Au^{k_i}\|^2}{\beta_{k_i}} \\ & - \frac{(1-\beta_{k_i})\sigma\delta}{2(1-\alpha)} \frac{\|y^{k_i} - v^{k_i}\|^2}{\beta_{k_i}} - \frac{(1-\beta_{k_i})\mu_{k_i}(\alpha_{k_i} - \mu_{k_i})}{2(1-\alpha)} \frac{\|f(\hat{v}^{k_i}) - x^{k_i}\|^2}{\beta_{k_i}} \\ & - \frac{(1-\beta_{k_i})\hat{\varepsilon}(\sigma_{k_i} - \tau_{k_i})\tau_{k_i}}{2(1-\alpha)} \frac{\|g(\hat{t}^{k_i}) - t^{k_i}\|^2}{\beta_{k_i}} \Bigg] \end{aligned} \tag{3.45}$$

exists. It results in that

$$\begin{cases} \lim_{i\to+\infty} \|q^{k_i} - Au^{k_i}\| = 0, & (3.46) \\ \lim_{i\to+\infty} \|f(\hat{v}^{k_i}) - x^{k_i}\| = 0, & (3.47) \\ \lim_{i\to+\infty} \|g(\hat{t}^{k_i}) - t^{k_i}\| = 0, & (3.48) \\ \lim_{i\to+\infty} \|y^{k_i} - v^{k_i}\| = 0, & (3.49) \\ \lim_{i\to+\infty} \|w^{k_i} - Au^{k_i}\| = 0. & (3.50) \end{cases}$$

From (3.49) and Lipschitz continuity of ϕ, we have $\|\phi(v^{k_i}) - \phi(y^{k_i})\| \to 0$ as $i \to \infty$. According to (3.11) and (3.49), we get $\|u^{k_i} - v^{k_i}\| \to 0$ as $i \to \infty$. From (3.9) and (3.47), we conclude that $\|x^{k_i} - v^{k_i}\| \to 0 (i \to \infty)$. Since

$$\begin{aligned} \|z^{k_i} - \mathrm{proj}_C[u^{k_i}]\| &= \|\mathrm{proj}_C[u^{k_i} + \hat{\varepsilon}A^*(q^{k_i} - Au^{k_i})] - \mathrm{proj}_C[u^{k_i}]\| \\ &\leq \hat{\varepsilon}\|A\|\|q^{k_i} - Au^{k_i}\|, \end{aligned}$$

it follows from (3.46) that $\lim_{i\to+\infty} \|z^{k_i} - \mathrm{proj}_C[u^{k_i}]\| = 0$. By (3.15), $\|x^{k_i+1} - z^{k_i}\| \to 0 (i \to \infty)$. Therefore, $\|x^{k_i} - x^{k_i+1}\| \to 0$ as $i \to \infty$. This asserts that $x^{k_i} \rightharpoonup p^\dagger$ as well. By the L_1-Lipschitz continuity of f, we have

$$\begin{aligned} \|f(x^{k_i}) - x^{k_i}\| &\leq \|f(x^{k_i}) - f(\hat{v}^{k_i})\| + \|f(\hat{v}^{k_i}) - x^{k_i}\| \\ &\leq L_1\alpha_{k_i}\|f(x^{k_i}) - x^{k_i}\| + \|f(\hat{v}^{k_i}) - x^{k_i}\|, \end{aligned}$$

which yields $\|f(x^{k_i}) - x^{k_i}\| \leq \frac{1}{1-L_1\alpha_{k_i}}\|f(\hat{v}^{k_i}) - x^{k_i}\|$. This together with (3.47) implies that

$$\lim_{i\to+\infty} \|f(x^{k_i}) - x^{k_i}\| = 0. \tag{3.51}$$

Observe that $y^{k_i} \rightharpoonup p^\dagger$ and $v^{k_i} \rightharpoonup p^\dagger$ as $i \to \infty$. In view of (2.8) and $y^{k_i} = \mathrm{proj}_C[v^{k_i} - \eta_{k_i}\phi(v^{k_i})]$, we achieve

$$\langle y^{k_i} - v^{k_i} + \eta_{k_i}\phi(v^{k_i}), y^{k_i} - u\rangle \le 0, \ \forall u \in C.$$

It follows that

$$\frac{1}{\eta_{k_i}}\langle v^{k_i} - y^{k_i}, u - y^{k_i}\rangle + \langle \phi(v^{k_i}), y^{k_i} - v^{k_i}\rangle \le \langle \phi(v^{k_i}), u - v^{k_i}\rangle, \ \forall u \in C. \tag{3.52}$$

Since $\{y^{k_i}\}$ and $\{\phi(v^{k_i})\}$ are bounded, by (3.49) and (3.52), we deduce

$$\liminf_{i\to\infty}\langle \phi(v^{k_i}), u - v^{k_i}\rangle \ge 0, \ \forall u \in C. \tag{3.53}$$

Next, we prove $p^\dagger \in \mathrm{Sol}(C, \phi)$ by considering two cases: (1) $\liminf_{i\to\infty}\|\phi(v^{k_i})\| = 0$, and (2) $\liminf_{i\to\infty}\|\phi(v^{k_i})\| > 0$. In the case where $\liminf_{i\to\infty}\|\phi(v^{k_i})\| = 0$, it follows from $v^{k_i} \rightharpoonup p^\dagger (i \to \infty)$ and ϕ satisfying condition (adc1) that $\phi(p^\dagger) = 0$. In this case, we have $p^\dagger \in \mathrm{Sol}(C, \phi)$.

Now, we consider the case (2) $\liminf_{i\to\infty}\|\phi(v^{k_i})\| > 0$. In terms of (3.53), we obtain

$$\liminf_{i\to\infty}\left\langle \frac{\phi(v^{k_i})}{\|\phi(v^{k_i})\|}, u - v^{k_i}\right\rangle \ge 0. \tag{3.54}$$

Thanks to (3.54), we can choose a positive real numbers sequence $\{\tilde{\epsilon}_i\}$ satisfying $\tilde{\epsilon}_i \to 0$ as $i \to \infty$. For each $\tilde{\epsilon}_i$, there exists the smallest positive integer N_i such that $\left\langle \frac{\phi(v^{k_i})}{\|\phi(v^{k_i})\|}, u - v^{k_i}\right\rangle + \tilde{\epsilon}_i \ge 0, \ \forall i \ge N_i$. It follows that

$$\langle \phi(v^{k_i}), u - v^{k_i}\rangle + \tilde{\epsilon}_i\|\phi(v^{k_i})\| \ge 0, \ \forall i \ge N_i. \tag{3.55}$$

Set $\tilde{v}^{k_i} = \frac{\phi(v^{k_i})}{\|\phi(v^{k_i})\|^2}$ and hence $\langle \phi(v^{k_i}), \tilde{v}^{k_i}\rangle = 1$ for each i. By (3.55), we deduce

$$\langle \phi(v^{k_i}), u + \tilde{\epsilon}_i\|\phi(v^{k_i})\|\tilde{v}^{k_i} - v^{k_i}\rangle \ge 0, \ \forall i \ge N_i. \tag{3.56}$$

Since ϕ is pseudomonotone, it follows from (3.56) that

$$\langle \phi(u + \tilde{\epsilon}_i\|\phi(v^{k_i})\|\tilde{v}^{k_i}), u + \tilde{\epsilon}_i\|\phi(v^{k_i})\|\tilde{v}^{k_i} - v^{k_i}\rangle \ge 0, \ \forall i \ge N_i. \tag{3.57}$$

Note that $\lim_{i\to\infty}\tilde{\epsilon}_i\|\phi(v^{k_i})\|\|\tilde{v}^{k_i}\| = \lim_{i\to\infty}\tilde{\epsilon}_i = 0$. Letting $i \to \infty$ in (3.57), we obtain

$$\langle \phi(u), u - p^\dagger\rangle \ge 0. \tag{3.58}$$

Applying Lemma 2.2 to (3.58), we conclude that $p^\dagger \in \mathrm{Sol}(C, \phi)$. On the other hand, according to (3.51), $x^{k_i} \rightharpoonup p^\dagger$ and Lemma 2.3, we deduce that $p^\dagger \in \mathrm{Fix}(f)$. Therefore, $p^\dagger \in \mathrm{Fix}(f) \cap \mathrm{Sol}(C, \phi)$.

Next, we show that $Ap^\dagger \in \mathrm{Fix}(g) \cap \mathrm{Sol}(Q, \varphi)$. Since

$$\begin{aligned}\|g(t^{k_i}) - t^{k_i}\| &\le \|g(t^{k_i}) - g(\hat{t}^{k_i})\| + \|g(\hat{t}^{k_i}) - t^{k_i}\| \\ &\le L_2\sigma_{k_i}\|g(t^{k_i}) - t^{k_i}\| + \|g(\hat{t}^{k_i}) - t^{k_i}\|,\end{aligned}$$

it follows that

$$\|g(t^{k_i}) - t^{k_i}\| \le \frac{1}{1 - L_2\sigma_{k_i}}\|g(\hat{t}^{k_i}) - t^{k_i}\|,$$

which together with (3.48) implies that

$$\lim_{k\to+\infty}\|g(t^{k_i}) - t^{k_i}\| = 0. \tag{3.59}$$

Thanks to (3.14) and (3.48), we have $q^{k_i} - t^{k_i} \to 0$ as $i \to \infty$. Note that $u^{k_i} \rightharpoonup p^\dagger$ and $p^{k_i} \rightharpoonup Ap^\dagger$ as $i \to \infty$. Thus, $t^{k_i} \rightharpoonup Ap^\dagger$ as $i \to \infty$. Applying Lemma 2.3 to (3.59), we obtain that $Ap^\dagger \in \mathrm{Fix}(g)$.

Next, we show that $Ap^\dagger \in \mathrm{Sol}(Q,\varphi)$. In view of (2.7) and $w^{k_i} = \mathrm{proj}_Q[Au^{k_i} - \zeta_{k_i}\varphi(Au^{k_i})]$, we achieve

$$\langle w^{k_i} - Au^{k_i} + \zeta_{k_i}\varphi(Au^{k_i}), w^{k_i} - v\rangle \le 0, \forall v \in Q.$$

It follows that

$$\frac{1}{\zeta_{k_i}}\langle w^{k_i} - Au^{k_i}, w^{k_i} - v\rangle + \langle \varphi(Au^{k_i}), w^{k_i} - Au^{k_i}\rangle \le \langle \varphi(Au^{k_i}), v - Au^{k_i}\rangle,\ \forall v \in Q. \tag{3.60}$$

Based on (3.50) and (3.60), we deduce

$$\liminf_{i\to\infty}\langle \varphi(Au^{k_i}), v - Au^{k_i}\rangle \ge 0,\ \forall v \in Q. \tag{3.61}$$

By the similar argument as that of f, we can prove $Ap^\dagger \in \mathrm{Sol}(Q,\varphi)$. So, $p^\dagger \in \Gamma$ and $\omega_w(x^k) \subset \Gamma$. □

Finally, with the help of Lemmas 3.5 and 3.6, we show that the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to a solution of the split problem (1.4).

Theorem 3.1. *Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges strongly to $\hat{x} = \mathrm{proj}_\Gamma F(\hat{x})$.*

Proof. From (3.43), we have

$$\begin{aligned}\|x^{k+1} - \hat{x}\|^2 \le &\left[1 - \frac{2(1-\alpha)\beta_k}{1+(1-\alpha)\beta_k}\right]\|x^k - \hat{x}\|^2\\ &+ \frac{2(1-\alpha)\beta_k}{1+(1-\alpha)\beta_k}\left(\frac{1}{1-\alpha}\langle F(\hat{x}) - \hat{x}, x^{k+1} - \hat{x}\rangle\right).\end{aligned} \tag{3.62}$$

Note that

$$\begin{aligned}\limsup_{k\to\infty} b_k &= \lim_{i\to\infty} b_{k_i}\\ &\le \lim_{i\to\infty}\frac{1}{1-\alpha}\langle F(\hat{x}) - \hat{x}, x^{k_i+1} - \hat{x}\rangle\\ &= \frac{1}{1-\alpha}\langle F(\hat{x}) - \hat{x}, p^\dagger - \hat{x}\rangle\\ &\le 0.\end{aligned} \tag{3.63}$$

According to Lemma 2.4, (3.62) and (3.63), we conclude that $x^k \to \hat{x}$ as $k \to \infty$. This completes the proof. □

Algorithm 3.2. Choose an initial guess $x^0 \in C$ arbitrarily. Let the sequence $\{x^k\}$ be generated by

$$\begin{cases}\hat{v}^k = (1-\alpha_k)x^k + \alpha_k f(x^k) \textit{ and } v^k = (1-\mu_k)x^k + \mu_k f(\hat{v}^k),\\ \hat{q}^k = (1-\sigma_k)Av^k + \sigma_k g(Av^k) \textit{ and } q^k = (1-\tau_k)Av^k + \tau_k g(\hat{q}^k),\\ x^{k+1} = \beta_k F(x^k) + (1-\beta_k)\mathrm{proj}_C[v^k + \hat{\varepsilon}A^*(q^k - Av^k)].\end{cases}$$

Corollary 3.1. *Suppose that $\Gamma_1 \ne \emptyset$. Then the sequence $\{x^k\}$ generated by Algorithm 3.2 converges strongly to $p_1 = \mathrm{proj}_{\Gamma_1}F(p_1)$.*

Algorithm 3.3. Choose an initial guess $x^0 \in C$ arbitrarily. Select two initial constants $\eta_0 > 0$ and $\zeta_0 > 0$. Set $k = 0$.

Step 1. Let x^k, η_k and ζ_k be known. Calculate

$$\begin{cases} y^k = \mathrm{proj}_C[x^k - \eta_k \phi(x^k)], \\ u^k = (1-\delta)x^k + \delta y^k + \delta\eta_k[\phi(x^k) - \phi(y^k)], \\ w^k = \mathrm{proj}_Q[Au^k - \zeta_k \varphi(Au^k)], \\ t^k = (1-\lambda)Au^k + \lambda w^k + \lambda\zeta_k[\varphi(Au^k) - \varphi(w^k)]. \end{cases}$$

Step 2. Calculate x^{k+1} via the following form

$$x^{k+1} = \beta_k F(x^k) + (1-\beta_k)\mathrm{proj}_C[u^k + \hat{\varepsilon}A^*(t^k - Au^k)].$$

Step 3. Set $k := k+1$ and update

$$\eta_{k+1} = \begin{cases} \min\left\{\eta_k, \frac{\omega\|y^k - x^k\|}{\|\phi(y^k) - \phi(x^k)\|}\right\}, & \phi(y^k) \neq \phi(x^k), \\ \eta_k, & else. \end{cases}$$

and

$$\zeta_{k+1} = \begin{cases} \min\left\{\zeta_k, \frac{\mu\|w^k - Au^k\|}{\|\varphi(w^k) - \varphi(Au^k)\|}\right\}, & \varphi(w^k) \neq \varphi(Au^k), \\ \zeta_k, & else. \end{cases}$$

Then go back to Step 1.

Corollary 3.2. *Suppose that* $\Gamma_2 \neq \emptyset$*. Then the sequence* $\{x^k\}$ *generated by Algorithm 3.3 converges strongly to* $p_2 = \mathrm{proj}_{\Gamma_2} F(p_2)$*.*

Appendix

In this appendix, we demonstrate a proposition and an example which indicate that the conditions (adc1) and (adc2) are strictly weaker than "the sequential weak-to-weak continuity" imposed on ϕ and φ, respectively.

Proposition 3.1. *Let* H *be a real Hilbert space. Let* $\psi : H \to H$ *be an operator. If* ψ *is sequentially weak-to-weak continuous, then* ψ *satisfies the following relation*

$$(\mathrm{con}): \qquad \left.\begin{array}{r} \textit{for any given sequence } \{u^k\} \subset H \\ u^k \rightharpoonup u^\dagger \in H \textit{ as } k \to +\infty \\ \liminf\limits_{k\to+\infty} \|\psi(u^k)\| = 0 \end{array}\right\} \Rightarrow \psi(u^\dagger) = 0.$$

Proof. Let $\{u^k\}$ be a sequence in H. Suppose that $u^k \rightharpoonup u^\dagger \in H$ as $k \to +\infty$ and $\liminf_{k\to+\infty} \|\psi(u^k)\| = 0$. First, we have the following equality

$$\liminf_{k\to+\infty} \|\psi(u^k)\|^2 = \liminf_{k\to+\infty} \|\psi(u^k) - \psi(u^\dagger\|^2 + \|\psi(u^\dagger)\|^2. \tag{3.64}$$

As a matter of fact, we have

$$\|\psi(u^k)\|^2 = \|\psi(u^k) - \psi(u^\dagger)\|^2 + 2\langle\psi(u^k) - \psi(u^\dagger), \psi(u^\dagger)\rangle + \|\psi(u^\dagger)\|^2. \tag{3.65}$$

Since $u^k \rightharpoonup u^\dagger \in H$ as $k \to +\infty$ and ψ is sequentially weak-to-weak continuous, $\psi(u^k) \rightharpoonup \psi(u^\dagger)(k \to +\infty)$. Taking the inferior limit on both sides of (3.65), we concluded the desired result (3.64).

Note that $\liminf_{k\to+\infty} \|\psi(u^k)\| = 0$. This together with (3.64) implies that

$$\liminf_{k\to+\infty} \|\psi(u^k) - \psi(u^\dagger\|^2 + \|\psi(u^\dagger)\|^2 = 0.$$

It follows that

$$0 \leq \liminf_{k\to\infty} \|\psi(^k) - \psi(u^\dagger\|^2 = -\|\psi(u^\dagger)\|^2,$$

which implies that $\psi(u^\dagger) = 0$, i.e., ψ satisfies the relation (con). □

Next, we give an example below which shows that

(i) ψ is continuous;
(ii) ψ is not sequentially weak-to-weak continuous;
(iii) ψ satisfies assumption (con).

Example 3.1. Let $H = \ell^2(\mathbb{N})$ with $\{e_n\}$ as its standard orthogonal basis. Define

$$\psi : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), x \mapsto \|x\|e_1. \tag{3.66}$$

(i) It is obvious that ψ is a norm continuous function.

(ii) Note that $e_k \rightharpoonup 0 (k \to +\infty)$ and $\psi(0) = 0$. But $\forall k \geq 1$, $\psi(e_k) = \|e_k\|e_1 \equiv e_1$ which does not weakly converge to 0. This fact indicates that ψ is not sequentially weak-to-weak continuous.

(iii) Next, we show that ψ satisfies assumption (con). In fact, let $u^k \in \ell^2(\mathbb{N})$ and $u^k \rightharpoonup u^\dagger (k \to +\infty)$. Assume that $\liminf_{k\to+\infty} \|\psi(u^k)\| = 0$. Then, there exists a subsequence $\{u^{k_i}\} \subset \{u^k\}$ such that

$$\liminf_{k\to+\infty} \|\psi(u^k)\| = \lim_{i\to+\infty} \|\psi(u^{k_i})\| = 0.$$

Note that $\|\psi(u^{k_i})\| = \|u^{k_i}\|\|e_1\| = \|u^{k_i}\|$. Then, we have $u^{k_i} \rightharpoonup u^\dagger$ and $\|u^{k_i}\| \to 0$ as $i \to +\infty$. Since

$$\begin{aligned} \|u^{k_i} - u^\dagger\|^2 &= \|u^{k_i}\|^2 - 2\langle u^{k_i}, u^\dagger\rangle + \|u^\dagger\|^2 \\ &\to -\|u^\dagger\|^2 \; as \; i \to +\infty, \end{aligned}$$

it follows that $u^\dagger = 0$ and thus $\psi(u^\dagger) = \|u^\dagger\|e_1 = 0$. Therefore, ψ satisfies the relation (con).

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