

Consequences of the product rule in Stieltjes differentiability

F. JAVIER FERNÁNDEZ^{1,2,*}, IGNACIO MÁRQUEZ ALBÉS^{3,†}, AND F. ADRIÁN F. TOJO^{1,2,‡}

ABSTRACT. This work revolves around the study of differentiability in the Stieltjes sense of a product of functions. A formula for the first order derivative has been obtained in the past, which is similar to the usual one with some extra terms in its expression. The aim of this paper is to take this behavior into account to study under which conditions we can guarantee the existence of higher order derivatives, while obtaining some other interesting results for the Stieltjes derivative along the way. We also investigate the regularity of the product of two functions.

1. INTRODUCTION

The notion of differentiating with respect to a function (Stieltjes differentiation) is a rather classical concept—see [3, 6, 17]—that has its roots at the genesis of calculus when, in an intuitive way, it was usual to think of a quantity varying continuously with respect to another (dependent or independent). More recently, a new important application of Stieltjes calculus was found: it serves as a powerful bridge between continuous and discrete calculus and, importantly, differential equations. In particular, it was shown that other methods of unifying discrete and differential calculus, such as time scales or equations with impulses, can be thought of as particular instances of Stieltjes calculus [11]. This theory has the advantage of building on the powerful results of measure theory but, at the same time, allowing for classical solutions and explicit computations. From there, several authors have revitalized the theory by providing a solid theoretical framework [5, 8] and different applications of Stieltjes differential equations such as models of silk worm populations [8], fishing models with open and closed seasons [7], culture and growth of cyanobacteria [9, 10], solvent solution and water evaporation models [5], among others.

In those previous works there was no definition of the Stieltjes derivative which made sense at every point of the domain. Instead, they provided a definition at almost every point for a certain measure, being thus comparable to the Radon-Nikodym derivative. Recently, in [4], the definition was extended in order to include all points of the domain, which also had an important repercussion in previously known results such as the product rule, which, in this more general context, reads

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t) (f_2)'_g(t) \Delta g(t^*),$$

where g is the function defining the Stieltjes derivative and t^* is a point that depends on t .

This expression has immediate consequences that make Stieltjes Calculus different from the usual setting. It is evident from the expression that the regularity of the functions f_1

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Corresponding author: F. Adrián F. Tojo; fernandoadrian.fernandez@usc.es

and f_2 in the expression is not enough to guarantee the same level of regularity to the product as, once we differentiate for a first time, the terms $f_1(t^*)$, $f_2(t^*)$ and $\Delta g(t^*)$ appear. This fact begs a few questions: *given a map f that is Stieltjes differentiable at a point t , is the map $f^*(t) := f(t^*)$ also Stieltjes differentiable at t ? Is there a relation between the derivatives in the Stieltjes sense of the two functions?* Similarly, we need to concern ourselves with the study of the differentiability in the Stieltjes sense of the map Δg^* and its role in the product rule. It would seem that, at least, Δg ought to be always Stieltjes differentiable as its regularity is intrinsically defined by the same function g with respect to which we are differentiating, but this is far from reality, as previously known examples show –cf. [4, Remark 3.16]. Hence, we need to ask ourselves, *when is Δg^* differentiable in the Stieltjes sense?*

Having a clear understanding of these issues is vital in order to deepen in the study of the classes of continuously Stieltjes differentiable functions and, in particular, in order to ascertain the role of the product of functions in these classes for, as it is already known, we will not enjoy the usual algebra structure with the product of functions as algebra operation.

With these objectives in mind, we set our course to better understand the differentiability of Δg and the role of t^* throughout the following sections. In Section 2 we establish some basic definitions and results that will be necessary in order to understand and develop further results. It is with Section 3 that we derive the Stieltjes differentiability properties of Δg and provide counterexamples that show the optimality of our results. In Section 4 we show under which conditions the product of two functions is, at least, two times g -differentiable. To that end, we study how evaluating a function at t^* instead of t affects a such function with regard to its differentiability in the Stieltjes sense and, in particular, the function Δg . Finally, in Section 5 we study the continuity of the Stieltjes derivative of a product of two functions.

2. PRELIMINARIES

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and left-continuous function, which we call *derivator*, and denote by \mathbb{F} the field \mathbb{R} or \mathbb{C} . We shall write as μ_g the Lebesgue-Stieltjes measure associated to g given by

$$\mu_g([c, d)) = g(d) - g(c), \quad c, d \in \mathbb{R}, \quad c < d,$$

see [2, 15, 16]. It is important to remark that μ_g is a Borel measure. We will use the term “ g -measurable” for a set or function to refer to μ_g -measurability in the corresponding sense, and we denote by $\mathcal{L}_g^1(X, \mathbb{F})$ the set of Lebesgue-Stieltjes μ_g -integrable functions on a g -measurable set X with values in \mathbb{F} , whose integral we write as

$$\int_X f(s) \, d\mu_g(s), \quad f \in \mathcal{L}_g^1(X, \mathbb{F}).$$

Similarly, we will talk about properties holding *g -almost everywhere* in a set X (shortened to *g -a.e.* in X), or holding for *g -almost all* (or, simply, *g -a.a.*) $x \in X$, as a simplified way to express that they hold μ_g -almost everywhere in X or for μ_g -almost all $x \in X$, respectively.

Define the sets

$$\begin{aligned} C_g &= \{t \in \mathbb{R} : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\}, \\ D_g &= \{t \in \mathbb{R} : \Delta g(t) > 0\}, \end{aligned}$$

where $\Delta g(t) := g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right handside limit of g at t . First, observe that $C_g \cap D_g = \emptyset$. Furthermore, as pointed out in [11], the set C_g is open in

the usual topology of the real line, so it can be uniquely expressed as a countable union of open disjoint intervals, say

$$(2.1) \quad C_g = \bigcup_{n \in \Lambda} (a_n, b_n).$$

where $\Lambda \subset \mathbb{N}$. With this notation, we introduce the sets N_g^- and N_g^+ in [9], defined as

$$N_g^- = \{a_n : n \in \Lambda\} \setminus D_g, \quad N_g^+ = \{b_n : n \in \Lambda\} \setminus D_g, \quad N_g = N_g^- \cup N_g^+.$$

Remark 2.1. It is important for the work ahead to remark that, by definition, for any $t \in \mathbb{R} \setminus C_g$, at least one of the following conditions must hold:

$$(2.2) \quad g(s) < g(t), \quad \text{for all } s < t,$$

$$(2.3) \quad g(s) > g(t), \quad \text{for all } s > t.$$

In particular, for $t \in N_g^-$ only (2.2) holds; and, similarly, for $t \in N_g^+$ only (2.3) holds. On the other hand, if $t \in D_g$, then (2.3) always holds and (2.2) fails only when $t = b_n$, $n \in \mathbb{N}$, for some b_n as in (2.1). For the remaining cases, i.e. when $t \in \mathbb{R} \setminus (C_g \cup N_g \cup D_g)$, both (2.2) and (2.3) hold.

Let us recall the following definition of Stieltjes derivative in [4, Definition 3.7]. To that end, we consider $a, b \in \mathbb{R}$, $a < b$, such that $a \notin N_g^-$ and $b \notin D_g \cup C_g \cup N_g^+$. A careful reader might observe that throughout the entirety of [4] it is also required that $g(a) = 0$ and $a \notin D_g$. The first of these conditions can easily be avoided by redefining the map g if necessary; whereas the condition $a \notin D_g$ can be imposed without loss of generality, as pointed out by [4,5] and [12, Proposition 4.28], where the focus of the study is the existence and uniqueness of solution of differential problems, which is not our case. Nevertheless, this condition is not required for the following definition.

Definition 2.2 ([4, Definition 3.7]). We define the *Stieltjes derivative*, or *g-derivative*, of a map $f : [a, b] \rightarrow \mathbb{F}$ at a point $t \in [a, b]$ as

$$f'_g(t) = \begin{cases} \lim_{s \rightarrow t} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \notin D_g \cup C_g, \\ \lim_{s \rightarrow t^+} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in D_g, \\ \lim_{s \rightarrow b_n^+} \frac{f(s) - f(b_n)}{g(s) - g(b_n)}, & t \in C_g, t \in (a_n, b_n), \end{cases}$$

where a_n, b_n are as in (2.1), provided the corresponding limits exist. In that case, we say that f is *g-differentiable at t*.

Remark 2.3. For $t \in N_g \cup \{a, b\}$, the corresponding limit in the definition of *g-derivative* at t must be understood in the sense explained in [13, Remark 2.2], that is, the Stieltjes derivative in such points is computed as

$$f'_g(t) = \begin{cases} \lim_{s \rightarrow t^+} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in N_g^+ \cup \{a\}, \\ \lim_{s \rightarrow t^-} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in N_g^- \cup \{b\}, \end{cases}$$

provided the corresponding limit exists.

Remark 2.4. It follows from the definition that, for $t \in D_g$, $f'_g(t)$ exists if and only if $f(t^+)$ exists and, in that case,

$$f'_g(t) = \frac{f(t^+) - f(t)}{\Delta g(t)}.$$

Similarly, if $t \in (a_n, b_n) \subset C_g$ for some a_n, b_n as in (2.1), we have that $f'_g(t)$ exists if and only if $f'_g(b_n)$ exists and, in that case, $f'_g(t) = f'_g(b_n)$.

It is possible to further simplify the definition of the Stieltjes derivative at a point $t \in [a, b]$ by defining

$$(2.4) \quad t^* = \begin{cases} t, & t \notin C_g, \\ b_n, & t \in (a_n, b_n) \subset C_g, \end{cases}$$

with $a_n, b_n, n \in \Lambda$, as in (2.1). With this notation, we have that

$$f'_g(t) = \begin{cases} \lim_{s \rightarrow t} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \notin D_g \cup C_g, \\ \lim_{s \rightarrow t^{**}} \frac{f(s) - f(t^*)}{g(s) - g(t^*)}, & t \in D_g \cup C_g, \end{cases}$$

provided the corresponding limit exists. Note that the information in Remark 2.3 should still be taken into account.

For completeness, we include the following result containing some basic properties of t^* that follow directly from the definition and the assumptions on the point b .

Lemma 2.5. *Let $t \in [a, b]$ and let t^* be the corresponding point in (2.4). Then, we have that $t^* \in [t, b]$ and $g(t) = g(t^*)$. Furthermore,*

- if $t \in C_g$, then $t^* \notin C_g$ (and so $t^* \neq t$);
- if $t \in [a, b] \setminus C_g$, then $t = t^*$;
- if $t^* \in [a, b] \setminus \{b_n : n \in \Lambda\}$, then $t = t^*$.

In particular, if we denote by t^{**} the corresponding point in (2.4) for t^* , it holds that $t^{**} = t^*$.

The following result can be found in [4, Proposition 3.9] and it includes some basic properties of the Stieltjes derivative. Observe that this is a generalization of [12, Proposition 3.13], which is only stated for real-valued functions which are g -differentiable at a point of $\mathbb{R} \setminus C_g$.

Proposition 2.6. *Let $t \in [a, b]$. If $f_1, f_2 : [a, b] \rightarrow \mathbb{F}$ are g -differentiable at t , then:*

- The function $\lambda_1 f_1 + \lambda_2 f_2$ is g -differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

- The product $f_1 f_2$ is g -differentiable at t and

$$(2.5) \quad (f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t) (f_2)'_g(t) \Delta g(t^*).$$

- If $f_2(t^*) (f_2(t^*) + (f_2)'_g(t) \Delta g(t^*)) \neq 0$, the quotient f_1/f_2 is g -differentiable at t and

$$\left(\frac{f_1}{f_2} \right)'_g(t) = \frac{(f_1)'_g(t) f_2(t^*) - (f_2)'_g(t) f_1(t^*)}{f_2(t^*) (f_2(t^*) + (f_2)'_g(t) \Delta g(t^*))}$$

Naturally, it is possible to extend the product rule for a finite number of functions. To that end, we introduce the following notation: for $n, k \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \{0, 1\}^n$ define

$$|(x_1, \dots, x_n)| := \text{card}\{k \in \{1, \dots, n\} : x_k = 1\}, \quad F_{n,k} := \{\sigma \in \{0, 1\}^n : |\sigma| = k\}.$$

We also denote $f_g^{(0)} \equiv f, f_g^{(1)} \equiv f'_g$.

Proposition 2.7 (Product rule). *Let $n \in \mathbb{N}, n \geq 2$, and $f_j : [a, b] \rightarrow \mathbb{F}, j = 1, 2, \dots, n$, be g -differentiable at $t \in [a, b]$. Then, the product $\prod_{j=1}^n f_j$ is also g -differentiable at t and*

$$(2.6) \quad \left(\prod_{j=1}^n f_j \right)'_g(t) = \sum_{k=0}^{n-1} (\Delta g(t^*))^k \left(\sum_{\sigma \in F_{n,k+1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)}(t^*) \right).$$

Remark 2.8. Observe that the term $\Delta g(t^*)^k$ might not be properly defined when $k = 0$. This is because (2.6) is just a concise way of writing

$$\left(\prod_{j=1}^n f_j \right)'_g(t) = \sum_{\sigma \in F_{n,1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)}(t^*) + \sum_{k=1}^{n-1} (\Delta g(t^*))^k \left(\sum_{\sigma \in F_{n,k+1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)}(t^*) \right).$$

In other words, in (2.6) we implicitly treat $\Delta g(t^*)^0$ as 1. We adopt this notation for the rest of the section.

Proof. We prove the result by induction on $n \in \mathbb{N}$. For $n = 2$, Proposition 2.6 ensures that $f_1 \cdot f_2$ is g -differentiable at t and

$$(f_1 \cdot f_2)'_g(t) = (f_1)_g^{(1)}(t) f_2^{(0)}(t^*) + f_1^{(0)}(t^*) (f_2)_g^{(1)}(t) + (f_1)_g^{(1)}(t) (f_2)_g^{(1)}(t) \Delta g(t^*).$$

Now, by definition of t^* , we have that $(f_j)_g^{(1)}(t^*) = (f_j)_g^{(1)}(t), j = 1, 2$, so it follows that (2.6) holds.

Assume that the result is true for $n - 1$. Then, Proposition 2.6 guarantees that $\prod_{j=1}^{n-1} f_j$ is g -differentiable at t and, since (2.6) holds for $n = 2$, omitting the evaluation at t^* , we have that

$$\begin{aligned} \left(\prod_{j=1}^n f_j \right)'_g &= \left(\prod_{j=1}^{n-1} f_j \right)'_g \cdot f_n^{(0)} + \left(\prod_{j=1}^{n-1} f_j \right)^{(0)} \cdot (f_n)_g^{(1)} + \left(\prod_{j=1}^{n-1} f_j \right)^{(1)} \cdot (f_n)_g^{(1)} \cdot \Delta g \\ &= \left(\sum_{k=0}^{n-2} (\Delta g)^k \left(\sum_{\sigma \in F_{n-1,k+1}} \prod_{j=1}^{n-1} (f_j)_g^{(\sigma_j)} \right) \right) \cdot f_n^{(0)} + \left(\prod_{j=1}^{n-1} f_k^{(0)} \right) \cdot (f_n)_g^{(1)} \\ &\quad + \left[\sum_{k=0}^{n-2} (\Delta g)^k \left(\sum_{\sigma \in F_{n-1,k+1}} \prod_{j=1}^{n-1} (f_j)_g^{(\sigma_j)} \right) \right] \cdot (f_n)_g^{(1)} \cdot \Delta g \\ &= \sum_{k=0}^{n-2} (\Delta g)^k \left(\sum_{\substack{\sigma \in F_{n,k+1} \\ \sigma_n=0}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right) + \left(\prod_{j=1}^{n-1} f_k^{(0)} \right) \cdot (f_n)_g^{(1)} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=0}^{n-2} (\Delta g)^{k+1} \left(\sum_{\substack{\sigma \in F_{n,k+2} \\ \sigma_n=1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right) \\
& = \sum_{k=0}^{n-2} (\Delta g)^k \left(\sum_{\substack{\sigma \in F_{n,k+1} \\ \sigma_n=0}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right) + \left(\prod_{j=1}^{n-1} f_k^{(0)} \right) \cdot (f_n)_g^{(1)} \\
& \quad + \sum_{k=1}^{n-1} (\Delta g)^k \left(\sum_{\substack{\sigma \in F_{n,k+1} \\ \sigma_n=1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right) \\
& = \sum_{k=0}^{n-2} (\Delta g)^k \left(\sum_{\substack{\sigma \in F_{n,k+1} \\ \sigma_n=0}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right) + \sum_{k=0}^{n-1} (\Delta g)^k \left(\sum_{\substack{\sigma \in F_{n,k+1} \\ \sigma_n=1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right) \\
& = \sum_{k=0}^{n-1} (\Delta g)^k \left(\sum_{\sigma \in F_{n,k+1}} \prod_{j=1}^n (f_j)_g^{(\sigma_j)} \right),
\end{aligned}$$

where in the last equality we have use the fact that $F_{n,n} = \{(1, \dots, 1)\}$ and thus $\sigma_n = 1$ for $\sigma \in F_{n,n}$. \square

Remark 2.9. Given that $(f_j)'_g(t^*) = (f_j)'_g(t)$, $j = 1, 2, \dots, n$, $t \in [a, b]$, we can write (2.6) as

$$\left(\prod_{j=1}^n f_j \right)'_g(t) = \sum_{k=0}^{n-1} (\Delta g(t^*))^k \left(\sum_{\sigma \in F_{n,k+1}^*} \prod_{j=1}^n (f_j)_g^{(\sigma_j)}(t) \right),$$

where $F_{n,k}^* := \{\sigma \in \{*, 1\}^n : |\sigma| = k\}$ with $|\sigma| := \text{card}\{k \in \{1, \dots, n\} : \sigma_k = 1\}$ and $f_g^{(*)}(t) \equiv f(t^*)$. Observe that, in this form, for $n = 2$, we recover (2.5).

In this last part of the section, we shall focus on the concept of g -continuity, as presented in [5, Definition 3.1] which we include below.

Definition 2.10 (g -continuous function). A function $f : X \subset \mathbb{R} \rightarrow \mathbb{F}$ is g -continuous at a point $t \in X$, or *continuous with respect to g at t* , if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon, \quad \text{for every } s \in X \text{ such that } |g(t) - g(s)| < \delta;$$

otherwise, we say that f is g -discontinuous at t . If f is g -continuous at every point $t \in X$, we say that f is g -continuous on X .

Remark 2.11. In [14, Remark 3.10], the authors presented the concept of lateral g -continuity. We say that f is g -continuous from the left at t if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon, \quad \text{for every } s \in X, s \leq t, \text{ such that } 0 \leq g(t) - g(s) < \delta.$$

Similarly, f is g -continuous from the right at t if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon, \quad \text{for every } s \in X, s \geq t, \text{ such that } 0 \leq g(s) - g(t) < \delta.$$

Naturally, f is g -continuous at t if and only if f is g -continuous from the left and right at t .

Remark 2.12. It is important to note that, as indicated in [5, Section 3], g -continuity can be understood as continuity between pseudometric spaces, which are sequential spaces (see [1]). Thus, we have that f is g -continuous at $t \in X$ if and only if

$$(2.7) \quad f(t_n) \rightarrow f(t), \quad \text{for every sequence } \{t_n\}_{n \in \mathbb{N}} \subset X \text{ such that } g(t_n) \rightarrow g(t).$$

The following result, [5, Proposition 3.2], describes some properties of g -continuous functions that are relevant for the work ahead.

Proposition 2.13. *If $f : [a, b] \rightarrow \mathbb{F}$ is g -continuous on $[a, b]$, then:*

- f is continuous from the left at every $t \in (a, b]$;
- if g is continuous at $t \in [a, b)$, then so is f ;
- if g is constant on some $[\alpha, \beta] \subset [a, b]$, then so is f .

In [5] the reader can find more information on the properties of g -continuous functions. In the following, we present some results that, to the best of our knowledge, are not available in the current literature.

Lemma 2.14. *Let $f, h : X \subset \mathbb{R} \rightarrow \mathbb{F}$ be g -continuous at $t \in [a, b]$ and $h(t) \neq 0$. Then f/h is g -continuous at t .*

Proof. Let $t \in X$ and $\varepsilon > 0$. Since h is g -continuous at t , there exists $\delta_1 \in \mathbb{R}^+$ such that

$$|h(s) - h(t)| < |h(t)|/2, \quad \text{for } s \in X \text{ such that } |g(s) - g(t)| < \delta_1.$$

Hence, $|h(t)|/2 < h(s) < 3|h(t)|/2$ for every $s \in X$ such that $|g(s) - g(t)| < \delta_1$. Now, since f, h are g -continuous and $h(t) \neq 0$, we can find $\delta \in (0, \delta_1)$ such that

$$|f(s) - f(t)|, |h(s) - h(t)| < \frac{|h(t)|^2}{2(|h(t)| + |f(t)|)} \varepsilon,$$

for $s \in X$ such that $|g(s) - g(t)| < \delta$. Thus,

$$\left| \frac{f(s)}{h(s)} - \frac{f(t)}{h(t)} \right| = \frac{|f(s)h(t) - f(t)h(s)|}{|h(t)h(s)|} \leq \frac{|f(s) - f(t)||h(t)| + |f(t)||h(s) - h(t)|}{\frac{1}{2}|h(t)|^2} \leq \varepsilon,$$

for every $s \in X$ such that $|g(s) - g(t)| < \delta$, that is, f/h is g -continuous at t . \square

Lemma 2.15. *Let $f : X \subset \mathbb{R} \rightarrow \mathbb{F}$ and $t_1, t_2 \in X$ be such that $g(t_1) = g(t_2)$. Then, f is g -continuous at t_1 if and only if f is g -continuous at t_2 . Furthermore, in that case, we have that $f(t_1) = f(t_2)$.*

Proof. Let $t_1, t_2 \in X$ be such that $g(t_1) = g(t_2)$. Observe that it is enough to prove that if f is g -continuous at t_1 then $f(t_1) = f(t_2)$ as, in that case, the equivalence for the g -continuity follows from the definition.

Since f is g -continuous at t_1 , for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $s \in X$ satisfies $|g(s) - g(t_1)| < \delta$, then $|f(s) - f(t_1)| < \varepsilon$. Hence, given that $g(t_1) = g(t_2)$ we have that $|f(t_2) - f(t_1)| < \varepsilon$ for every $\varepsilon > 0$ so, necessarily, $f(t_1) = f(t_2)$, which finishes the proof. \square

It is important to note that, as presented in [5, Example 3.3], g -continuous functions need not be regulated or even locally bounded. With this idea in mind, given $a, b \in \mathbb{R}$, $a < b$, we define

$$\mathcal{BC}_g([a, b], \mathbb{F}) = \{f : [a, b] \rightarrow \mathbb{F} : f \text{ bounded and } g\text{-continuous on } [a, b]\}.$$

Similarly, [12, Example 3.23] shows that, unlike in the usual setting, g -differentiability does not imply g -continuity, which lead to the following definition.

Definition 2.16 ([4, Definition 3.12]). Given $a, b \in \mathbb{R}$, $a < b$, such that $a \notin N_g^- \cup D_g$ and $b \notin C_g \cup N_g^+ \cup D_g$, we define

$$\mathcal{BC}_g^1([a, b], \mathbb{F}) := \{f \in \mathcal{BC}_g([a, b], \mathbb{F}) : f'_g \in \mathcal{BC}_g([a, b], \mathbb{F})\}.$$

3. SOME PROPERTIES OF Δg

We now turn our attention to the study of the function $\Delta g : \mathbb{R} \rightarrow \mathbb{R}$. In what follows, given a set $X \subset \mathbb{R}$, we denote by X' the set of limit points of X .

We start with a simple result showing that Δg is a regulated function, which is enough to ensure measurability with respect to g .

Proposition 3.1. *For each $t \in \mathbb{R}$, we have that*

$$(3.1) \quad \lim_{s \rightarrow t^-} \Delta g(s) = \lim_{s \rightarrow t^+} \Delta g(s) = 0.$$

In particular, Δg is a regulated function, Borel-measurable and g -measurable.

Proof. In order to show that (3.1) holds, let $t \in \mathbb{R}$. We first show that $\Delta g(t^+) = 0$.

If $t \notin (D_g \cap (t, +\infty))'$, then there exists $r > 0$ such that $D_g \cap (t, +\infty) \cap (t - r, t + r) = D_g \cap (t, t + r) = \emptyset$, so g is continuous on $(t, t + r)$, which implies that $\Delta g = 0$ on that same interval and the result follows. Otherwise, we have that $t \in (D_g \cap (t, +\infty))'$ and we need to show that $\Delta g(t^+) = 0$. This happens if and only if

$$\lim_{\substack{s \rightarrow t^+ \\ s \in D_g}} \Delta g(s) = \lim_{\substack{s \rightarrow t^+ \\ s \notin D_g}} \Delta g(s) = 0,$$

where the second limit can be taken because D_g is countable, so $\mathbb{R} \setminus D_g$ is dense in \mathbb{R} . However, noting that $\Delta g = 0$ on $\mathbb{R} \setminus D_g$, we have that the second of these limits is null, so it is enough to show that

$$\lim_{\substack{s \rightarrow t^+ \\ s \in D_g}} \Delta g(s) = 0.$$

Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence in $D_g \cap (t, +\infty)$ converging to t . Then, there exists $N \in \mathbb{N}$ such that $t_n \in (t, t + 1)$ for all $n \in \mathbb{N}$ such that $n \geq N$. Hence,

$$0 < \sum_{n=N}^{\infty} \Delta g(t_n) \leq \sum_{s \in (t, t+1) \cap D_g} \Delta g(s) = \int_{(t, t+1) \cap D_g} dg(s) \leq \int_{[t, t+1]} dg(s) = \mu_g([t, t + 1)).$$

As a consequence, the series $\sum_{n \in \mathbb{N}} \Delta g(t_n)$ is convergent, which guarantees that the sequence $\{\Delta g(t_n)\}_{n \in \mathbb{N}}$ must converge to 0. Since $\{t_n\}_{n \in \mathbb{N}}$ was arbitrarily chosen, it follows that $\Delta g(t^+) = 0$.

The reasoning for $\Delta g(t^-)$ is analogous and we omit it. Observe that this is enough to guarantee that Δg is regulated, which in turn makes it Borel measurable and, thus, g -measurable. \square

Taking into account the previous result it is only natural to wonder what these limits imply with regard to the g -continuity of Δg . This is covered in the next result.

Proposition 3.2. *Consider the set*

$$(3.2) \quad A_g = \{t \in \mathbb{R} : \sup\{s \in [a, b] : g(s) = g(t)\} \in D_g\}.$$

The map Δg is g -continuous at every point of $\mathbb{R} \setminus A_g$ and g -discontinuous at every point of A_g .

Proof. Throughout this proof, given $t \in \mathbb{R}$, we denote

$$(3.3) \quad \widehat{t} = \sup\{s \in [a, b] : g(s) = g(t)\}.$$

First, we prove that Δg is g -discontinuous at every point of $D_g \subset A_g$. To that end, let $t \in D_g$. Then, either (2.2) holds or

$$(3.4) \quad g(s) = g(t), \quad \text{for all } s \in [t_0, t] \text{ for some } t_0 < t.$$

Suppose (3.4) holds. Then $\Delta g(t_0) = g(t_0^+) - g(t_0) = g(t) - g(t) = 0$, while $\Delta g(t) > 0$, so for every $\delta \in \mathbb{R}^+$ we have that $|g(t_0) - g(t)| < \delta$, but $|\Delta g(t_0) - \Delta g(t)| = \Delta g(t) > \Delta g(t)/2$, so Δg is not g -continuous at t .

If (2.2) holds instead, since $\lim_{s \rightarrow t^-} \Delta g(s) = 0$, there exists $\rho > 0$ such that

$$\Delta g(s) < \frac{\Delta g(t)}{2}, \quad s \in \mathbb{R}, \quad 0 < t - s < \rho.$$

Since (2.2) holds, so we can find $t_1 < t$ such that $t - t_1 < \rho$ and $g(t_1) < g(t)$. Let $\delta = g(t) - g(t_1)$. Observe that, if $0 < g(t) - g(s) < \delta$, since g is nondecreasing, then $0 < t - s < t - t_1 < \rho$, and so $\Delta g(s) < \Delta g(t)/2$. But this implies that, if $0 < g(t) - g(s) < \delta$, then

$$|\Delta g(s) - \Delta g(t)| \geq \Delta g(t) - \Delta g(s) > \Delta g(t) - \frac{\Delta g(t)}{2} = \frac{\Delta g(t)}{2},$$

and, therefore, Δg is not continuous at t .

Now, let $t \in A_g \setminus D_g$. Since g is left-continuous, we have that $g(t) = g(\widehat{t})$ so it follows from Lemma 2.15 that Δg cannot be g -continuous at t since it is g -discontinuous at $\widehat{t} \in D_g$.

Now, we shall study the g -continuity of Δg in the remaining points. Let $t \in \mathbb{R} \setminus A_g$. In that case, $t \notin D_g$ so $\Delta g(t) = 0$. We claim that

$$(3.5) \quad \Delta g \text{ is } g\text{-continuous from the left at } t \text{ if (2.2) holds;}$$

$$(3.6) \quad \Delta g \text{ is } g\text{-continuous from the right at } t \text{ if (2.3) holds.}$$

Suppose that (2.2) holds and let $\varepsilon > 0$. Since $\lim_{s \rightarrow t^-} \Delta g(s) = 0$ and $\Delta g(t) = 0$, there exists $\rho > 0$ such that $\Delta g(s) < \varepsilon$, for all $s \in (t - \rho, t]$. Since (2.2) holds, we can take $\delta = g(t) - g(t - \rho) > 0$. If $s \leq t$ is such that $0 \leq g(t) - g(s) < \delta$, since g is nondecreasing, we necessarily have that $s \in (t - \rho, t]$ so

$$|\Delta g(s) - \Delta g(t)| = \Delta g(s) < \varepsilon,$$

which proves that Δg is also g -continuous from the left at t .

The proof of (3.6) is analogous, and we omit it. Since (3.5)-(3.6) hold, it follows that Δg is g -continuous if $t \notin C_g \cup N_g \cup A_g$. Similarly, (3.5) guarantees that Δg is g -continuous from the left if $t \in N_g^- \setminus A_g$, while (3.6) ensures the g -continuity from the right if $t \in N_g^+ \setminus A_g$.

Let us investigate what happens in the remaining cases, namely, when $t \in (C_g \cup N_g) \setminus A_g$.

Suppose $t \in (N_g^- \cup C_g) \setminus A_g$ and let us show that Δg is g -continuous from the right at t . Let $\varepsilon > 0$ and consider \widehat{t} as in (3.3). Observe that $g(s) = g(t)$ for all $s \in [t, \widehat{t}]$, which means $\Delta g(s) = 0$ for all $s \in [t, \widehat{t}]$. Furthermore, since $t \notin A_g$, we have that $\widehat{t} \notin D_g$, so we

necessarily have that $g(\widehat{t}) = g(t)$ and $\widehat{t} \in N_g^+$. In that case, Δg is g -continuous from the right at \widehat{t} , which means that there exists $\delta > 0$ such that

$$|\Delta g(s)| = |\Delta g(\widehat{t}) - \Delta g(s)| < \varepsilon, \quad \text{for each } s \geq \widehat{t} \text{ such that } 0 \leq g(s) - g(\widehat{t}) < \delta.$$

Now, if $s \geq t$ is such that $0 \leq g(s) - g(t) < \delta$, then either $s \in [t, \widehat{t})$, or $s \geq \widehat{t}$ and $0 \leq g(s) - g(\widehat{t}) < \delta$. In either case, it follows that

$$|\Delta g(t) - \Delta g(s)| = |\Delta g(s)| < \varepsilon,$$

proving that Δg is g -continuous from the right at t . In particular, this proves that Δg is g -continuous if $t \in N_g^- \setminus A_g$.

Finally, suppose $t \in (N_g^+ \cup C_g) \setminus A_g$ and let us show that Δg is g -continuous from the left at t . Let $\varepsilon > 0$ and consider

$$\widetilde{t} = \inf\{s \in [a, b] : g(s) = g(t)\}.$$

Observe that $g(s) = g(t)$ for all $s \in (\widetilde{t}, t]$, which means $\Delta g(s) = 0$ for all $s \in (\widetilde{t}, t]$. Now, if $\widetilde{t} \notin D_g$, then $g(\widetilde{t}) = g(t)$ and $\widetilde{t} \in N_g^-$. In that case, Δg is g -continuous from the left at \widetilde{t} so there exists $\delta > 0$ such that

$$|\Delta g(s)| = |\Delta g(\widetilde{t}) - \Delta g(s)| < \varepsilon, \quad \text{for each } s \leq \widetilde{t} \text{ such that } 0 \leq g(\widetilde{t}) - g(s) < \delta.$$

Now, if $s \leq t$ is such that $0 \leq g(t) - g(s) < \delta$, then either $s \in (\widetilde{t}, t]$, or $s \leq \widetilde{t}$ and $0 \leq g(\widetilde{t}) - g(s) < \delta$. In either case, it follows that

$$|\Delta g(t) - \Delta g(s)| = |\Delta g(s)| < \varepsilon.$$

Otherwise, $\widetilde{t} \in D_g$, so taking $\delta \in (0, \Delta g(\widetilde{t}))$, it follows that if $s \leq t$ is such that $0 \leq g(t) - g(s) < \delta$, then $s \in (\widetilde{t}, t]$, so

$$|\Delta g(t) - \Delta g(s)| = 0 < \varepsilon,$$

which finishes the proof. \square

Example 3.3. To illustrate the set A_g , consider the function represented in Figure 3.1. Here $A_g = \{-1\} \cup [0, 2]$.

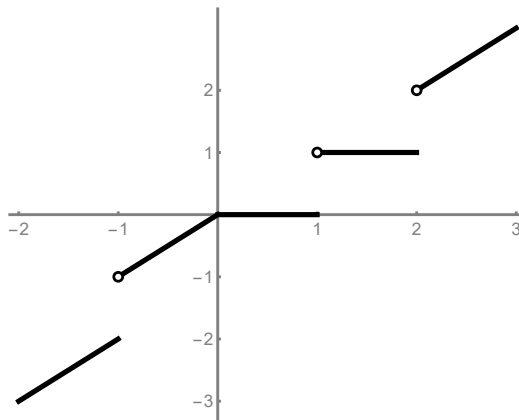


FIGURE 3.1. The graph of function g in Example 3.3.

We now consider the g -differentiability of Δg . In order to do so, we present the following more general result from which we can deduce some information about the differentiability in the Stieltjes sense of Δg . To that end, we introduce the following notation: given $X \subset \mathbb{R}$, we denote by χ_X the characteristic function of the set X , namely,

$$\chi_X(t) = \begin{cases} 1, & t \in X, \\ 0, & t \notin X. \end{cases}$$

Proposition 3.4. *Let $f : [a, b] \rightarrow \mathbb{F}$ be a map and $h : [a, b] \rightarrow \mathbb{F}$ be defined as $h(t) = f(t)\Delta g(t)$. Consider the sets*

$$(3.7) \quad D_1 = \{t \in [a, b] \cap N_g^- : t \in (D_g \cap [a, t])'\},$$

$$(3.8) \quad D_2 = \{t \in [a, b] \cap N_g^+ : t \in (D_g \cap (t, b])'\},$$

$$(3.9) \quad D_3 = \{t \in [a, b] \setminus (N_g \cup D_g) : t \in (D_g \cap [a, b])'\}.$$

For $t \in [a, b]$, and denoting by t^* the corresponding point in (2.4), we have the following properties:

1. If $t^* \in D_1 \cup D_2 \cup D_3$, then h is g -differentiable at t if and only if

$$(3.10) \quad \lim_{\substack{s \rightarrow t^* \\ s \in D_g}} \frac{f(s)\Delta g(s)}{g(s) - g(t)} = 0,$$

where we might be considering the corresponding lateral limit according to the definition of Stieltjes derivative, see Remark 2.3.

2. If $t^* \in D_g \cap (D_g \cap (t, b])'$, then h is g -differentiable at t if and only if

$$\lim_{\substack{s \rightarrow t^{*+} \\ s \in D_g}} f(s)\Delta g(s) = 0.$$

3. In any other case, h is g -differentiable at t .

Moreover, if h is g -differentiable at t , then

$$(3.11) \quad h'_g(t) = -f(t^*)\chi_{D_g}(t^*).$$

Proof. We will first prove 1-3 in order for $t \in [a, b] \setminus C_g$ while showing that (3.11) holds for each case.

Let $t \in [a, b] \setminus C_g$. Observe that, in that case, $t^* = t$.

First, suppose $t \in D_1 \cup D_2 \cup D_3$. Observe that, in that case, $t \notin D_g$ so $\Delta g(t) = 0$. Now, if h is g -differentiable at t , by definition,

$$h'_g(t) = \lim_{s \rightarrow t} \frac{h(s) - h(t)}{g(s) - g(t)} = \lim_{s \rightarrow t} \frac{f(s)\Delta g(s)}{g(s) - g(t)}.$$

Observe that, since such limit exists, the following limits exist and are equal and, since $\Delta g = 0$ in $[a, b] \setminus D_g$, we must have that

$$\lim_{\substack{s \rightarrow t \\ s \in D_g}} \frac{f(s)\Delta g(s)}{g(s) - g(t)} = \lim_{\substack{s \rightarrow t \\ s \notin D_g}} \frac{f(s)\Delta g(s)}{g(s) - g(t)} = 0,$$

where the second limit can be taken because D_g is countable, so $[a, b] \setminus D_g$ is dense in $[a, b]$. This demonstrates that if h is g -differentiable at t then (3.10) holds. Furthermore, this also shows that (3.11) holds in this case. Conversely, assume that (3.10) holds. It is clear now that we need to show that

$$\lim_{s \rightarrow t} \frac{f(s)\Delta g(s)}{g(s) - g(t)} = 0.$$

Once again, such limit exists provided that

$$\lim_{\substack{s \rightarrow t \\ s \in D_g}} \frac{f(s)\Delta g(s)}{g(s) - g(t)} = \lim_{\substack{s \rightarrow t \\ s \notin D_g}} \frac{f(s)\Delta g(s)}{g(s) - g(t)} = 0.$$

Note that this is the case thanks to condition (3.10) and the fact that $\Delta g = 0$ in $[a, b] \setminus D_g$.

Now, suppose $t \in D_g \cap (D_g \cap (t, b])'$. In that case, the hypotheses guarantee that $t \neq b$ and $h = 0$ on $(t, b] \setminus D_g$, which is dense on $(t, b]$ as D_g is countable. Hence, we have that

$$\lim_{\substack{s \rightarrow t^+ \\ s \in D_g}} h(s) = 0.$$

Now, Remark 2.4 ensures that h is g -differentiable at t if and only if $h(t^+)$ exists which, at the same time, exists if and only if

$$\lim_{\substack{s \rightarrow t^+ \\ s \notin D_g}} h(s) = 0.$$

Observe that this proves 2 and, furthermore, equality (3.11) as, under these circumstances, $h(t^+) = 0$ and

$$h'_g(t) = \frac{h(t^+) - h(t)}{\Delta g(t)} = \frac{-f(t)\Delta g(t)}{\Delta g(t)} = -f(t) = -f(t^*)\chi_{D_g}(t^*).$$

Finally, for 3, we first consider the case when $t \in [a, b] \setminus (D'_g \cup C_g)$. In that case, there is $\varepsilon > 0$ such that $((t - \varepsilon, t + \varepsilon) \setminus \{t\}) \cap D_g = \emptyset$, which implies that the map g is continuous on $(t - \varepsilon, t + \varepsilon) \setminus \{t\}$ and, in particular, $\Delta g = 0$ on that set. This means that $h = 0$ on $(t - \varepsilon, t + \varepsilon) \setminus \{t\}$. Therefore, if $t \notin D_g$,

$$\lim_{s \rightarrow t} \frac{h(s) - h(t)}{g(s) - g(t)} = \lim_{s \rightarrow t} \frac{h(s)}{g(s) - g(t)} = 0,$$

which means that h is g -differentiable at t and $h'_g(t) = 0 = -f(t^*)\chi_{D_g}(t^*)$. Otherwise, we have that $t \in D_g$ and, in that case, it follows that $h(t^+) = 0$. Remark 2.4 guarantees that h is g -differentiable at t and

$$h'_g(t) = \frac{h(t^+) - h(t)}{\Delta g(t)} = \frac{-f(t)\Delta g(t)}{\Delta g(t)} = -f(t) = -f(t^*)\chi_{D_g}(t^*).$$

Hence, all that is left to do to prove 3 for $t \in [a, b] \setminus C_g$ is to show that h is g -differentiable if $t \in D'_g$ but does not satisfy the conditions in 1 and 2, that is, if $t \in D_g \cap (D_g \cap [a, b])'$ and $t \notin D_g \cap (D_g \cap (t, b])'$. In this setting, there exists $r > 0$ such that $(t, t + r) \cap D_g = \emptyset$, which guarantees that $\Delta g = 0$ on $(t, t + r)$. This implies that $h = 0$ on $(t, t + r)$ and, thus, $h(t^+) = 0$. As a consequence, h is g -differentiable at t and

$$h'_g(t) = \frac{h(t^+) - h(t)}{\Delta g(t)} = \frac{-f(t)\Delta g(t)}{\Delta g(t)} = -f(t) = -f(t^*)\chi_{D_g}(t^*).$$

Lastly, we prove the result for $t \in [a, b] \cap C_g$. Let $t \in (a_n, b_n) \subset [a, b] \cap C_g$ for some a_n, b_n in (2.1). In that case, $t^* = b_n$ and, as pointed out in Remark 2.4, h is g -differentiable at t if and only if h is g -differentiable at t^* . Hence, 1, 2 and 3 follow from the previous cases. Furthermore, noting that $t^* \in D_g$ if and only if $b_n \in D_g$ and using (3.11), we have that

$$h'_g(t) = h'_g(b_n) = -f(b_n^*)\chi_{D_g}(b_n^*) = -f(b_n)\chi_{D_g}(b_n) = -f(t^*)\chi_{D_g}(t^*),$$

that is, (3.10) also holds for these cases. \square

Remark 3.5. To visualize the sets D_1 , D_2 and D_3 , we provide an illustration in Figure 3.2.

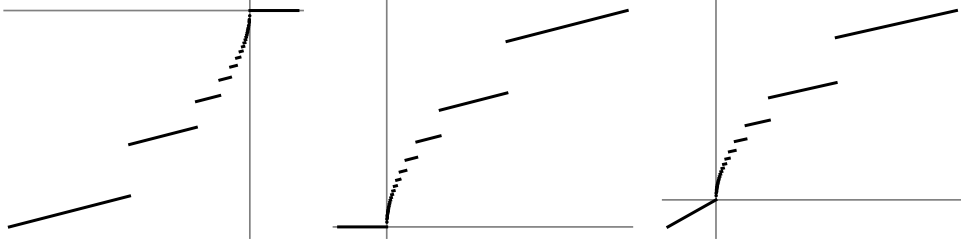


FIGURE 3.2. From left to right, representation of derivators for which zero belongs to D_1 , D_2 and D_3 respectively.

Combining Propositions 3.1 and 3.4 we can obtain the following result about the existence of the Stieltjes derivative of Δg .

Corollary 3.6. Consider the sets D_1, D_2, D_3 in (3.7)-(3.9) and the restriction $\Delta g|_{[a,b]}$. For $t \in [a, b]$, and denoting by t^* the corresponding point in (2.4), we have the following properties:

1. If $t^* \in D_1 \cup D_2 \cup D_3$ then $\Delta g|_{[a,b]}$ is g -differentiable at t if and only if

$$(3.12) \quad \lim_{\substack{s \rightarrow t^* \\ s \in D_g}} \frac{\Delta g|_{[a,b]}(s)}{g(s) - g(t)} = 0,$$

where we might be considering the corresponding lateral limit according to the definition of Stieltjes derivative, see Remark 2.3.

2. In any other case, $\Delta g|_{[a,b]}$ is g -differentiable at t .

Furthermore, if $\Delta g|_{[a,b]}$ is g -differentiable at t , then

$$(3.13) \quad (\Delta g|_{[a,b]})'_g(t) = -\chi_{D_g}(t^*).$$

In particular, $\Delta g|_{[a,b]}$ is g -differentiable on $[a, b]$ and (3.13) holds on $[a, b]$ if

$$(3.14) \quad \lim_{\substack{s \rightarrow t \\ s \in D_g}} \frac{\Delta g|_{[a,b]}(s)}{g(s) - g(t)} = 0, \quad \text{for all } t \in D_1 \cup D_2 \cup D_3,$$

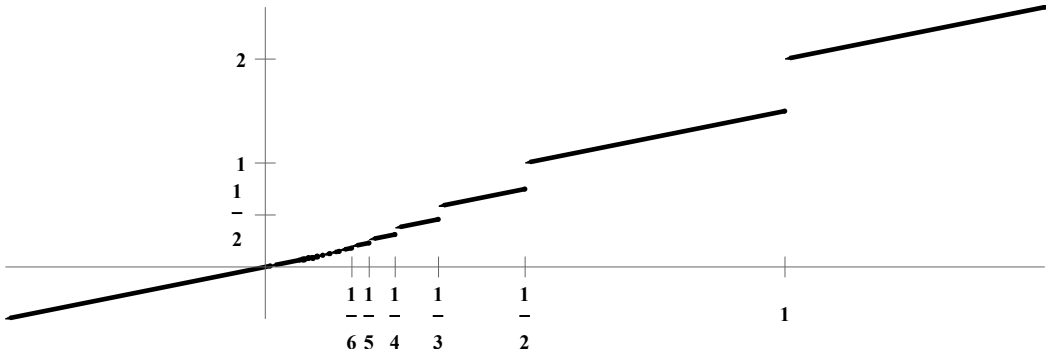
with the same consideration for the lateral limits, see Remark 2.3.

Remark 3.7. Note condition (3.14) becomes vacuous when $[a, b] \cap D_g$ is closed, as in that case $([a, b] \cap D_g)' \subset [a, b] \cap D_g$, so $D_1 = D_2 = D_3 = \emptyset$. Furthermore, condition (3.14) can be satisfied when $([a, b] \cap D_g)' \setminus ([a, b] \cap D_g) \neq \emptyset$. Indeed, the function

$$g(t) = t + \sum_{n=1}^{\infty} 2^{-n} \chi_{(1/n, +\infty)}(t), \quad t \in \mathbb{R},$$

represented in Figure 3.3 is nondecreasing, left-continuous with $D_g = \{1/n : n \in \mathbb{N}\}$ and

$$\Delta g(t) = \begin{cases} 2^{-n}, & t = 1/n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

FIGURE 3.3. Graph of the function g in Remark 3.7.

Observe that $-1 \notin D_g$, $2 \notin D_g \cup C_g \cup N_g^+$ and $([-1, 2] \cap D_g)' \setminus D_g = \{0\}$. Hence, we have that $D_1 = D_2 = \emptyset$ and $D_3 = \{0\}$ and, furthermore,

$$\begin{aligned} \lim_{\substack{s \rightarrow 0 \\ s \in D_g}} \frac{\Delta g|_{[-1,2]}(s)}{g(s) - g(0)} &= \lim_{n \rightarrow +\infty} \frac{\Delta g|_{[-1,2]}(1/n)}{g(1/n)} \\ &= \lim_{n \rightarrow +\infty} \frac{2^{-n}}{1/n + \sum_{k=1}^{\infty} 2^{-k} \chi_{(1/k, +\infty)}(1/n)} \\ &= \lim_{n \rightarrow +\infty} \frac{2^{-n}}{1/n + \sum_{k=n+1}^{\infty} 2^{-k}} = \lim_{n \rightarrow +\infty} \frac{2^{-n}}{1/n + 2^{-n}} \\ &= \lim_{n \rightarrow +\infty} \frac{1}{2^n/n + 1} = 0, \end{aligned}$$

that is, (3.14) is satisfied on $[-1, 2]$. As a consequence, we have that $\Delta g|_{[-1,2]}$ is g -differentiable and, in this case, $(\Delta g|_{[-1,2]})'_g(t) = \chi_{D_g}(t)$, $t \in [-1, 2]$. This example also shows that $(\Delta g|_{[-1,2]})'_g$ needs not be g -differentiable as $(\Delta g|_{[-1,2]})''_g(0)$ cannot exist since

$$\lim_{\substack{s \rightarrow 0 \\ s \in D_g}} \frac{(\Delta g|_{[-1,2]})'_g(s) - (\Delta g|_{[-1,2]})'_g(0)}{g(s) - g(0)} = \lim_{\substack{s \rightarrow 0 \\ s \in D_g}} \frac{1 - 0}{g(s) - g(0)} = +\infty.$$

Nevertheless, we should highlight that there are nondecreasing and left-continuous functions satisfying all the conditions but (3.14). Indeed, it is enough to consider a small modification of the previous example, namely,

$$(3.15) \quad \tilde{g}(t) = t\chi_{(1, +\infty)}(t) + \sum_{n=1}^{\infty} 2^{-n} \chi_{(1/n, +\infty)}(t), \quad t \in \mathbb{R},$$

represented in Figure 3.4.

Once again, this is a nondecreasing and left-continuous map such that $D_{\tilde{g}} = D_g$ and $\Delta \tilde{g} = \Delta g + \chi_{\{1\}}$. Furthermore,

$$(3.16) \quad C_{\tilde{g}} = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right),$$

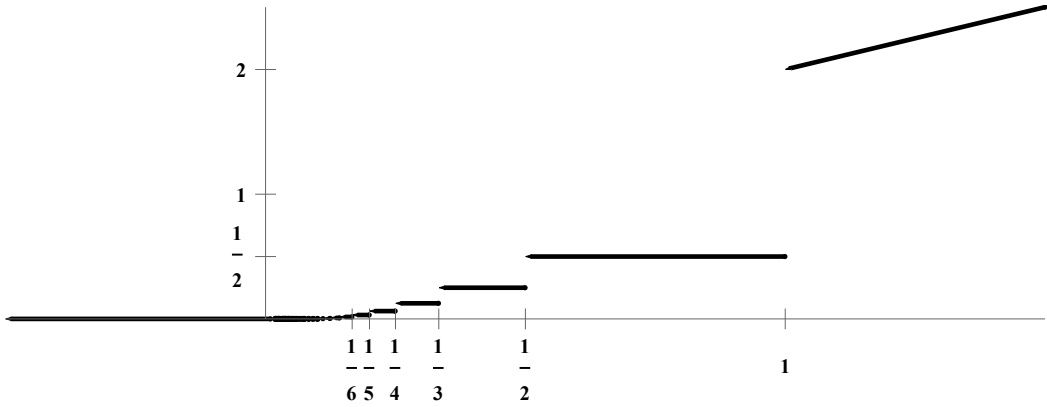


FIGURE 3.4. Graph of the function \tilde{g} in Remark 3.7.

so it follows that $-1 \notin D_{\tilde{g}} \cup N_{\tilde{g}}^-, 2 \notin D_{\tilde{g}} \cup C_{\tilde{g}} \cup N_{\tilde{g}}^+$. However, condition (3.14) cannot be satisfied as, in this case, $D_1 = D_3 = \emptyset, D_2 = \{0\}, \tilde{g}(0) = 0$ and

$$\lim_{n \rightarrow +\infty} \frac{\Delta \tilde{g}|_{[-1,2]}(1/n)}{\tilde{g}(1/n)} = \lim_{n \rightarrow +\infty} \frac{2^{-n}}{\sum_{k=1}^{\infty} 2^{-k} \chi_{(1/k, +\infty)}(1/n)} = \lim_{n \rightarrow +\infty} \frac{2^{-n}}{2^{-n}} = 1.$$

4. DIFFERENTIATING THE PRODUCT RULE

The aim of this section is to study the differentiability of the product of two differentiable functions beyond the first derivative, as this case is already covered by Proposition 2.6.

Observe that Proposition 2.6 provides a good starting point for our research. Indeed, given that in (2.5) some of the maps involved are evaluated at t^* , we need to consider how this affects the Stieltjes differentiability of functions. With this idea in mind, we present the following result establishing some relations between a differentiable map and its corresponding counterpart evaluated at t^* .

Proposition 4.1. *Let $f : [a, b] \rightarrow \mathbb{F}$ and define $f^* : [a, b] \rightarrow \mathbb{F}$ as $f^*(t) = f(t^*)$ with t^* as in (2.4).*

Consider the sets

$$\begin{aligned} C_1 &= \{t \in [a, b] \cap N_g^- : t \in (C_g \cap [a, t])'\}, \\ C_2 &= \{t \in [a, b] \cap (N_g^+ \cup D_g) : t \in (C_g \cap (t, b])'\}, \\ C_3 &= \{t \in [a, b] \setminus (C_g \cup N_g \cup D_g) : t \in (C_g \cap [a, b])'\}. \end{aligned}$$

Then, for $t \in [a, b]$:

1. If $t^* \in C_1 \cup C_2 \cup C_3$, then f is g -differentiable at t if and only if f^* is g -differentiable at t and

$$(4.1) \quad \lim_{\substack{s \rightarrow t^* \\ s \in C_g}} \frac{f(s) - f(t^*)}{g(s) - g(t^*)} = (f^*)'_g(t),$$

where we might be considering the corresponding lateral limit according to the definition of Stieltjes derivative, see Remark 2.3.

2. If $t^* \notin C_1 \cup C_2 \cup C_3$, f is g -differentiable at t if and only if f^* is g -differentiable at t .

Furthermore, if f and f^* are g -differentiable at $t \in [a, b]$, then $(f^*)'_g(t) = f'_g(t) = f'_g(t^*)$.

Proof. First, observe that given the definition of the Stieltjes derivative at the points of C_g , it is enough to prove the result for $t \in [a, b] \setminus C_g$, for which we will use the information in Remark 2.1.

Let $t \in [a, b] \setminus C_g$. We define $A_t = \{s \in [a, b] : g(s) \neq g(t)\}$ and $F_t, F_t^* : A_t \rightarrow \mathbb{F}$ as

$$F_t(s) = \frac{f(s) - f(t)}{g(s) - g(t)}, \quad F_t^*(s) = \frac{f^*(s) - f^*(t)}{g(s) - g(t)} = \frac{f^*(s) - f(t)}{g(s) - g(t)},$$

where the last equality holds since $t^* = t$ for such point. Now, since $t \in [a, b] \setminus C_g$, Remark 2.1 guarantees that (2.2) and/or (2.3) must hold.

Let us assume that (2.2) holds.

We shall assume that $t \neq a$ as the g -derivatives in such point are computed as the right handside limit so it is irrelevant if (2.2) holds in that case. In these conditions, we have that $[a, t) \subset A_t$. We claim that

$$(4.2) \quad \lim_{s \rightarrow t^-} F_t^*(s) = f'_g(t), \quad \text{if } f'_g(t) \text{ exists,}$$

$$(4.3) \quad \lim_{s \rightarrow t^-} F_t(s) = (f^*)'_g(t), \quad \text{if } (f^*)'_g(t) \text{ exists and,}$$

$$\text{if } t \in ([a, t) \cap C_g)', \text{ then } \lim_{\substack{s \rightarrow t^- \\ s \in C_g}} F_t(s) = (f^*)'_g(t).$$

If $t \notin ([a, t) \cap C_g)'$, there exists $r \in (0, t - a)$ such that $(t - r, t) \cap C_g = \emptyset$. Hence, by definition of f^* , we have that $f^* = f$ on $(t - r, t)$ which, in turn, implies that $F_t^* = F_t$ on that set, from which (4.2) and (4.3) follow.

Otherwise, we have that $t \in ([a, t) \cap C_g)'$. In that case, $t \in ([a, t) \setminus C_g)'$ as well as, if this was not the case, there would be $\varepsilon \in (0, t - a)$ such that $(t - \varepsilon, t) \cap C_g = \emptyset$, which would imply that $(t - \varepsilon, t) \subset C_g$ and would contradict (2.2). Now, it is clear that (4.2) and (4.3) hold if the following statements are true:

$$\lim_{\substack{s \rightarrow t^- \\ s \notin C_g}} F_t^*(s) = \lim_{\substack{s \rightarrow t^- \\ s \in C_g}} F_t^*(s) = f'_g(t), \quad \text{if } f'_g(t) \text{ exists,}$$

$$\lim_{\substack{s \rightarrow t^- \\ s \notin C_g}} F_t(s) = (f^*)'_g(t), \quad \text{if } (f^*)'_g(t) \text{ exists and } \lim_{\substack{s \rightarrow t^- \\ s \in C_g}} F_t(s) = (f^*)'_g(t),$$

where we can consider all the limits above because $t \in ([a, t) \cap C_g)' \cap ([a, t) \setminus C_g)'$. Observe that $F_t^* = F_t$ on $A_t \setminus C_g$, so it is clear that $\lim_{s \rightarrow t^-} F_t^*|_{A_t \setminus C_g}(s) = \lim_{s \rightarrow t^-} F_t|_{A_t \setminus C_g}(s)$. Thus, it suffices to show that

$$(4.4) \quad \lim_{\substack{s \rightarrow t^- \\ s \in C_g}} F_t^*(s) = f'_g(t), \quad \text{if } f'_g(t) \text{ exists.}$$

Let $\{s_n\}_{n \in \mathbb{N}}$ be a sequence in $[a, t) \cap C_g$ converging to t . For each $n \in \mathbb{N}$, denote by s_n^* the corresponding number assigned to s_n by (2.4). Observe that, for each $n \in \mathbb{N}$, we have that $s_n^* \in (s_n, t)$, which implies that the sequence $\{s_n^*\}_{n \in \mathbb{N}}$ is contained in $[a, t)$ and converges to t . Furthermore, we have that, by definition and the left-continuity of g ,

$$g(s_n) = g(s_n^*), \quad f^*(s_n) = f(s_n^*), \quad n \in \mathbb{N}.$$

Hence,

$$\lim_{n \rightarrow +\infty} F_t^*(s_n) = \lim_{n \rightarrow +\infty} \frac{f^*(s_n) - f^*(t)}{g(s_n) - g(t)} = \lim_{n \rightarrow +\infty} \frac{f(s_n^*) - f(t)}{g(s_n^*) - g(t)} = f'_g(t),$$

where the last equality follows from the fact that $f'_g(t)$ exists. Since $\{s_n\}_{n \in \mathbb{N}}$ was arbitrarily chosen, we have that (4.4) holds. Hence, we have that (4.2) and (4.3) hold. Observe that, in particular, this proves the result if $t \in C_1$.

We now assume that (2.3) holds. Once again, we can assume that $t \neq b$ as the g -derivatives at such point are computed as the left handside limit so it is irrelevant if (2.3) holds in that case. In these conditions, $(t, b] \subset A_t$ and we claim that

$$(4.5) \quad \lim_{s \rightarrow t^+} F_t^*(s) = f'_g(t), \quad \text{if } f'_g(t) \text{ exists,}$$

$$(4.6) \quad \lim_{s \rightarrow t^+} F_t(s) = (f^*)'_g(t), \quad \text{if } (f^*)'_g(t) \text{ exists and,}$$

$$\text{if } t \in ((t, b] \cap C_g)', \quad \lim_{\substack{s \rightarrow t^+ \\ s \in C_g}} F_t(s) = (f^*)'_g(t).$$

If $t \notin ((t, b] \cap C_g)'$, there exists $r \in (0, b - t)$ such that $(t, t + r) \cap C_g = \emptyset$, which once again implies that $F_t^* = F_t$ on that set, from which (4.5) and (4.6) follow. Otherwise, $t \in ((t, b] \cap C_g)'$ and, following a similar reasoning as before, we can see that $t \in ((t, b] \cap C_g)'$ and $\lim_{s \rightarrow t^+} F_t^*|_{A_t \setminus C_g}(s) = \lim_{s \rightarrow t^+} F_t|_{A_t \setminus C_g}(s)$ so it is enough to show that

$$(4.7) \quad \lim_{\substack{s \rightarrow t^+ \\ s \in C_g}} F_t^*(s) = f'_g(t), \quad \text{if } f'_g(t) \text{ exists.}$$

Let $\{s_n\}_{n \in \mathbb{N}}$ be a sequence in $(t, b] \cap C_g$ converging to t . For each $n \in \mathbb{N}$, denote by s_n^* the corresponding number assigned to s_n by (2.4). Note that $s_n^* \in (s_n, b]$, $n \in \mathbb{N}$. We claim that

$$(4.8) \quad \text{for each } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } 0 < s_n^* - s_n < \varepsilon \text{ for all } n \geq N.$$

Suppose that (4.8) is not true. In that case, we can find $\varepsilon_0 > 0$ such that $s_n^* - s_n \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Since $\{s_n\}_{n \in \mathbb{N}} \subset (t, b]$ converges to t , we can find $p \in \mathbb{N}$ such that if $n \geq p$, then $s_n < s_p$ and

$$0 < s_n - t < \frac{\varepsilon_0}{2}.$$

Hence, for any $n \geq p$, we have that

$$s_n^* - s_p = s_n^* - s_n + s_n - t + t - s_p \geq s_n^* - s_n + t - s_p \geq \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2} > 0,$$

so $s_n < s_p < s_n^*$. This implies that, for every $n \geq p$, s_n and s_p are in the same connected component of C_g and, as a consequence, $s_n^* = s_p^*$. Hence, $\bigcup_{n \geq p} (s_n, s_n^*) = (t, s_p^*) \subset C_g$, which contradicts that $t \in ((t, b] \cap C_g)'$. Thus, (4.8) must be true.

We can now show that $\{s_n^*\}_{n \in \mathbb{N}}$ converges to t . Indeed, let $\varepsilon > 0$. On the one hand, (4.8) guarantees that there exists $n_1 \in \mathbb{N}$ such that

$$0 < s_n^* - s_n < \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}, n \geq n_1.$$

On the other hand, since $\{s_n\} \subset (t, b]$ converges to t , there exists $n_2 \in \mathbb{N}$ such that

$$0 < s_n - t < \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}, n \geq n_2.$$

Hence, if we take $M = \max\{n_1, n_2\}$, for any $n \in \mathbb{N}$ such that $n \geq M$, we have that

$$0 < s_n^* - t = s_n^* - s_n + s_n - t \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

that is, $\{s_n^*\}$ converges to t . Furthermore, by definition,

$$g(s_n) = g(s_n^*), \quad f^*(s_n) = f(s_n^*), \quad n \in \mathbb{N},$$

from which we have that

$$\lim_{n \rightarrow +\infty} F_t^*(s_n) = \lim_{n \rightarrow +\infty} \frac{f^*(s_n) - f^*(t)}{g(s_n) - g(t)} = \lim_{n \rightarrow +\infty} \frac{f(s_n^*) - f(t)}{g(s_n^*) - g(t)} = f'_g(t),$$

where the last equality follows from the fact that $f'_g(t)$ exists. Since $\{s_n\}_{n \in \mathbb{N}}$ was arbitrarily chosen, we have that (4.7) holds. Hence, we have that (4.5) and (4.6) hold. Observe that, in particular, this proves the result if $t \in C_2$.

The remaining cases for $t \in [a, b] \setminus C_g$, namely when (2.2) and (2.3) hold simultaneously, now follow since, in that case, (4.2)-(4.3) and (4.5)-(4.6) hold. \square

Remark 4.2. To visualize the sets C_1, C_2 and C_3 , we provide an illustration in Figure 4.1.

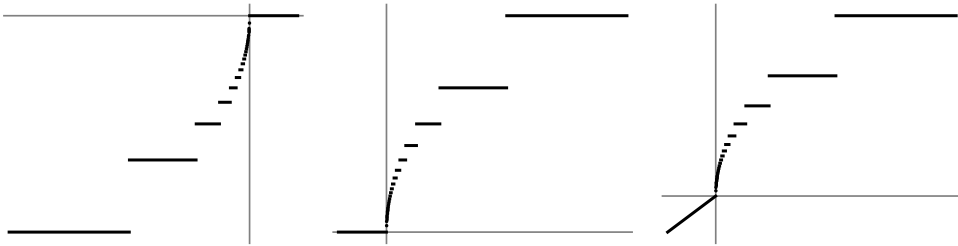


FIGURE 4.1. From left to right, representation of derivators for which zero belongs to C_1, C_2 and C_3 respectively.

Remark 4.3. Observe that, in particular, we have that f^* is g -differentiable at every point that f is g -differentiable, regardless of which type of points it might be. The converse is not necessarily true because, by considering the map f^* , we are losing information on the behavior of f along the points of C_g . This is reflected in the extra condition that is required for the converse implication, namely, condition (4.1). Without it, we cannot ensure the differentiability of f . Indeed, consider, for example, the map \tilde{g} in (3.15) and $f : [-1, 1] \rightarrow \mathbb{R}$ defined as

$$f(t) = \begin{cases} 1, & \text{if } t \in \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right), \\ 0, & \text{otherwise,} \end{cases}$$

and shown in Figure 4.2.

Given (3.16), it is easy to see that $f^*(t) = 0, t \in [-1, 1]$, so it follows that f is g -differentiable on $[-1, 1]$. In particular, it is g -differentiable at $0 \in N_g^+ \cap ((0, 1] \cap C_g)'$, which belongs to C_2 .

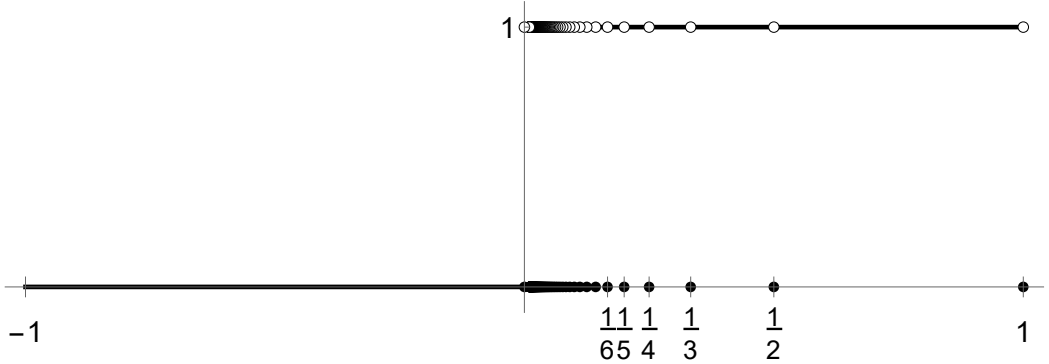


FIGURE 4.2. Graph of the function f .

Now, consider the sequence $\{t_n\}_{n \in \mathbb{N}} = \left\{ \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right) \right\}_{n \in \mathbb{N}}$. It is easy to see that it belongs to $(0, 1] \cap C_g$, converges to 0 and, furthermore,

$$\lim_{n \rightarrow +\infty} \frac{f(t_n) - f(0)}{g(t_n) - g(0)} = \lim_{n \rightarrow +\infty} \frac{1}{\sum_{k=1}^{\infty} 2^{-k} \chi_{(1/k, +\infty)}(t_n)} = \lim_{n \rightarrow +\infty} \frac{1}{\sum_{k=n+1}^{\infty} 2^{-k}} = +\infty.$$

This implies that f is not g -differentiable at 0 and shows that (4.1) is not satisfied for $t = 0$. Similar counterexamples can be constructed for the remaining conditions.

The following result can be directly deduced from Proposition 4.1 and Corollary 3.6 and it gives some conditions under which the map Δg^* is g -differentiable.

Corollary 4.4. Consider the sets D_1, D_2, D_3 in (3.7)-(3.9) and define $\Delta g^* : [a, b] \rightarrow \mathbb{R}$ as

$$\Delta g^*(t) = \Delta g|_{[a,b]}(t^*).$$

Then, for $t \in [a, b]$:

1. If $t^* \in D_1 \cup D_2 \cup D_3$ and (3.12) holds, then Δg^* is g -differentiable at t .
2. If $t^* \notin D_1 \cup D_2 \cup D_3$, Δg^* is g -differentiable at t .

In particular, Δg^* is g -differentiable on $[a, b]$ if (3.14) holds. Furthermore, if $(\Delta g^*)_g'(t)$ exists, we have that

$$(\Delta g^*)_g'(t) = -\chi_{D_g}(t^*).$$

Since we have some conditions guaranteeing the Stieltjes differentiability of all the maps involved in (2.5), we can finally obtain a formula for the second Stieltjes derivative of the product of two functions, as presented in the following result.

Proposition 4.5. Consider the sets D_1, D_2, D_3 in (3.7)-(3.9) and let $t \in [a, b]$ and $f_1, f_2 : [a, b] \rightarrow \mathbb{F}$ be two times g -differentiable at t , then:

- If $t^* \in D_1 \cup D_2 \cup D_3$ and (3.12) holds, then $f_1 f_2$ is two times g -differentiable at t and

$$(f_1 f_2)_g''(t) = (f_1)_g''(t) f_2(t^*) + f_1(t^*) (f_2)_g''(t) + (2 - \chi_{D_g}(t^*)) (f_1)_g'(t) (f_2)_g'(t) + \Delta g(t^*) ((f_1)_g'(t) (f_2)_g''(t) + (f_2)_g'(t) (f_1)_g''(t)).$$

(4.9)

- If $t^* \notin D_1 \cup D_2 \cup D_3$ then $f_1 f_2$ is two times g -differentiable at t and (4.9) holds.

Proof. First, note that Proposition 2.6 ensures that $f_1 f_2$ is g -differentiable at t and

$$\begin{aligned} (f_1 f_2)'_g(t) &= (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t) (f_2)'_g(t) \Delta g(t^*) \\ &= (f_1)'_g(t) f_2^*(t) + (f_2)'_g(t) f_1^*(t) + (f_1)'_g(t) (f_2)'_g(t) \Delta g^*(t). \end{aligned}$$

Furthermore, since f_1 and f_2 are g -differentiable at t , Proposition 4.1 guarantees that f_1^* and f_2^* are also g -differentiable at t . Now, Corollary 4.4 ensures that Δg^* is g -differentiable under the corresponding conditions. Hence, Proposition 2.6 guarantees that $(f_1 f_2)'_g$ is g -differentiable at t and, differentiating the expression in (2.5), and denoting $\chi_{D_g}^*(t) = \chi_{D_g}(t^*)$, we have that

$$\begin{aligned} (f_1 f_2)''_g(t) &= (f_1)''_g(t) f_2^*(t) + (f_1)'_g(t) (f_2^*)'_g(t) + (f_1)''_g(t) (f_2^*)'_g(t) \Delta g^*(t) \\ &\quad + (f_1^*)'_g(t) (f_2)'_g(t) + f_1^*(t) (f_2)''_g(t) + (f_1^*)'_g(t) (f_2)''_g(t) \Delta g^*(t) \\ &\quad + ((f_1)''_g(t) (f_2)'_g(t) + (f_1)'_g(t) (f_2)''_g(t) + (f_1)''_g(t) (f_2)''_g(t) \Delta g^*(t)) \Delta g^*(t) \\ &\quad - ((f_1)''_g(t) (f_2)'_g(t) + (f_1)'_g(t) (f_2)''_g(t) + (f_1)''_g(t) (f_2)''_g(t) \Delta g^*(t)) \chi_{D_g}^*(t) \Delta g^*(t) \\ &\quad - (f_1)'_g(t) (f_2)'_g(t) \chi_{D_g}^*(t). \end{aligned}$$

Now, noting that, $\chi_{D_g}^*(t) \Delta g^*(t) = \Delta g^*(t)$ and $(f_i^*)'_g(t) = (f_i)'_g(t)$, $i = 1, 2$, it follows that

$$\begin{aligned} (f_1 f_2)''_g(t) &= (f_1)''_g(t) f_2^*(t) + (f_1)'_g(t) (f_2)'_g(t) + (f_1)'_g(t) (f_2)''_g(t) \Delta g^*(t) \\ &\quad + (f_1^*)'_g(t) (f_2)'_g(t) + f_1^*(t) (f_2)''_g(t) + (f_1^*)'_g(t) (f_2)''_g(t) \Delta g^*(t) \\ &\quad - (f_1)'_g(t) (f_2)'_g(t) \chi_{D_g}^*(t) \\ &= (f_1)''_g(t) f_2^*(t) + f_1^*(t) (f_2)''_g(t) + (2 - \chi_{D_g}^*(t)) (f_1)'_g(t) (f_2)'_g(t) \\ &\quad + \Delta g^*(t) ((f_1)'_g(t) (f_2)''_g(t) + (f_1)'_g(t) (f_2)''_g(t)), \end{aligned}$$

which finishes the proof. \square

Remark 4.6. Observe that (4.9) yields the usual expression of the second derivative of a product of two functions when $D_g = \emptyset$ as, in that case, $\chi_{D_g}(t^*) = \Delta g(t^*) = 0$ for all $t \in [a, b]$. In particular, this means that (4.9) is, in fact, a generalization of the corresponding expression in the setting of the usual derivative, which corresponds to $g = \text{Id}$.

This result is enough to shed some light upon the question of higher order derivatives of a product of two functions. Given the expression in (4.9), in order to have the product of two three-times differentiable functions be three-times differentiable, we would need the map $\chi_{D_g}^*$ to be g -differentiable. Given Corollary 4.4, this would imply that Δg^* would be two-times differentiable in the Stieltjes sense. However, this is not the case, as shown in the first example in Remark 3.7 (observe that in that case $C_g = \emptyset$ so $\Delta g^* = \Delta g$). This means that, in order to have g -differentiability for $\chi_{D_g}^*$, condition (3.12) is not enough and, thus, further conditions would be required to ensure the existence of higher order derivatives of a product of functions. To that end, we include the following result from which we will derive some information about the differentiability of χ_{D_g} and, as a consequence, of $\chi_{D_g}^*$.

Proposition 4.7. *Consider the sets*

$$(4.10) \quad \tilde{D}_1 = \{t \in [a, b] \cap N_g^- : t \notin (D_g \cap [a, t])'\},$$

$$(4.11) \quad \tilde{D}_2 = \{t \in [a, b] \cap (N_g^+ \cup D_g) : t \notin (D_g \cap (t, b])'\},$$

$$(4.12) \quad \tilde{D}_3 = \{t \in [a, b] \setminus (C_g \cup N_g \cup D_g) : t \notin (D_g \cap [a, b])'\},$$

and a map $h : [a, b] \rightarrow \mathbb{F}$. Given $t \in [a, b]$, the map $f : [a, b] \rightarrow \mathbb{F}$ defined as

$$f(t) = h(t)\chi_{D_g}(t), \quad t \in [a, b],$$

is g -differentiable at t if $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$ and

$$(4.13) \quad f'_g(t) = \begin{cases} 0, & \text{if } t^* \notin D_g, \\ -\frac{h(t^*)}{\Delta g(t^*)}, & \text{if } t^* \in D_g. \end{cases}$$

Similarly, $f^* : [a, b] \rightarrow \mathbb{F}$ defined as

$$f^*(t) = h(t^*)\chi_{D_g}(t^*), \quad t \in [a, b],$$

is g -differentiable at every $t \in [a, b]$ such that $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$ and, in that case,

$$(f^*)'_g(t) = \begin{cases} 0, & \text{if } t^* \notin D_g, \\ -\frac{h(t^*)}{\Delta g(t^*)}, & \text{if } t^* \in D_g. \end{cases}$$

Proof. Observe that, thanks to Proposition 4.1, it is enough to prove the result for f to obtain the desired property for f^* . Hence, we shall focus on showing that f can only be g -differentiable at those $t \in [a, b]$ such that $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$. Furthermore, given the definitions of t^* and the Stieltjes derivative, it is enough to show that, given $t \in [a, b] \setminus C_g$, f is g -differentiable at every $t \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$.

Let $t \in [a, b] \setminus C_g$. We shall study some cases separately.

First, suppose $t \in N_g^-$. In that case, $t \neq a$ and $t \notin D_g$, so $f(t) = 0$ and the g -differentiability of f depends on the existence of

$$\lim_{s \rightarrow t^-} \frac{f(s)}{g(s) - g(t)}.$$

Hence, if $t \in \tilde{D}_1$, we have that $t \notin (D_g \cap [a, t])'$, so there exists $r > 0$ such that $D_g \cap (t-r, t) = \emptyset$, which means that $f = 0$ on $(t-r, t)$. Thus, the previous limit is zero, and so f is g -differentiable at t and (4.13) holds.

Suppose now that $t \in N_g^+$. We have that $t \neq b$ and $f(t) = 0$, so f is g -differentiable at t if and only if the following limit exists:

$$\lim_{s \rightarrow t^+} \frac{f(s)}{g(s) - g(t)}.$$

If $t \in \tilde{D}_2$, then $t \notin (D_g \cap (t, b])'$, so we find $r > 0$ such that $D_g \cap (t, t+r) = \emptyset$, which guarantees that $f = 0$ on $(t, t+r)$ so the previous limit equals zero, i.e., f is g -differentiable at t and (4.13) holds.

Next, suppose $t \in [a, b] \setminus (D_g \cup N_g)$. Then $f(t) = 0$ so the g -differentiability of f comes from the existence of

$$\lim_{s \rightarrow t} \frac{f(s)}{g(s) - g(t)}.$$

If $t \in \tilde{D}_3$, it follows that $t \notin (D_g \cap [a, b])'$, so there is $r > 0$ such that $D_g \cap (t-r, t+r) \setminus \{t\} = \emptyset$. This means that $f = 0$ on $(t, t+r) \setminus \{t\}$, so the previous limit equals zero, that is, f is g -differentiable at t and (4.13) holds.

Finally, suppose $t \in D_g$. In this case, $t \neq b$ and $f(t) = h(t)$. Observe that, as pointed out in Remark 2.4, it is enough to check if $f(t^+)$ exists. If $t \in \tilde{D}_2$, we have that $t \notin (D_g \cap (t, b])'$,

so we find $r > 0$ such that $D_g \cap (t, t+r) = \emptyset$, which guarantees that $f = 0$ on $(t, t+r)$ and thus $f(t^+) = 0$. It follows now that f is g -differentiable at t and

$$f'_g(t) = \frac{f(t^+) - f(t)}{\Delta g(t)} = -\frac{h(t)}{\Delta g(t)},$$

so (4.13) holds. \square

Remark 4.8. To visualize the sets \tilde{D}_1 , \tilde{D}_2 and \tilde{D}_3 , we provide an illustration in Figure 4.3.

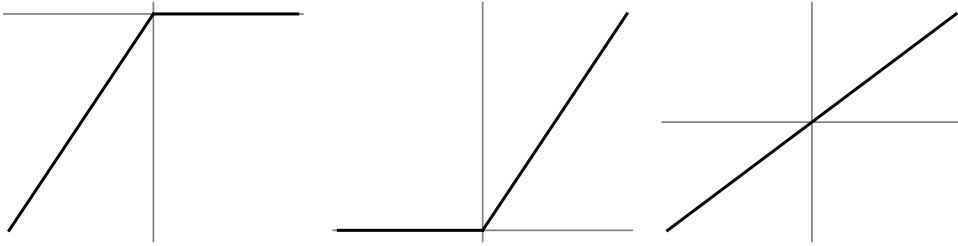


FIGURE 4.3. From left to right, representation of derivators for which zero belongs to \tilde{D}_1 , \tilde{D}_2 and \tilde{D}_3 respectively.

As a direct consequence, we have the following result characterizing in depth the g -differentiability of χ_{D_g} and $\chi_{D_g}^*$.

Corollary 4.9. Consider the sets \tilde{D}_1 , \tilde{D}_2 , \tilde{D}_3 in (4.10)-(4.12). The maps χ_{D_g} , $\chi_{D_g}^*$ are g -differentiable at every $t \in [a, b]$ such that $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$ and, for such points,

$$(\chi_{D_g})'_g(t) = (\chi_{D_g}^*)'_g(t) = \begin{cases} 0, & \text{if } t^* \notin D_g, \\ -(\Delta g(t))^{-1}, & \text{if } t^* \in D_g. \end{cases}$$

If $t \in [a, b]$ is such that $t^* \notin \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$, then χ_{D_g} is not g -differentiable at t .

Proof. The first part of the result follows directly from Proposition 4.7, so it only remains to show that χ_{D_g} fails to be g -differentiable for every $t \in [a, b]$ such that $t^* \notin \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$. To that end, it is enough to prove that it is not g -differentiable at every $t \in [a, b] \setminus C_g$ such that $t \notin \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$.

First, note that if $t \in [a, b] \setminus C_g$ such that $t \notin \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$ belongs to D_g , then necessarily $t \in (D_g \cap (t, b])'$ (for otherwise, it would belong to \tilde{D}_2) and, in that case, χ_{D_g} cannot be g -differentiable at t as $\chi_{D_g}(t^+)$ does not exist since

$$\lim_{\substack{s \rightarrow t^+ \\ s \in D_g}} \chi_{D_g}(s) = 1 \neq 0 = \lim_{\substack{s \rightarrow t^+ \\ s \notin D_g}} \chi_{D_g}(s).$$

Therefore, it remains to show that χ_{D_g} is not g -differentiable at every $t \in [a, b] \setminus (C_g \cup D_g)$ such that $t \notin \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$.

Let $t \in [a, b] \setminus (C_g \cup D_g)$ be such that $t \notin \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$. Note that, since $t \in [a, b] \setminus (C_g \cup D_g)$, we have that $t \in [a, b] \cap N_g^-$, $t \in [a, b] \cap N_g^+$ or $t \in [a, b] \setminus (C_g \cup N_g \cup D_g)$. Furthermore, $\chi_{D_g}(t) = 0$.

First, suppose $t \in [a, b] \cap N_g^-$. In that case, $t \in (D_g \cap [a, t])'$ (for otherwise, it would belong to \tilde{D}_1) and so χ_{D_g} is not g -differentiable at t since

$$(4.14) \quad \lim_{\substack{s \rightarrow t^- \\ s \in D_g}} \frac{\chi_{D_g}(s) - \chi_{D_g}(t)}{g(s) - g(t)} = \lim_{\substack{s \rightarrow t^- \\ s \in D_g}} \frac{\chi_{D_g}(s)}{g(s) - g(t)} = -\infty.$$

Next, suppose $t \in [a, b] \cap N_g^+$. Similarly, $t \in (D_g \cap (t, b])'$ (for otherwise, it would belong to \tilde{D}_2) and so χ_{D_g} is not g -differentiable at t since

$$(4.15) \quad \lim_{\substack{s \rightarrow t^+ \\ s \in D_g}} \frac{\chi_{D_g}(s) - \chi_{D_g}(t)}{g(s) - g(t)} = \lim_{\substack{s \rightarrow t^+ \\ s \in D_g}} \frac{\chi_{D_g}(s)}{g(s) - g(t)} = +\infty.$$

Finally, suppose $t \in [a, b] \setminus (C_g \cup N_g \cup D_g)$. Then $t \in (D_g \cap [a, b])'$ (for otherwise, it would belong to \tilde{D}_3) and thus χ_{D_g} is not g -differentiable at t as (4.14) and/or (4.15) hold. \square

Remark 4.10. Observe that we do not discuss the g -differentiability of $\chi_{D_g}^*$ beyond the set $\tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$. This is because if $C_g = \emptyset$, $\chi_{D_g}^* = \chi_{D_g}$ which is not g -differentiable outside of the mentioned set. Thus, we limit ourselves to the set in which we can guarantee differentiability for all possible nondecreasing and left-continuous functions g .

We are now in position to obtain the formula for the third derivative of a product of two three-times g -differentiable functions. To that end, we first need to note that the set D_1, D_2, D_3 in (3.7)-(3.9), and the sets $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3$ in (4.10)-(4.12) satisfy that $D_i \cap \tilde{D}_i = \emptyset$, $i = 1, 2, 3$. With these relations, we can establish the following result.

Proposition 4.11. *Let $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3$ be as in (4.10)-(4.12) and $f_1, f_2 : [a, b] \rightarrow \mathbb{F}$ be three times g -differentiable at $t \in [a, b]$ such that $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$. Then $f_1 f_2$ is three times g -differentiable at t and*

$$(4.16) \quad \begin{aligned} (f_1 f_2)_g'''(t) &= (f_1)_g'''(t) f_2(t^*) + f_1(t^*) (f_2)_g'''(t) + (3 - \chi_{D_g}(t^*)) ((f_1)'_g(t) (f_2)''_g(t) \\ &\quad + (f_2)'_g(t) (f_1)''_g(t)) + \Delta g(t^*) ((f_1)'''_g(t) (f_2)'_g(t) \\ &\quad + 2(f_1)''_g(t) (f_2)''_g(t) + (f_1)'_g(t) (f_2)'''_g(t)) + Q(t^*) (f_1)'_g(t) (f_2)'_g(t) \end{aligned}$$

with

$$(4.17) \quad Q(t) = \begin{cases} 0, & \text{if } t \notin D_g, \\ (\Delta g(t))^{-1}, & \text{if } t \in D_g. \end{cases}$$

Proof. Let $t \in [a, b]$ be such that $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$. Then, $t^* \notin D_1 \cup D_2 \cup D_3$ for D_1, D_2, D_3 as in (3.7)-(3.9), so Proposition 4.5 ensures that $f_1 f_2$ is two times g -differentiable at t and $(f_1 f_2)_g''(t) = \alpha(t) + \beta(t) + \gamma(t)$ with

$$\begin{aligned} \alpha(t) &:= (f_1)''_g(t) f_2^*(t) + f_1^*(t) (f_2)''_g(t), & t \in [a, b], \\ \beta(t) &:= (2 - \chi_{D_g}^*(t)) (f_1)'_g(t) (f_2)'_g(t), & t \in [a, b], \\ \gamma(t) &:= \Delta g^*(t) ((f_1)'_g(t) (f_2)''_g(t) + (f_2)'_g(t) (f_1)''_g(t)), & t \in [a, b]. \end{aligned}$$

Now, Proposition 4.1 ensures that f_1^* and f_2^* are g -differentiable at t ; whereas Corollary 4.4 and Corollary 4.9 guarantee the same property for Δg^* and $\chi_{D_g}^*$, so it follows from Proposition 2.6 that $f_1 f_2$ is three times g -differentiable at t . Hence, we need to check that (4.16) holds.

First, using (2.5) it is easy to see that

$$\begin{aligned} \alpha'_g(t) &= (f_1)'''_g(t)f_2(t^*) + f_1(t^*)(f_2)'''_g(t) + (f_1)''_g(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)''_g(t) \\ &\quad + \Delta g(t^*)((f_1)'''_g(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)'''_g(t)). \end{aligned}$$

Next, it follows from Propositions 3.4 and 4.1 that

$$\gamma'_g(t) = -\chi_{D_g}(t^*)((f_1)'_g(t)(f_2)''_g(t) + (f_2)'_g(t)(f_1)''_g(t)).$$

Finally, Corollary 4.9 and (2.5) ensure that

$$\begin{aligned} \beta'_g(t) &= Q(t^*)(f_1)'_g(t)(f_2)'_g(t) \\ &\quad + (2 - \chi_{D_g}(t^*)((f_1)''_g(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)''_g(t) + (f_1)''_g(t)(f_2)''_g(t)\Delta g(t^*))) \\ &\quad + Q(t^*)((f_1)''_g(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)''_g(t) + (f_1)''_g(t)(f_2)''_g(t)\Delta g(t^*))\Delta g(t^*) \\ &= Q(t^*)(f_1)'_g(t)(f_2)'_g(t) \\ &\quad + 2((f_1)''_g(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)''_g(t) + (f_1)''_g(t)(f_2)''_g(t)\Delta g(t^*)), \end{aligned}$$

where the last equality follows from the fact that $Q(t^*)\Delta g(t^*) = \chi_{D_g}^*(t)$, $t \in [a, b]$. Thus, by the linearity of the Stieltjes derivative, $(f_1 f_2)'''_g(t) = \alpha'_g(t) + \beta'_g(t) + \gamma'_g(t)$ so basic computations yield (4.16). \square

Given (4.16), it is easy to see that the existence of derivatives of a product of order higher than three depends on the Stieltjes derivative of the map Q in (4.17). Note that Proposition 4.7 shows that Q is g -differentiable at t whenever $t^* \in \tilde{D}_1 \cup \tilde{D}_2 \cup \tilde{D}_3$ and, furthermore, its derivative is another function the same characteristics. Hence, restricting ourselves to such points, it follows that the existence of higher order derivatives of the product of two functions depends exclusively on the order of differentiability of the functions in the product. Observe that this happens, in particular, if $D'_g = \emptyset$, which gives an easy condition for the product rule to preserve the order of differentiability.

5. REGULARITY OF THE PRODUCT

In this final section, we shall focus on the issues with the regularity of the product of two functions that arise as a consequence of the product rule, (2.5). Specifically, we aim to find minimal conditions that ensure that the product of two functions in $\mathcal{BC}_g^1([a, b], \mathbb{F})$ remains in such set. As a consequence, throughout this section, we shall assume that $a, b \in \mathbb{R}$, $a < b$, are such that $a \notin N_g^- \cup D_g$ and $b \notin C_g \cup N_g^+ \cup D_g$ as these are the conditions under which the set $\mathcal{BC}_g^1([a, b], \mathbb{F})$ is defined.

Given $f_1, f_2 \in \mathcal{BC}_g^1([a, b], \mathbb{F})$ and (2.5), it is clear that the term preventing their product from being another function in $\mathcal{BC}_g^1([a, b], \mathbb{F})$ is $(f_1)'_g(f_2)'_g\Delta g^*$. In particular, it is the g -continuity of Δg^* that is interfering with the regularity. Thus, we start this section by studying the set of points in which Δg^* is g -continuous, which we do through the following result that connects the g -continuity of a function, f , with the g -continuity of the modified map f^* .

Proposition 5.1. *If $f : [a, b] \rightarrow \mathbb{F}$ is g -continuous at $t \in [a, b]$, then f^* is also g -continuous at t .*

Proof. We prove the result in terms of the sequential formulation of the g -continuity, (2.7).

Let $\{t_n\}_{n \in \mathbb{N}} \subset [a, b]$ be such that $g(t_n) \rightarrow g(t)$. Since $g(s) = g(s^*)$ for all $s \in \mathbb{R}$, it follows that $g(t_n^*) \rightarrow g(t^*)$. Now, Lemma 2.15 ensures that f is g -continuous at t^* , so it follows

that $f(t_n^*) \rightarrow f(t^*)$, i.e., $f^*(t_n) \rightarrow f^*(t)$. Since the sequence was arbitrarily chosen, we have that f^* is g -continuous at t . \square

Remark 5.2. The converse of Proposition 5.1 is not necessarily true. Indeed, consider the map

$$g(t) = \begin{cases} t, & \text{if } t \in (-\infty, 0] \cup (2, +\infty), \\ 1, & \text{if } t \in (0, 2] \end{cases}$$

In that case, denoting $f(t) = \Delta g|_{[-1,3]}(t)$, we have that

$$f(t) = \begin{cases} 1, & \text{if } t \in \{0, 2\}, \\ 0, & \text{if } t \in [-1, 3] \setminus \{0, 2\}, \end{cases} \quad f^*(t) = \begin{cases} 1, & \text{if } t \in [0, 2] \\ 0, & \text{if } t \in [-1, 0) \cup (2, 3]. \end{cases}$$

Observe that Proposition 3.2 ensures that f is not g -continuous at 1. However, f^* is g -continuous at 1 as, given $\varepsilon > 0$, taking $\delta \in (0, 1)$, we have that if $s \in [-1, 3]$ is such that $|g(s) - g(1)| < \delta$ then, necessarily, $s \in (0, 2]$, in which case $|f^*(s) - f^*(1)| = 0 < \varepsilon$.

The following result provides a partial converse to Proposition 5.1.

Proposition 5.3. *Let $f : [a, b] \rightarrow \mathbb{F}$ and assume f^* is g -continuous on $[a, b]$. Then f is g -continuous on $[a, b]$ if and only if $f(t) = f(s)$ for every $t, s \in [a, b]$ such that $g(t) = g(s)$.*

Proof. Observe that Lemma 2.15 ensures that if f is g -continuous at t and $g(t) = g(s)$ then $f(t) = f(s)$, so if f is g -continuous on $[a, b]$, we have that $f(t) = f(s)$ for every $t, s \in [a, b]$ such that $g(t) = g(s)$.

Conversely, suppose that $f(t) = f(s)$ for every $t, s \in [a, b]$ such that $g(t) = g(s)$. We claim that $f = f^*$. Indeed, let $s \in [a, b]$. If $s \notin C_g$, we have that $f(s) = f^*(s)$ as $s = s^*$. Otherwise, $s \in C_g$ and we have that $g(s) = g(s^*)$ since g is left-continuous, which implies that $f(s) = f(s^*) = f^*(s)$. Since $f = f^*$ and f^* is g -continuous, the result follows. \square

Remark 5.4. It might be tempting to state Proposition 5.3 in a pointwise fashion, but this does not work as the example in Remark 4.3 shows. The map f there satisfies that $f(t) = f(0) = 0$ for every $t \in \mathbb{R}$ such that $\tilde{g}(t) = \tilde{g}(0)$ (that is, $t \leq 0$), but it is not \tilde{g} -continuous at 0 as the sequence $\{t_n\}_{n \in \mathbb{N}} = \left\{ \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n+1} \right) \right\}_{n \in \mathbb{N}}$ is such that $\tilde{g}(t_n) \rightarrow 0 = \tilde{g}(0)$ but $f(t_n) \rightarrow 1 \neq 0 = f(0)$. However, $f^* = 0$ is clearly \tilde{g} -continuous at 0.

Note that, unfortunately, Proposition 5.3 does not apply in the context of Δg and Δg^* . However, combining the information in Propositions 3.2 and 5.1, we can obtain the following result about the g -continuity of Δg^* .

Proposition 5.5. *Consider the set A_g in (3.2) and*

$$H_g = \{t \in \mathbb{R} : t \in (s_1, s_2] \text{ for some } s_1, s_2 \in D_g \text{ such that } (s_1, s_2) \subset C_g\}.$$

Then, the restriction $\Delta g^|_{[a,b]}$ is g -continuous at every point of $((a^*, b] \setminus A_g) \cup (H_g \cap [a, b])$ and g -discontinuous at every point of $((a^*, b) \cap A_g) \setminus H_g$. Furthermore, $\Delta g^*|_{[a,b]}$ is g -continuous on $[a, a^*]$, where we are also considering the case where $[a, a^*]$ is a degenerate interval, i.e., $[a, a^*] = \{a\}$.*

Proof. Given Propositions 3.2 and 5.1, it is clear that $\Delta g^*|_{[a,b]}$ is g -continuous at every $t \in (a^*, b) \setminus A_g$. Now, given that $b \notin C_g \cup N_g^+ \cup D_g$, it follows that $b \notin A_g$, so Propositions 3.2 and 5.1 are again enough to guarantee the g -continuity at b .

Consider $t \in (a^*, b) \cap H_g$ and let us show that $\Delta g^*|_{[a,b]}$ is g -continuous at t . Let $\varepsilon > 0$. Since $t \in (a^*, b) \cap H_g$ and $b \notin D_g$, there exist $s_1, s_2 \in D_g \cap [a^*, b)$ such that $t \in (s_1, s_2)$ and $(s_1, s_2) \subset C_g$. Taking $\delta = \min\{\Delta g(s_1), \Delta g(s_2)\} > 0$, we have that, if $s \in \mathbb{R}$ is such that $|g(s) - g(t)| < \delta$, then $s \in (s_1, s_2]$, so

$$|\Delta g^*(s) - \Delta g^*(t)| = |\Delta g(s^*) - \Delta g(t^*)| = |\Delta g(s_2) - \Delta g(s_2)| = 0 < \varepsilon,$$

i.e. $\Delta g^*|_{[a,b]}$ is g -continuous at t .

Next, we show that $\Delta g^*|_{[a,b]}$ is g -discontinuous at every point of $((a^*, b) \cap A_g) \setminus H_g$, distinguishing between the points that belong to D_g , from those which do not.

First, let $t \in ((a^*, b) \cap A_g) \setminus H_g$ be such that $t \in D_g$. In this case, $\Delta g^*(t) = \Delta g(t)$. Proceeding in a similar fashion to the proof of Proposition 3.2, we can have that either (2.2) or (3.4) holds.

If (2.2) holds, then, necessarily, $s \leq s^* < t = t^*$ for every $s < t$. This implies that, given a sequence $\{t_n\}_{n \in \mathbb{N}}$ converging to t such that $t_n < t$, $n \in \mathbb{N}$, the sequence $\{t_n^*\}_{n \in \mathbb{N}}$ also converges to t and is such that $t_n^* < t$, $n \in \mathbb{N}$, so Proposition 3.1 guarantees that

$$\lim_{n \rightarrow \infty} \Delta g^*(t_n) = \lim_{n \rightarrow \infty} \Delta g(t_n^*) = 0.$$

Since the sequence $\{t_n\}_{n \in \mathbb{N}}$ was arbitrarily chosen, $\lim_{s \rightarrow t^-} \Delta g^*(s) = 0$, so we can find $\rho \in (0, t - a)$ such that

$$\Delta g^*(s) = \Delta g(s^*) < \frac{\Delta g(t)}{2}, \quad s \in \mathbb{R}, \quad 0 < t - s < \rho.$$

Since (2.2) holds, so we can find $t_1 < t$ such that $t - t_1^* < \rho$ and $g(t_1^*) < g(t)$. Let $\delta = g(t) - g(t_1^*)$. Observe that, if $0 < g(t) - g(s) < \delta$, since g is nondecreasing, then $0 < t - s < t - t_1^* < \rho$, and so $\Delta g^*(s) < \Delta g(t)/2$. But this implies that, if $0 < g(t) - g(s) < \delta$, then

$$|\Delta g^*(s) - \Delta g^*(t)| \geq \Delta g^*(t) - \Delta g^*(s) = \Delta g(t) - \Delta g^*(s) > \Delta g(t) - \frac{\Delta g(t)}{2} = \frac{\Delta g(t)}{2},$$

and, therefore, Δg^* is not g -continuous at t .

On the other hand, if (3.4) holds, let us consider

$$\tilde{t} = \inf\{s \in [a^*, b] : g(s) = g(t)\}.$$

Observe that $\tilde{t} \notin D_g$, since $t \notin H_g$, and $\tilde{t} > a^*$, given that $a^* \in N_g^+ \cup D_g$ by definition. Furthermore, we also have that $g(s) \leq g(\tilde{t}) \leq g(t)$ for every $s < t$ and $g(s) = g(t)$ for every $s \in [\tilde{t}, t]$. Now, reasoning as in the previous case, we find $\delta \in (0, t - a)$ such that, if $0 < g(\tilde{t}) - g(s) < \delta$, then $\Delta g^*(s) < \Delta g(t)/2$, which, again, is enough to show that $\Delta g^*|_{[a,b]}$ is not g -continuous at t .

Consider now $t \in ((a^*, b) \cap A_g) \setminus H_g$ such that $t \notin D_g$ and consider \hat{t} as in (3.3). Since $t \in A_g$, we have that $\hat{t} \in D_g$ so, given that $t, b \notin D_g$, it follows that $t < \hat{t} < b$. In particular, this implies that $\hat{t} \in ((a^*, b) \cap A_g) \setminus H_g$ and $\hat{t} \in D_g$, so we already know $\Delta g^*|_{[a,b]}$ is not g -continuous at \hat{t} . Observe that, since g is left-continuous, we have that $g(t) = g(\hat{t})$, so Lemma 2.15 is enough to ensure that $\Delta g^*|_{[a,b]}$ is not g -continuous at t , which finishes the proof of the first part of the result.

Let us focus now on the behaviour of $\Delta g^*|_{[a,b]}$ on $[a, a^*]$. Observe that it is enough to study the g -continuity at a^* . Indeed, this is trivial when $a^* = a$. When $a^* > a$, it is enough to note that, in that case, $a \in C_g$, which means that

$$g(t) = g(a^*), \quad \Delta g^*(t) = \Delta g^*(a) = \Delta g^*(a^*), \quad t \in [a, a^*],$$

so the g -continuity at $t \in [a, a^*]$ can always be deduced from the g -continuity at a^* by Lemma 2.15.

First, suppose that $a^* \in D_g$. Given $\varepsilon > 0$, take $\delta \in (0, \Delta g(a^*))$. In that case, if $t \in [a, b]$ is such that $|g(t) - g(a^*)| < \delta$ we necessarily have that $t \in [a, a^*]$, so

$$|\Delta g^*(t) - \Delta g^*(a^*)| = 0 < \varepsilon.$$

This means that $\Delta g^*|_{[a,b]}$ is g -continuous at a^* and, thus, g -continuous on $[a, a^*]$.

Finally, if $a^* \notin D_g$, then $a^* \notin A_g$ so Δg is g -continuous at a^* . Thus, by Proposition 5.1, $\Delta g^*|_{[a,b]}$ is g -continuous at a^* which, again, is enough to obtain that it is g -continuous on $[a, a^*]$. \square

Remark 5.6. To visualize the kind of points in H_g , observe that in Example 3.3, $A_g = \{-1\} \cup [0, 2]$, $H_g = (0, 2]$ and $A_g \setminus H_g = \{-1, 0\}$.

We are now finally in position to determine under which conditions the product of two functions in $\mathcal{BC}_g^1([a, b], \mathbb{F})$ remains in this set. This is the information in the next result.

Theorem 5.7. *Let $f_1, f_2 \in \mathcal{BC}_g^1([a, b], \mathbb{F})$. Then, $f_1 f_2 \in \mathcal{BC}_g^1([a, b], \mathbb{F})$ if and only if*

$$(5.1) \quad (f_1)'_g(t)(f_2)'_g(t) = 0, \quad \text{for all } t \in ((a^*, b) \cap A_g) \setminus H_g.$$

Proof. First, observe that, since $f_1, f_2 \in \mathcal{BC}_g^1([a, b], \mathbb{F})$, Proposition 2.5 ensures that

$$(f_1 f_2)'_g(t) = (f_1)'_g(t)f_2(t^*) + (f_2)'_g(t)f_1(t^*) + (f_1)'_g(t)(f_2)'_g(t)\Delta g(t^*), \quad t \in [a, b],$$

so, given the definition of t^* and Proposition 2.13, we have that

$$(5.2) \quad (f_1 f_2)'_g(t) = (f_1)'_g(t)f_2(t) + (f_2)'_g(t)f_1(t) + (f_1)'_g(t)(f_2)'_g(t)\Delta g(t^*), \quad t \in [a, b].$$

Hence, since all the functions involved in this expression of $(f_1 f_2)'_g$ are bounded, it is clear that we only need to concern ourselves with its g -continuity. Furthermore, Proposition 5.5 guarantees that $\Delta g^*|_{[a,b]}$ is g -continuous on $[a, b] \setminus (((a^*, b) \cap A_g) \setminus H_g)$, so it follows that $(f_1 f_2)'_g$ is g -continuous on that set. As a consequence, it is enough to show that (5.1) holds if and only if

$$(5.3) \quad (f_1 f_2)'_g \text{ is } g\text{-continuous at every } t \in ((a^*, b) \cap A_g) \setminus H_g.$$

First, suppose (5.3) holds. Reasoning by contradiction, suppose (5.1) did not hold. In that case, we would find $t \in ((a^*, b) \cap A_g) \setminus H_g$ such that

$$(f_1)'_g(t)(f_2)'_g(t) \neq 0.$$

Since $(f_1)'_g$ and $(f_2)'_g$ are g -continuous by hypothesis, we can find $\delta > 0$ such that

$$(f_1)'_g(s)(f_2)'_g(s) \neq 0, \quad s \in I := \{r \in [a, b], |g(r) - g(t)| < \delta\}.$$

Thus, given (5.2), we have that

$$\Delta g^*(s) = \frac{(f_1 f_2)'_g(s) - (f_1)'_g(s)f_2(s) - (f_2)'_g(s)f_1(s)}{(f_1)'_g(s)(f_2)'_g(s)}, \quad s \in I.$$

Everything on the right hand side of the equation is g -continuous at t , so, thanks to Lemma 2.14, we conclude that $\Delta g^*|_{[a,b]}$ is g -continuous at t , which contradicts Proposition 5.5.

Conversely, suppose (5.1) holds. Consider $t \in ((a^*, b) \cap A_g) \setminus H_g$ and let us show that $(f_1 f_2)'_g$ is g -continuous at t . Given (5.2), it is clear that it is enough to prove that $h := (f_1)'_g (f_2)'_g \Delta g^*$ is g -continuous at t .

By hypothesis, $h(t) = 0$. Assume, by contradiction, that h is not g -continuous at t . Then there exists $\varepsilon_0 > 0$ and a sequence $\{t_n\}_{n \in \mathbb{N}} \subset [a, b]$ such that $g(t_n) \rightarrow g(t)$ and $|h(t_n)| \geq \varepsilon_0$ for every $n \in \mathbb{N}$. Observe that this fact, together with (5.1), imply that $\{t_n\}_{n \in \mathbb{N}} \subset [a, b] \setminus (((a^*, b) \cap A_g) \setminus H_g)$. Furthermore, since $t^* \notin D_g$ for any $t \notin A_g$, we have that $\Delta g^*|_{[a, b]} = 0$ on $[a, b] \setminus A_g$, so

$$\{t_n\}_{n \in \mathbb{N}} \subset A_g \setminus (((a^*, b) \cap A_g) \setminus H_g) = [a, a^*] \cup H_g.$$

Now, since $t \in ((a^*, b) \cap A_g) \setminus H_g$, for any $\delta \in (0, g(t) - g(a^*))$ we have that

$$|g(t) - g(s)| = g(t) - g(s) \geq g(t) - g(a^*) > \delta, \quad s \in [a, a^*].$$

Therefore, since $g(t_n) \rightarrow g(t)$, we conclude that $t_n \in H_g$ for n sufficiently big, so we can assume without loss of generality that $\{t_n\}_{n \in \mathbb{N}} \subset H_g$. Furthermore, since $h(t_n) = h(t_n^*)$ and $g(t_n) = g(t_n^*)$ for all $n \in \mathbb{N}$, we can also assume that $t_n = t_n^*$ and, thus, $\{t_n\}_{n \in \mathbb{N}} \subset D_g$. Now, we necessarily have that there are infinitely many distinct terms in $\{t_n\}_{n \in \mathbb{N}}$ for, otherwise, the sequence would be eventually constant which would imply that $t \in H_g$, which we know is not the case. Finally, since $f_1, f_2 \in \mathcal{BC}_g^1([a, b], \mathbb{F})$, there exists $M > 0$ such that $|(f_1)'_g(t_n)|, |(f_2)'_g(t_n)| \leq M, n \in \mathbb{N}$. Therefore,

$$\varepsilon_0 \leq |h(t_n)| = |(f_1)'_g(t_n)(f_2)'_g(t_n)\Delta g(t_n)| \leq M^2 \Delta g(t_n), \quad n \in \mathbb{N},$$

that is, $\Delta g(t_n) \geq \varepsilon/M^2$ for every $n \in \mathbb{N}$. This means that there are infinitely many points $s \in [a, b] \cap D_g$ such that $\Delta g(s) \geq \varepsilon/M^2$, which is impossible since

$$0 \leq \sum_{s \in [a, b] \cap D_g} \Delta g(s) = \int_{[a, b] \cap D_g} dg(s) \leq \int_{[a, b]} dg(s) = \mu_g([a, b]) = g(b) - g(a) < \infty. \quad \square$$

Remark 5.8. Observe that Theorem 5.7 generalizes, in some sense, [4, Proposition 3.17], where a sufficient condition for the continuity of the derivative of the product was used: that one of the functions involved was continuous in the usual sense. Observe that if f_1 is continuous this automatically implies that $(f_1)'_g(t) = 0$ for all $t \in [a, b] \cap D_g$, in which case (5.2) becomes the usual product rule formula. Theorem 5.7, on the other hand, provides a necessary and sufficient condition for the product of two $\mathcal{BC}_g^1([a, b], \mathbb{F})$ functions to remain in that set without requiring a simpler version of the product rule.

Remark 5.9. Observe that, under the conditions of Theorem 5.7, we have that

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + (f_2)'_g(t) f_1(t),$$

for $t \in ([a, a^*] \setminus A_g) \cup [(a^*, b) \setminus H_g]$. Furthermore, if $f_1 \cdot f_2$ is two times g -differentiable at t (that is, under the hypotheses of Proposition 4.5), then, by (4.9),

$$(f_1 f_2)''_g(t) = (f_1)''_g(t) f_2(t^*) + f_1(t^*) (f_2)''_g(t) + \Delta g(t^*) ((f_1)'_g(t) (f_2)''_g(t) + (f_2)'_g(t) (f_1)''_g(t))$$

for $t \in ((a^*, b) \cap A_g) \setminus H_g$. This can serve as a basis to prove a characterization of a product of two class two functions being in the same class similar to Theorem 5.7.

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¹ DEPARTAMENTO DE ESTADÍSTICA, ANÁLISE MATEMÁTICA E OPTIMIZACIÓN, UNIVERSIDADE DE SANTIAGO DE COMPOSTELA, 15782, FACULDADE DE MATEMÁTICAS, SANTIAGO DE COMPOSTELA, SPAIN.

² CITMAGA, 15782, SANTIAGO DE COMPOSTELA, SPAIN.

³ INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC.

* FJAVIER.FERNANDEZ@USC.ES

† MARQUEZ@MATH.CAS.CZ

‡ FERNANDOADRIAN.FERNANDEZ@USC.ES