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Fixed point theorems for basic θ-contraction and applications

MARIJA CVETKOVIĆ[1], ERDAL KARAPINAR[2,3], AND ADRIAN PETRUŞEL[4,5]

ABSTRACT. The main aim of this paper is omitting some superfluous assumptions in the definition of the class of functions Θ, by means of which were defined and studied various classes of θ-contractions, and still obtaining the uniqueness of the fixed point for this new type of contractive mappings. Several generalizations of continuous θ-contractions are presented along with their applications to the study of integral equations.

1. INTRODUCTION

M. Jleli and B. Samet in [14] presented a new generalization of Banach contractive condition. They through defining a new class of contractions known as θ-contractions in the setting of the generalized metric space in the sense of Branciari [7]. The concept of Branciari's generalized metric space is based on the modification of the triangle inequality with $d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$ for any pairwise distinct points which is known as rectangular or quadrilateral inequality. It was shown in [14] that a θ-contraction has a unique fixed point on a complete generalized metric space in the sense of Branciari. M. Jleli, E. Karapınar and B. Samet [13] proved several θ-contraction type results assuming that θ-function is continuous. Additionally, more general contractive condition was considered. Afterwards, several different generalizations of the θ-contraction were presented and existence and uniqueness of a fixed point of these classes of contractive mappings was proved in several settings (Branciari's metric space, metric space, b-metric space, partial metric space, cone metric space, etc.). It is important to mention that the focus of many papers was on the applications in the area od image processing, differential and integral equations, fractional calculus, etc. (see e.g. [1, 4, 5, 12], [15]-[18]). For related notions and results see [6] and [9].

Several questions regarding the definition of θ-functions have arisen. First of them was the question of necessity for θ to fulfill such a strict condition as (θ_3). Moreover, do we and in which occasions, need to add a continuity assumption? Can the properties of θ be redefined in order to obtain larger class of mappings? Are some of these conclusions different depending on the setting-Branciari's metric space, metric space or some other? Some of these questions have already been partially answered, as in [13, 16], and some of them will be the main point of interest of this article. It is our intention to present some applications and also some possible research problems demanding a different approach like in [17].

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Corresponding author: Erdal Karapınar, erdalkarapinar@yahoo.com

2. Preliminaries

As previously stated, M. Jleli and B. Samet defined a new class of functions denoted with Θ and related contractive condition.

Definition 2.1. *Let Θ be a set of functions $\theta : (0, \infty) \mapsto (1, \infty)$ such that*

(θ_1) *θ is nondecreasing, i.e., $x \leq y \implies \theta(x) \leq \theta(y)$;*

(θ_2) *for each sequence $(x_n) \subseteq (0, \infty)$*

$$\lim_{n\to\infty} \theta(x_n) = 1 \Leftrightarrow \lim_{n\to\infty} x_n = 0;$$

(θ_3) *there exist $0 < k < 1$ and $l \in (0, \infty]$ such that*

$$\lim_{x\to 0} \frac{\theta(x) - 1}{x^k} = l.$$

Question of the existence of a fixed point for the class of θ-contractions, along with the uniqueness, was discussed in a setting of a generalized metric space in the sense of Branciari. (see [7])

Definition 2.2. *Let X be a non-empty set and $d : X \times X \mapsto [0, \infty)$ a mapping such that (d_1) and (d_2) hold for all $x, y \in X$ and the inequality*

$$(d_3^*) d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$$

holds for all pairwise distinct points $x, y, z, w \in X$.

Remark 2.1. *Note that every metric space is a generalized metric space. The inequality (d_3^*) is known as rectangular or quadrilateral inequality. There are several examples of mappings satisfying (d_1), (d_2) and the rectangular inequality, but not the triangle inequality.*

Definition 2.3. *Let (X, d) be a generalized metric space in the sense of Branciari. A mapping $T : X \mapsto X$ is a θ-contraction if there exists a function $\theta \in \Theta$ and $k \in (0, 1)$ such that for all $x, y \in X$*

$$Tx \neq Ty \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k. \tag{2.1}$$

The θ-contraction can be also found in the literature as a ϕ-contraction. In [14] the authors have shown that a θ-contraction on a complete generalized metric space in the sense of Branciari and, as a consequence on a complete metric space, has a unique fixed point.

Theorem 2.1. [14] *Let (X, d) be a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ a θ-contraction. A mapping T has a unique fixed point on X.*

As a modification of the previous result, in [13] was presented a new definition of a class Θ, which will be denoted with Θ' in the sequence.

Definition 2.4. *Let Θ' be a set of functions $\theta : (0, \infty) \mapsto (1, \infty)$ such that*

(θ_1) *θ is nondecreasing, i.e., $x \leq y \implies \theta(x) \leq \theta(y)$;*

(θ_2) *for each sequence $(x_n) \subseteq (0, \infty)$*

$$\lim_{n\to\infty} \theta(x_n) = 1 \Leftrightarrow \lim_{n\to\infty} x_n = 0;$$

(θ_3) *there exist $0 < k < 1$ and $l \in (0, \infty]$ such that*

$$\lim_{x\to 0} \frac{\theta(x) - 1}{x^k} = l;$$

(θ_4) *θ is continuous.*

In a same manner, they have defined a new type of θ-contraction depending on a more general contractive condition than (2.1).

Theorem 2.2. [13] *Let (X,d) be a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ a mapping. If there exist a function $\theta \in \Theta'$ and $k \in (0,1)$ such that for all $x, y \in X$*

$$\theta(d(Tx,Ty)) \leq [\theta(M(x,y))]^k,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$, then T has a unique fixed point.

As a corollary, we obtain that each mapping on a complete (generalized) metric space for which there exists some $\theta \in \Theta'$ and $k \in (0,1)$ such that (2.1) holds for any $x, y \in X$ has a unique fixed point with the additional continuity assumption.
With the appropriate definition of distance d and mappings T and θ, we can discuss also on some examples fulfilling the assumptions of Theorem 2.2, but not the contractive condition (2.1). Observe also that the techniques in the proofs of Theorem 2.1 and Theorem 2.2 differ due to the part of the proof concerning the Cauchy property of the sequence of successive approximations for arbitrary starting point $x_0 \in X$. Evidently, $\Theta' \subset \Theta$. J. Ahmad et al. [4] omitted (θ_3) in the definition of class Θ' and gave the proof of existence and uniqueness of a fixed point for a modified θ-contraction on a complete metric space where observed class of θ-type functions gathers (θ_1), (θ_2) and (θ_4). We will denote that class with Θ^*. It is notable that the classes Θ and Θ^* do not coincide since (θ_3) and (θ_4) do not generate same classes of functions as confirmed in the following example.

Example 2.1. *The conditions (θ_3) and (θ_4) are independent. If $\theta(x) = e^{e^x-1}$, for $x \in (0,\infty)$, then $\lim\limits_{x\to 0+} \frac{\theta(x)-1}{x^k} = 0$ for any $k \in (0,1)$, but it is a continuous function. On the other hand, $\theta(x) = 1 + \sqrt{x}(1+[x])$, for any $x > 0$,is not continuous, but (θ_4) holds for any $k \in [\frac{1}{2}, 1)$. Also note that (θ_1) and (θ_2) do hold in both cases. Nevertheless, a set $\Theta^* \cap \Theta$ is non-empty since we have, par example, $\theta(x) = e^{\sqrt{x}}$, $x > 0$.*

To make a difference, we will name the new type of contraction, in accordance with the class of functions Θ^*, a θ^*-contraction.

Definition 2.5. *Let (X,d) be a generalized metric space in the sense of Branciari. A mapping $T : X \mapsto X$ is a θ^*-contraction if there exists a function $\theta \in \Theta^*$ and $k \in (0,1)$ such that (2.1) holds for any $x, y \in X$.*

The main result of [4] is the following

Theorem 2.3. [4] *If (X,d) is a complete metric space and $T : X \mapsto X$ a θ^*-contraction for some $\theta \in \Theta^*$ and $k \in (0,1)$, then T has a unique fixed point in X.*

Further results on this topic do mainly focus on the definition of θ-contraction including (θ_4) rather than (θ_3). Contractive conditions and types of settings vary (Kannan contraction, Hardy-Rogers contraction, φ-contraction, Meir-Keeler contraction,etc; b-metric space, cone metric space, generalized metric space in the sense of Branciari etc.
Liu et al, [16] presented a different approach to the class of θ functions by introducing an equivalent to (θ_2):

$$(\theta_2^*)\ \inf_{x>0} \theta(x) = 1.$$

They proved existence and uniqueness of a fixed point for a Suzuki θ-contraction and, as a direct corollary, for a class of modified θ^*-contractions where θ functions satisfies (θ_1), (θ_2^*) and (θ_4). Than class will be denoted with $\tilde{\Theta}$ as in [16].
In [11], the authors added one more condition to (θ_1)-(θ_3):

$$(\theta_5)\ \theta(x+y) \leq \theta(x)\theta(y).$$

Denote with Θ^+ family of all functions $\theta : (0,\infty) \to (1,\infty)$ fulfilling $(\theta_1) - (\theta_3)$ and (θ_5). In [10] it was shown that the θ^+-contraction is a Banach contraction on a equivalent metric

space, hence this type of θ-contraction will not be analyzed in the sequel.
The main idea of this article is removing unnecessary assumptions in the definition of the class of function Θ, but retaining the existence and uniqueness of the fixed point of θ-contraction. In this way, we intend to unify the results of [4, 11, 14, 13] and many others in the setting of a complete metric space.

3. Main results

We start this section by introducing the refined class of functions denoted with Θ and a contractive condition including this type of functions.
Let Θ_b be a set of functions $\theta : (0,\infty) \mapsto (1,\infty)$ such that θ is nondecreasing. Thus, the class of Θ_b is subset of all previous classes, e.g. Θ, Θ^+, Θ^*, Θ' and so on. Note that for none of the mentioned types of θ-functions is defined at zero. This can be easily repaired by adding $\theta(0) = 1$. In that way we will not influence the contractive condition and, if needed, respect $(\theta_1) - (\theta_4)$.

Example 3.2. *Define a mapping $\theta_1 : (0,\infty) \mapsto (1,\infty)$ by*

$$\theta_1(x) = \begin{cases} e^{e^{-\frac{1}{x}}}, & x \in (0,1) \\ e^{e^{-\frac{1}{2x}}}, & x \in [1,\infty) \end{cases}.$$

Evidently, the function θ is non-decreasing, so $\theta_1 \in \Theta_b$. The function is not continuous, so $\theta_1 \notin \Theta' \cup \Theta^*$. Moreover,

$$\lim_{x\to 0+} \frac{\theta_1(x) - 1}{x^k} = 0,$$

for arbitrary $k \in (0,1)$. Consequently, $\theta_1 \notin \Theta$. It is notable that θ_1 is not subadditive on $(0,1)$.

Example 3.3. *Define a mapping $\theta_2 : (0,\infty) \mapsto (1,\infty)$ by*

$$\theta_2(x) = \begin{cases} e^{e^{-3+x}}, & x \in (0,1) \\ e^{e^{-\frac{1}{x}}}, & x \in [1,\infty) \end{cases}.$$

In addition to the remarks made in previous example, $\theta_2 \notin \Theta \cup \Theta' \cup \Theta^*$, here we have that (θ_2) ((θ_2^*)) do not hold on $(0,\infty)$. Hence, θ_2 fulfills only (θ_1) and $\theta_2 \in \Theta_b$.

Definition 3.6. *Let (X,d) be a metric space. A mapping $T : X \mapsto X$ is a basic-θ-contraction if there exists a function $\theta \in \Theta_b$ and $k \in (0,1)$ such that for any $x, y \in X$ the following implication holds*

$$Tx \neq Ty \Longrightarrow \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k. \tag{3.2}$$

Theorem 3.4. *Let (X,d) be a complete metric space and $T : X \mapsto X$ be a basic-θ-contraction. Then T has a unique fixed point in X and the sequence $(T^n x_0)$ converges to the fixed point for any $x_0 \in X$, i.e., T is a Picard operator (see* [19]*).*

Proof. Assume that (X,d) is a complete metric space and $T : X \mapsto X$ a mapping such that there exists a nondecreasing function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that (3.2) holds. Let $x_0 \in X$ be an arbitrary point and define the sequence of successive approximations $(x_n) \subseteq X$ such that $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T. Otherwise, we will assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$.
Then, for any $n \in \mathbb{N}$, by the principle of mathematical induction, we may easily obtain

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \theta\left(d(x_{n-1}, x_n)\right)^k \\ &\vdots \\ &\leq \theta\left(d(x_0, x_1)\right)^{k^n}. \end{aligned} \tag{3.3}$$

As $n \to \infty$, $k^n \to 0$ implies that

$$1 \leq \lim_{n\to\infty} \theta(d(x_n, x_{n+1})) \leq \theta\left(d(x_0, x_1)\right)^{k^n} = 1.$$

Since θ is nondecreasing function, we can not claim directly that $\lim\limits_{n\to\infty} d(x_n, x_{n+1})$ exists and is equal to 0. In order to achieve that, observe

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\leq \theta(d(x_{n-1}, x_n))^k \\ &< \theta(d(x_{n-1}, x_n)),\end{aligned}$$

implying $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, for any $n \in \mathbb{N}$. Consequently, the sequence $(d(x_{n-1}, x_n))$ is a monotone decreasing sequence, so its limit when $n \to \infty$ exists and it is equal to the infimum of the sequence (x_n). If

$$a = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = \lim_{n\to\infty} d(x_{n-1}, x_n),$$

then $a \in (0, 1)$ leads to

$$\begin{aligned}\theta(a) &\leq \theta(d(x_n, x_{n+1})) \\ &\leq \theta\left(d(x_{n-1}, x_n)\right)^k \\ &\;\vdots \\ &\leq \theta\left(d(x_0, x_1)\right)^{k^n}.\end{aligned}$$

Letting $n \to \infty$, we get $\theta(a) = 1$ which is impossible due to the codomain of θ, so a must be equal to zero. Thus, $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Assume contrary of what we intend to prove, that (x_n) is not a Cauchy sequence meaning that there exists $\varepsilon > 0$ and strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$ such that $n_i < m_i$ for any $i \in \mathbb{N}$ and

$$d(x_{n_i}, x_{m_i}) \geq \varepsilon \text{ and } d(x_{n_i}, x_{m_i-1}) < \varepsilon,$$

where n_i is minimal such that those subsequences exist, meaning

$$n_i = \min\{j \geq i \mid d(x_j, x_m) \geq \varepsilon \wedge m > j\},$$

and

$$m_i = \min\{j > n_i \mid d(x_{n_i}, x_j) \geq \varepsilon\}.$$

Having in mind the definition of Cauchy sequence and the fact that monotone function has at most countable many discontinuities, ε may be chosen such that θ is continuous at ε. Further,

$$\begin{aligned}\varepsilon &\leq d(x_{n_i}, x_{m_i}) \\ &\leq d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{m_i-1}) + d(x_{m_i-1}, x_{m_i}),\end{aligned}$$

leads to $\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = \varepsilon$. Moreover,

$$\begin{aligned}d(x_{n_i-1}, x_{m_i-1}) &\geq d(x_{n_i}, x_{m_i}) - d(x_{n_i-1}, x_{n_i}) - d(x_{m_i}, x_{m_i-1}) \\ d(x_{n_i-1}, x_{m_i-1}) &\leq d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_i-1}),\end{aligned}$$

implies $\lim_{i\to\infty} d(x_{n_i-1}, x_{m_i-1}) = \varepsilon$.

Therefore,

$$\begin{aligned}\theta(\varepsilon) &\leq \theta\left(d(x_{n_i}, x_{m_i})\right) \\ &\leq \left(\theta\left(d(x_{n_i-1}, x_{m_i-1})\right)\right)^k\end{aligned}$$

as $i \to \infty$, we get the impossible inequality $\theta(\varepsilon) \le (\theta(\varepsilon))^k$. Consequently, our assumption is false, so (x_n) is a Cauchy sequence, thus convergent with the limit $x^* \in X$. Since $Tx^* \ne x_n$ starting from some n_1, then

$$\theta(d(Tx^*, Tx_n)) \le (\theta(d(x^*, x_n)))^k . \tag{3.4}$$

However, $(d(x^*, x_n))$ is a zero sequence and $\lim_{n\to\infty} \theta(d(x^*, x_n)) = \inf_{t>0} \theta(t)$ which was proven to be 1 by estimating $\lim_{n\to\infty} \theta(d(x_n, x_{n+1}))$. As a result, we get $\lim_{n\to\infty} \theta(d(Tx^*, Tx_n)) = 1$, suggesting $\lim_{n\to\infty} Tx_n = Tx^*$, i.e., $Tx^* = x^*$. Uniqueness easily follows, since if $Ty = y$ and $y \ne x^*$, then

$$\begin{aligned}\theta(d(x^*, y)) &= \theta(d(Tx^*, Ty) \\ &\le (\theta(d(x^*, y)))^k ,\end{aligned}$$

leads to the contradiction, so x^* is a unique fixed point of the mapping T.
If $y_0 \in X$ is arbitrary and $y_n = T^n y_0$ for any $n \in \mathbb{N}$, then we prove in a same manner that (y_n) is a Cauchy sequence, thus convergent and that its limit is a fixed point. Since fixed point of a mapping T is a unique, we conclude that any sequence of successive approximations converges to the fixed point.

□

This elements of this class of contractive mapping will be named basic θ-contractions. Since the class of nondecreasing functions $\theta : (0,\infty) \mapsto (1,\infty)$ is a strict superset of the classes Θ, Θ' and Θ^*, as a corollary of Theorem 3.4, we get Corollary 2.1 of [14].

Corollary 3.1. *Let (X, d) be a complete metric space and $T : X \mapsto X$ be a given mapping. If there exist $\theta \in \Theta'$ and $k \in (0, 1)$ such that*

$$Tx \ne Ty \implies \theta(d(Tx, Ty)) \le (\theta(d(x, y)))^k .$$

for all $x, y \in X$, then T has a unique fixed point.

The following corollary is exactly the main result of [4].

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X \mapsto X$ be a given mapping. If there exist $\theta \in \Theta^*$ and $k \in (0, 1)$ such that*

$$Tx \ne Ty \implies \theta(d(Tx, Ty)) \le (\theta(d(x, y)))^k ,$$

for all $x, y \in X$, then T has a unique fixed point in X.

In order to obtain the results of [13] in the setting of a complete metric space, we will analyze the generalization of the condition (3.2).

Theorem 3.5. *If (X, d) is a complete metric space and $T : X \mapsto X$ a mapping such that exists a nondecreasing function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ the following implication holds*

$$Tx \ne Ty \implies \theta(d(Tx, Ty)) \le (\theta(M(x, y)))^k , \tag{3.5}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

and jumps on the left at each discontinuity x of the function θ are less than $\theta(x) - (\theta(x))^k$, i.e.,

$$\theta(x) - \lim_{y\to x-} \theta(y) > (\theta(x))^k ,$$

then T has a unique fixed point in X.

Proof. If $x_0 \in X$ is arbitrary, define the sequence $(x_n) \subseteq X$ such that $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T. Otherwise, assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, so we will estimate the Cauchy property of the sequence (x_n) by using (3.5). Obviously, $M(x_{n-1}, x_n) \in \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$ for any $n \in \mathbb{N}$. The case $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ for some $n \in \mathbb{N}$ leads to the contradiction, since (3.5) implies

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_n, x_{n+1})))^k ,$$

meaning that $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$ and more the inequality

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_0, x_1)))^{k^n} ,$$

holds for any n, which is easily obtainable by the principle of mathematical induction. Letting $n \to \infty$, we get

$$1 \leq \lim_{n\to\infty} \theta(d(x_n, x_{n+1})) \leq \lim_{n\to\infty} (\theta(d(x_0, x_1)))^{k^n} = 1.$$

Moreover,

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\leq \theta(d(x_{n-1}, x_n))^k \\ &< \theta(d(x_{n-1}, x_n)),\end{aligned}$$

implies $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for any $n \in \mathbb{N}$. As the sequence $(d(x_{n-1}, x_n))$ is a monotone decreasing sequence, its limit exists and $a = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = \lim_{n\to\infty} d(x_{n-1}, x_n)$. Assuming that $a > 0$, we get

$$\theta(a) \leq \lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1.$$

It cannot be the case, meaning that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. In order to prove that (x_n) is a Cauchy sequence, we will assume contrary. Recall also that the function θ is monotone, so its set of discontinuities is countable. Hence, there exist $\varepsilon > 0$ such that is not a discontinuity of θ and strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$ such that $n_i < m_i$ for any $i \in \mathbb{N}$ and

$$d(x_{n_i}, x_{m_i}) \geq \varepsilon \text{ and } d(x_{n_i}, x_{m_i-1}) < \varepsilon,$$

where

$$n_i = \min\{j \geq i \mid d(x_j, x_m) \geq \varepsilon \wedge m > j\},$$

and

$$m_i = \min\{j > n_i \mid d(x_{n_i}, x_j) \geq \varepsilon\}.$$

Accordingly,

$$\begin{aligned}\varepsilon &\leq d(x_{n_i}, x_{m_i}) \\ &\leq d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{m_i-1}) + d(x_{m_i-1}, x_{m_i}),\end{aligned}$$

leads to $\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = \varepsilon$.
Similarly as in the proof of Theorem 3.4, we get $\lim_{i\to\infty} d(x_{n_i-1}, x_{m_i-1}) = \varepsilon$ and

$$\begin{aligned}\theta(\varepsilon) &\leq \theta(d(x_{n_i}, x_{m_i})) \\ &\leq (\theta(M(x_{n_i-1}, x_{m_i-1})))^k\end{aligned}$$

for $M(x_{n_i-1}, x_{m_i-1}) = \max\{d(x_{n_i-1}, x_{m_i-1}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{m_i})\}$. By previous estimations, both $\lim_{i\to\infty} \theta(d(x_{m_i-1}, x_{m_i}))$ and $\lim_{i\to\infty} \theta(d(x_{n_i-1}, x_{n_i}))$ are equal to 1, if there are infinitely many $i \in \mathbb{N}$ such that $M(x_{n_i-1}, x_{m_i-1}) \in \{d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{m_i})\}$, it follows that $\theta(\varepsilon) = 1$ which is impossible. It remains to assume that $M(x_{n_i-1}, x_{m_i-1}) = d(x_{n_i-1}, x_{m_i-1})$ starting from some $i_0 \in \mathbb{N}$. Hence,

$$\begin{aligned} \theta(\varepsilon) &\leq \lim_{i\to\infty} \theta\left(d(x_{n_i}, x_{m_i})\right) \\ &\leq \lim_{i\to\infty} \left(\theta\left(M(x_{n_i-1}, x_{m_i-1})\right)\right)^k \\ &= \lim_{i\to\infty} \left(\theta\left(d(x_{n_i-1}, x_{m_i-1})\right)\right)^k \\ &= \left(\theta(\varepsilon)\right)^k, \end{aligned} \tag{3.6}$$

again leading to the contradiction. In consequence, there exists some $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Then,

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, x_{n+1}) + d(x_{n+1}, x^*) \\ &\leq M(x^*, x_n) + d(x_{n+1}, x^*). \end{aligned}$$

where

$$M(x^*, x_n) = \max\{d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1})\}.$$

(i) If $M(x^*, x_n) = d(x^*, x_n)$ for infinitely many $n \in \mathbb{N}$, then

$$d(Tx^*, x^*) \leq d(x^*, x_n) + d(x_{n+1}, x^*).$$

Letting $n \to \infty$ leads to the conclusion that x^* is a fixed point of T.
(ii) In the case that $M(x^*, x_n) = d(x_n, x_{n+1})$ for infinitely many $n \in \mathbb{N}$, we have

$$d(Tx^*, x^*) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x^*),$$

that implies $Tx^* = x^*$.
(iii) Remaining, assume that $M(x^*, x_n) = d(x^*, Tx^*)$ starting from some $n_0 \in \mathbb{N}$. As $\lim_{n\to\infty} d(Tx^*, x_n) = d(Tx^*, x^*)$, from the estimation of

$$\theta(d(Tx^*, x_{n+1})) \leq \left(\theta(d(x^*, Tx^*))\right)^k, \tag{3.7}$$

we obtain the contradiction since it must be

$$\lim_{n\to\infty} \theta(d(Tx^*, x_n)) > \left(\theta(d(x^*, Tx^*))\right)^k.$$

From all of the above, x^* is a fixed point of the mapping T. If $Ty = y$ and $y \neq x^*$, then

$$\begin{aligned} \theta(d(x^*, y)) &= \theta(d(Tx^*, Ty) \\ &\leq \left(M(x^*, y)\right)\right)^k, \end{aligned}$$

where $M(x^*, y) = \max\{d(x^*, y), 0\}$ and x^* is a unique fixed point of the mapping T. □

Remark 3.2. *Notice that the cases (i) and (ii) could be excluded if we have assumed that $Tx^* \neq x^*$ due to the fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x^*) = 0$.*

Corollary 3.3. *If (X, d) is a complete metric space and $T : X \mapsto X$ a mapping such that exists a nondecreasing continuous on the left function $\theta : (0, \infty) \mapsto (1, \infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ the condition (3.5) holds, then T has a unique fixed point in X.*

The following corollary is still a generalization of Corollary 3.6 of [13] since (θ_3) is not requested by any means.

Corollary 3.4. *If (X, d) is a complete metric space and $T : X \mapsto X$ a mapping such that exists a nondecreasing continuous function $\theta : (0, \infty) \mapsto (1, \infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ the condition (3.5) holds, then T has a unique fixed point in X.*

We will present an example of a complete metric space and a basic θ-contraction which is not a Banach contraction as in [14].

Example 3.4. *Let $X = \{x_n = \frac{n(n+1)}{2} \mid n \in \mathbb{N}\}$ be a set equipped with a metric $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a complete metric space. We will define a mapping $T : X \mapsto X$ such that:*

$$Tx_n = \begin{cases} x_1, & \text{if } n = 1 \\ x_{n-1}, & \text{otherwise} \end{cases}.$$

Then a mapping T is a basic θ-contraction on (X, d) for $\theta(t) = e^{\sqrt{te^t}}$ for $t > 0$ and $k = e^{-1}$.

The concept of F-contraction was broadly investigated in the last decade. It originated in the paper of Wardowski [23].

Definition 3.7. [23] *Let $F : (0, \infty) \to \mathbb{R}$ be a function fulfilling the following conditions:*

(F_1) *F is increasing, meaning $0 < x < y \implies F(x) < F(y)$;*
(F_2) *For any sequence $(x_n) \subseteq (0, \infty)$,*

$$\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) *There exists $k \in (0, 1)$, such that $\lim_{x\to 0^+} x^k F(x) = 0$.*

Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying $(F_1) - (F_3)$. F-contraction is defined in a following way:

Definition 3.8. *Let (X, d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \le F(d(x, y)),$$

then a mapping T is called an F-contraction.

In [23] was proven that an F-contraction on a complete metric space has a unique fixed point and that the sequence $(T^n x_0)$ converges to the fixed point of a mapping T for any initial point $x_0 \in X$.
It is quite evident that any F-contraction is a θ-contraction for a mapping θ defined by $\theta(t) = e^{e^{F(t)}}$, $t > 0$. Thus, we may formulate more general result than in the case of [23] where all assumptions $(F_1) - (F_3)$ are replaced with a monotone increasing property of F.

Corollary 3.5. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a mapping such that the following implication holds:*

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \le F(d(x, y)),$$

for some $\tau > 0$ and $F : (0, \infty) \mapsto \mathbb{R}$ nondecreasing function. Then a mapping T has a unique fixed point $x^* \in X$ and the iterative sequence $(T^n x_0)$ converges to the fixed point x^* for any initial point $x_0 \in X$.

Proof. Define a function $\theta : (0, \infty) \mapsto (1, +\infty)$ such that $\theta(t) = e^{e^{F(t)}}$ for any $t > 0$. It is a well-defined function since $F(t) \in \mathbb{R} \implies e^{F(t)} > 0 \implies e^{e^{F(t)}} > 1$. Also, if F is nondecreasing, θ is nondecreasing and if $Tx \neq Ty$:

$$\begin{aligned}\theta(d(Tx,Ty)) &= e^{e^{F(d(Tx,Ty))}} \\ &\leq e^{e^{F(d(x,y))-\tau}} \\ &\leq e^{e^{F(d(x,y))}e^{-\tau}} \\ &= \left(e^{e^{F(d(x,y))}}\right)^{e^{-\tau}} \\ &= (\theta(d(x,y)))^{e^{-\tau}}.\end{aligned}$$

Hence, T is a basic θ-contraction for $k = e^{-\tau}$ and the conclusion follows from Theorem 3.4. □

4. Results in generalized metric spaces in a sense of Branciari

As mentioned in the Introduction and Preliminaries, first results on the topic of θ-contraction and many following results were in the setting of a generalized metric spaces in a sense of Branciari. The authors' choice in this article was to present the result in the setting of a complete metric space, but in order to give a complete picture concerning basic θ-contraction and that indeed this approach presents an improvement of results in [4, 14, 13] among others, we will present two main theorems in the setting of generalized metric space in a sense of Branicari. It is important to mention that the differences in the proofs are almost not existing with few additional comments.
In order to do so, we recall some basic properties of the generalized metric space in a sense of Branicari regarding Cauchy and convergent sequence.

Definition 4.9. *If (X,d) is a generalized metric space in a sense of Branicari and $(x_n) \subseteq X$ a sequence, then the sequence (x_n) is a Cauchy sequence if for any $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $d(x_n, x_m) < \varepsilon$.*

Definition 4.10. *If (X,d) is a generalized metric space in a sense of Branicari and $(x_n) \subseteq X$ a sequence, then (x_n) is a convergent sequence with a limit $x^* \in X$ if for any $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $d(x_n, x^*) < \varepsilon$.*

Obviously, same as in the case of metric space, we say that the generalized metric space in a sense of Branciari is complete if any Cauchy sequence is convergent.
We will state the result induced by Theorem 3.4 for the generalized metric space in the sense of Branciari.

Theorem 4.6. *If (X,d) is a complete generalized metric space in a sense of Branciari and $T: X \mapsto X$ a mapping such that exists a nondecreasing function $\theta : (0,\infty) \mapsto (1,\infty)$ and $k \in (0,1)$ such that (3.2) holds for any $x, y \in X$, then T has a unique fixed point in X and the sequence $(T^n x_0)$ converges to the fixed point for any $x_0 \in X$.*

Proof. Assume that (X,d) is a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ a mapping fulfilling all assumptions of the theorem. Let $x_0 \in X$ be arbitrary and define the sequence of successive approximations $(x_n) \subseteq X$ such that $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. We may assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ as it has been considered in the proof of Theorem 3.4.
Analogously, we get (3.3) and, as $n \to \infty$,

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = \lim_{n\to\infty} \theta\left(d(x_0, x_1)\right)^{k^n} = 1.$$

Following the same estimations as in the proof of Theorem 3.4, we get $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$ and assume that (x_n) is not a Cauchy sequence so there exist $\varepsilon > 0$ out of the set of discontinuities of function θ and strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$ such that $n_i < m_i$ for any $i \in \mathbb{N}$ and

$$d(x_{n_i}, x_{m_i}) \geq \varepsilon \text{ and } d(x_{n_i}, x_{m_i-1}) < \varepsilon,$$

constructed as already established. Further,

$$\begin{aligned}\varepsilon &\leq d(x_{n_i}, x_{m_i}) \\ &\leq d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{m_i-1}) + d(x_{m_i-1}, x_{m_i}),\end{aligned}$$

leads to $\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = \varepsilon$, and similarly, again by applying the quadrilateral inequality we get $\lim_{i\to\infty} d(x_{n_i-1}, x_{m_i-1}) = \varepsilon$. Therefore, we have a contradiction as $i \to \infty$, i.e., (x_n) is a Cauchy sequence. Let $x^* \in X$ be the limit of the sequence (x_n) and $n_1 \in \mathbb{N}$ such that $Tx^* \neq x_n$ for $n \geq n_1$, then (3.4) with some analysis that does not involve the application of triangle/quadrilateral inequality imply $Tx^* = x^*$. Uniqueness easily follows as in the proof of Theorem 3.4. □

As a corollary we get the Theorem 2.1 of [14] with a remark that the Theorem 4.6 presents an improvement of the main results of [14].

Corollary 4.6. *Let (X, d) be a complete generalized metric space in a sense of Branicari and $T : X \mapsto X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$Tx \neq Ty \implies \theta(d(Tx, Ty)) \leq (\theta(d(x, y)))^k .$$

Then T has a unique fixed point.

Corollary 4.7. *Let (X, d) be a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ be a given mapping. Suppose that there exist $\theta \in \Theta^*$ and $k \in (0, 1)$ such that*

$$Tx \neq Ty \implies \theta(d(Tx, Ty)) \leq (\theta(d(x, y)))^k .$$

Then T has a unique fixed point.

Some generalizations of the above contractive condition will be the discussed on a set equipped with the Branciari metric.

Theorem 4.7. *If (X, d) is a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ a mapping for which there exists a nondecreasing function $\theta : (0, \infty) \mapsto (1, \infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ (3.5) holds, where*

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

and jumps on the left at each discontinuity x of the function θ are less than $\theta(x) - (\theta(x))^k$, i.e.,

$$\theta(x) - \lim_{y\to x-} \theta(y) > (\theta(x))^k ,$$

then T has a unique fixed point in X.

Proof. If $x_0 \in X$ is arbitrary, define the sequence $(x_n) \subseteq X$ such that $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T. Otherwise, assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, so we will estimate the Cauchy property of the sequence (x_n) by using (3.5). Obviously, $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$ due to same considerations as in the proof of Theorem 3.5. As in the proof of Theorem 3.4 and Theorem 3.5, we come to the conclusion $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Once again we will assume that there exist $\varepsilon > 0$ such that it is not a discontinuity of θ and previously described strictly increasing sequences $(n_i), (m_i) \subseteq \mathbb{N}$ such that $n_i < m_i$ for any $i \in \mathbb{N}$ and

$$d(x_{n_i}, x_{m_i}) \geq \varepsilon \text{ and } d(x_{n_i}, x_{m_i-1}) < \varepsilon,$$

Instead of applying the triangle inequality two times as in the proof of Theorem 3.5, we will get the same conclusion $\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = \varepsilon$ through the one quadrilateral inequality

$$\begin{aligned}\varepsilon &\le d(x_{n_i}, x_{m_i})\\ &\le d(x_{n_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{m_i-1}) + d(x_{m_i-1}, x_{m_i}).\end{aligned}$$

Also we have

$$\lim_{i\to\infty} d(x_{n_i-1}, x_{m_i-1}) = \lim_{i\to\infty} \theta(d(x_{m_i-1}, x_{n_i})) = \lim_{i\to\infty} \theta(d(x_{n_i-1}, x_{m_i})) = \varepsilon$$

and

$$\begin{aligned}\theta(\varepsilon) &\le \theta\left(d(x_{n_i}, x_{m_i})\right)\\ &\le \left(\theta\left(M(x_{n_i-1}, x_{m_i-1})\right)\right)^k\end{aligned}$$

for $M(x_{n_i-1}, x_{m_i-1}) = \max\{d(x_{n_i-1}, x_{m_i-1}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{m_i})\}$. In the case that there are infinitely many $i \in \mathbb{N}$ with $M(x_{n_i-1}, x_{m_i-1})$ being equal to $d(x_{n_i-1}, x_{n_i})$ or $d(x_{m_i-1}, x_{m_i})$, it follows that $\theta(\varepsilon) = 1$ which is impossible. If it is $d(x_{n_i-1}, x_{m_i-1})$ starting from some $i_0 \in \mathbb{N}$, then the same inequalities as in (3.6) hold, so the sequence is Cauchy in a complete generalized metric space in a sense of Branciari and $\lim_{n\to\infty} x_n = x^* \in X$. The main difference between this proof and the proof of Theorem 3.5 lyes in the following lines

$$\begin{aligned}d(Tx^*, x^*) &\le d(Tx^*, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x^*)\\ &\le M(x^*, x_n) + d(x_{n+1}, x_n) + d(x_n, x^*).\end{aligned}$$

where

$$M(x^*, x_n) = \max\{d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1})\}.$$

(i) If $M(x^*, x_n) = d(x^*, x_n)$ for infinitely many $n \in \mathbb{N}$, then

$$d(Tx^*, x^*) \le d(x^*, x_n) + d(x_{n+1}, x_n) + d(x_n, x^*).$$

Letting $n \to \infty$ leads to the conclusion that x^* is a fixed point of T.
(ii) In the case that $M(x^*, x_n) = d(x_n, x_{n+1})$ for infinitely many $n \in \mathbb{N}$, we have

$$d(Tx^*, x^*) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x^*),$$

that implies $Tx^* = x^*$.
(iii) Assume that $M(x^*, x_n) = d(x^*, Tx^*)$ starting from some $n_0 \in \mathbb{N}$. As $\lim_{n\to\infty} d(Tx^*, x_n) = d(Tx^*, x^*)$, (3.7) and the assumption concerning jumps on the left imply the contradiction. Hence, x^* is a fixed point of the mapping T and uniqueness of the fixed point is deduced in a same manner as in the previous proofs. □

Corollary 4.8. *If (X, d) is a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ a mapping such that exists a nondecreasing continuous on the left function $\theta : (0, \infty) \mapsto (1, \infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ the condition (3.5) holds, then T has a unique fixed point in X.*

The main result of [13] stated in Theorem 2.1 is indeed directly implied by the following

Corollary 4.9. *If (X, d) is a complete generalized metric space in a sense of Branciari and $T : X \mapsto X$ a mapping such that exists a nondecreasing continuous function $\theta : (0, \infty) \mapsto (1, \infty)$ and $k \in (0, 1)$ such that for any $x, y \in X$ the condition (3.5) holds, then T has a unique fixed point in X.*

5. Applications

The modifications and the generalizations of θ-contractions have found numerous applications as seen in [1, 2, 5, 12, 15, 16] among others. Consider the nonlinear Hammerstein integral equation

$$x(t) = h(t) + \int_0^t K(t,s) f(s, x(s)) ds, \; t \in [0,1] \tag{5.8}$$

where $x : [0,1] \mapsto \mathbb{R}$ is continuous on $[0,1]$, $h : [0,1] \mapsto \mathbb{R}$, $K : [0,1] \times [0,1] \mapsto \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$.
Observe $X = C[0,1]$ is the set of all continuous real-valued functions with the domain $[0,1]$ equipped with the usual metric $d(x,y) = \max_{0 \le s \le 1} |x(s) - y(s)|$ and the metric

$$d_\lambda(x,y) = \max_{0 \le s \le 1} |x(s) - y(s)| e^{-\lambda s}, \tag{5.9}$$

for any $x, y \in X$ and $\lambda \ge 1$, then (X, d) and (X, d_λ) are complete metric spaces for any $\lambda \ge 1$. Define the mapping $T : X \mapsto X$ such that

$$Tx(t) = h(t) + \int_0^t K(t,s) f(s, x(s)) ds, \tag{5.10}$$

for any $t \in [0,1]$ and $x \in X$. Clearly, a fixed point of T is a solution of the integral equation (5.8) and vice-versa.

Corollary 5.10. *Assume that the functions $h \in X$, $K : [0,1] \times [0,1] \mapsto \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ are both integrable and $T : X \mapsto X$ is defined by (5.10). If $\max\limits_{0 \le s,t \le 1} |K(t,s)| = \alpha > 0$ and for some $\lambda \in [1, \infty)$*

$$Tx \neq Ty \implies |f(s, x(s)) - f(s, y(s))| \le \frac{1}{\alpha} e^{-\lambda s} |x(s) - y(s)|, \; s \in [0,1]$$

holds for any $x, y \in X$, then T has a unique fixed point in X.

Proof. A mapping T is a basic θ-contraction on a complete metric space (X, d_λ) for $\theta(t) = e^t$ and $k = \frac{1}{\lambda} e^{-1}$. □

Corollary 5.11. *Assume that the functions $h \in X$, $K : [0,1] \times [0,1] \mapsto \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ are both integrable and $T : X \mapsto X$ is defined by (5.10). If $\max\limits_{0 \le s,t \le 1} |K(t,s)| = \alpha > 0$ and for some $\lambda \in [1, \infty)$*

$$Tx \neq Ty \implies |f(s, x(s)) - f(s, y(s))| \le \frac{1}{\alpha} e^{-\lambda s} M_{x,y}(s)$$

holds for any $x, y \in X$, where

$$M_{x,y}(s) = \max \{ |x(s) - y(s)|, |x(s) - Tx(s)|, |y(s) - Ty(s)| \},$$

then T has a unique fixed point in X.

In the same metric setting, we will additionally observe the Volterra type integral equation:

$$Tx(t) = \int_0^t K(t, s, h(x(s))) \, \mathrm{ds} + g(t), \; t \in [0,1], \tag{5.11}$$

where $x, g : [0,1] \mapsto \mathbb{R}$, $K : [0,1] \times [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ and $h : \mathbb{R} \mapsto \mathbb{R}$ continuous functions. Then T is a well defined operator acting on $X = C[0,1]$ which will be equipped with the metric d_λ defined through (5.9) for any positive λ in a general case. The following result holds:

Theorem 5.8. *If:*

(1) $K : [0,1] \times [0,1] \times \mathbb{R} \mapsto \mathbb{R}$, $h : \mathbb{R} \mapsto \mathbb{R}$ *and* $g : [0,1] \mapsto \mathbb{R}$ *are continuous functions;*

(2) there exists $\alpha > 0$ *such that*

$$|K(t,s,h(x(s))) - K(t,s,h(y(s)))|^2 \leq |x(s) - y(s)|,$$

for all continuous real valued functions $x, y \in C[0,1]$ *and all* $s, t \in [0,1]$,

then the equation (5.11) has a unique solution in $C[0,1]$.

Proof. Observe a complete metric space (X,d) for $X = C[0,1]$ and suppose that T is defined by (5.11) If $x, y \in X$ are arbitrarily chosen, then:

$$\begin{aligned}
(d(Tx,Ty))^2 &= \left(\max_{0\leq t\leq 1} |Tx(t) - Ty(t)|\right)^2 \\
&= \max_{0\leq t\leq 1} |Tx(t) - Ty(t)|^2 \\
&= \max_{0\leq t\leq 1} \left| \left(\int_0^t K(t,s,h(x(s)))\,\mathrm{ds} + g(t)\right) - \left(\int_0^t K(t,s,h(y(s)))\,\mathrm{ds} + g(t)\right)\right|^2 \\
&\leq \max_{0\leq t\leq 1} \int_0^t |K(t,s,h(x(s))) - K(t,s,h(y(s)))|^2 \,\mathrm{ds} \\
&\leq \max_{0\leq t\leq 1} \int_0^t |x(s) - y(s)| \,\mathrm{ds} \\
&\leq td(x,y) \\
&\leq d(x,y).
\end{aligned}$$

Taking $\theta(t) = t^2$ for $t > 0$ and $k = \frac{1}{2}$ we get that T is a basic θ-contraction and the equation (5.11) has a unique solution in $C[0,1]$. □

Corollary 5.12. *If:*

(1) $K : [0,1] \times [0,1] \times \mathbb{R} \mapsto \mathbb{R}$, $h : \mathbb{R} \mapsto \mathbb{R}$ *and* $g : [0,1] \mapsto \mathbb{R}$ *are continuous functions;*

(2) there exists $\alpha > 0$ *such that*

$$|K(t,s,h(x(s))) - K(t,s,h(y(s)))| \leq \alpha |x(s) - y(s)|,$$

for all continuous real valued functions $x, y \in C[0,1]$ *and all* $s, t \in [0,1]$

then the equation (5.11) has a unique solution in $C[0,1]$.

Inspired by [8] we will discuss on another idea for possible applications of basic θ-contraction. In [8] the new approach to the fixed point theorems in the set of integrable functions was investigated. Observe that Corollary 5.10 may be applied on solving equation (3.1) of [8] in the set of continuous functions and requested conditions are less restrictive than in Proposition 3.1 of [8]. Additionally, we will discuss on another discussed problem of fractional equation:

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) u(s)\,\mathrm{ds}, \tag{5.12}$$

where $0 < \alpha \leq 1$, Γ is a usual Gamma function and in our case $f, u \in X = C[0,1]$. We obtain the following result:

Theorem 5.9. *If* $f \in X$ *and* $0 < \alpha \leq 1$, *then the equation (5.12) has a unique solution in* X.

Proof. Define $T : X \mapsto X$ as

$$Tu(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) u(s)\,\mathrm{ds},\ t \in [0,1],$$

for any $u \in X$. Additionally, we will observe X equipped with d_1 metric defined in (5.9) for $\lambda = 1$.
In that case, for $u, v \in X$, we have:

$$\begin{aligned}
d_1(Tu,Tv) &= \max_{0\le t\le 1}\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,(u(s)-v(s))\,\mathrm{ds}\right| e^{-t}\\
&\overset{x=\frac{s}{t}}{=} \max_{0\le t\le 1}\left|\frac{1}{\Gamma(\alpha)}\int_0^1 (t-xt)^{\alpha-1} f(xt)\,(u(xt)-v(xt))\,t\,\mathrm{dx}\right| e^{-t}\\
&= \max_{0\le t\le 1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-x)^{\alpha-1}|f(xt)|\,|u(xt)-v(xt)|\,e^{-tx}e^{tx}t^{\alpha}\,\mathrm{dx}\,e^{-t}\\
&\le d_1(u,v)\max_{0\le t\le 1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-x)^{\alpha-1}e^{-(1-x)}|f(xt)|e^{(1-t)(1-x)}t^{\alpha}\,\mathrm{dx}\\
&\le d_1(u,v)\max_{0\le t\le 1}\frac{t^{\alpha}e^{(1-t)}f_{max}}{\Gamma(\alpha)}\int_0^1 (1-x)^{\alpha-1}e^{-(1-x)}\,\mathrm{dx}\\
&\le a^a e^{(1-a)} f_{\max} d_1(u,v),
\end{aligned}$$

where $f_{\max}$ is a maximum of a function $|f|$ on $[0,1]$.
Consequently, if $b = a^a e^{(1-a)} f_{\max}$ and $b < 1$ then T is a contraction on (X, d_1), but if $b \ge 1$, then $e^{d_1(Tu,Tv)} \le \left(e^{d_1(u,v)}\right)^b$, and for $\theta(t) = e^{t^{\frac{1}{2b}}}$, we have that T is a basic θ-contraction. Hence, T has a unique fixed point in X. □

6. Conclusions

Obtained results are an improvement of all previously derived results concerning θ-contractions involving contractive conditions (2.1) and (3.5) on a complete metric space. As presented, the same proof techniques are applicable in the setting of the complete generalized metric space in the sense of Branciari and in that way we unify all results of [4, 13, 14]. It is our belief that the same approach may be applied in many other results concerning the concept of θ-contraction in the setting of b-metric space, extended b-metric space, cone metric space, partial metric space, uniform spaces, etc. But, up to know, that remains an open problem. Applications of presented theoretical results are numerous and would be of a great importance to make an appropriate comparison of quality of conclusions obtained in that way.

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[1] Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš,Višegradska 33, Niš, 18000, Serbia,
Email address: marija.cvetkovic@pmf.edu.rs

[2] Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan

[3] Department of Mathematics, Çankaya University,06796, Ankara, Turkey
Email address: erdalKarapinar@yahoo.com

[4] Babeş-Bolyai University, Cluj-Napoca, Romania

[5] Academy of Romanian Scientists, Bucharest, Romania
Email address: adrian.petrusel@ubbcluj.ro