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On the error estimates for the sequence of successive approximations for cyclic φ–contractions in metric spaces

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ABSTRACT. In this paper, in the setting of metric spaces, we introduce the notion of strongly cyclic Sehgal type φ-contraction of type one as generalization of the notions of cyclic φ-contraction map in the sense of Păcurar-Rus and cyclic contraction map in the sense of Suzuki-Kikawa-Vetro. Then we study the existence and uniqueness of the best proximity points for such mappings by using the WUC property. In the following, while presenting an algorithm to determine the best proximity points, we also find a priori and a posteriori error estimates of the best proximity point for this algorithm associated with a strongly cyclic Sehgal type φ-contraction of type one, which is defined on a uniformly convex Banach space with a modulus of convexity of power type. Also, we give a positive answer to Zlatanov's question ['Error estimates for approximating best proximity points for cyclic contractive maps', Carpathian J. Math. 32(2) (2016), 265-270] on error estimates for the sequence of successive approximations for cyclic φ-contraction maps in the sense of Păcurar-Rus. As an important result, we obtain a generalization of Ćirić's Theorem, which itself is a generalization of the Banach contraction principle in a particular case.

1. INTRODUCTION

Fixed point theory is an important tool to solve equation $Tx = x$ for mappings T defined on subsets of metric or normed spaces. The possibility of estimating the error of successive approximations and its convergence rate is one of the strengths of the fixed point theory. There are equations $Tx = x$ for which it is not easy or even impossible to find the exact solution. The error estimate is very useful in these cases. An extensive study about approximations of fixed points for self maps can be found in [2,3]. In 2010, Păcurar and Rus [11], in their main result, in the case that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function [2], proved the existence and uniqueness of the fixed point and found a priori and a posteriori error estimates of the fixed point of a cyclic φ-contraction [11].

In [6], the concept of a fixed point was extended to the best proximity point for cyclic maps. The problem of the existence of a best proximity point of cyclic mappings, have been extensively studied by many authors; see for instance [7, 12, 14, 16–20] and references therein. Therefore, it is important and necessary to estimate the error of successive approximations and the rate of convergence.

In 2016, Zlatanov [19] found a priori and a posteriori error estimates for approximation best proximity point associated to a cyclic contraction map, which is defined on a uniformly convex Banach space with modulus of convexity of power type. He posed a question on the possibility of calculating the error estimates for the sequence of successive approximations for for cyclic φ-contractions [11], in the case that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is only a comparison function [13] (i.e., φ is increasing function and the sequence $\{\varphi^n(t)\}$

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converges to 0 as $n \to \infty$, for all $t \in \mathbb{R}_+$). Note that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function, then φ is a comparison function, but the converse is not necessarily true [11].

In this work, we generalized the notion of cyclic φ-contraction maps in the sense of Păcurar-Rus to strongly cyclic Sehgal type φ-contraction on distinct subsets of a metric space (X, d), where the distance between them is not necessarily equal to zero. Therefore, we have the more general concept best proximity points for these mappings, so we study the existence and uniqueness of best proximity points for such mappings in the metric spaces. We also find a priori and a posteriori error estimates of the best proximity point for our proposed algorithm associated with a strongly cyclic Sehgal type φ-contraction of type one, which is defined on two distinct subsets of a uniformly convex Banach space with a modulus of convexity of power type. As consequence of our main results, we give a positive answer to Zlatanov's question [19] in the metric spaces and a generalization of Ćirić's Theorem [4].

2. Preliminaries

Let (X, d) be a metric space. Define a distance between two subset $A, B \subseteq X$ by $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. A self mapping $T : A \cup B \to A \cup B$ is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x^* \in A \cup B$ is called a best proximity point for X if $d(x^*, Tx^*) = d(A, B)$. If $d(A, B) = 0$, the above problems are equivalent to find a fixed point of X.

Definition 2.1. *[7,16] Let A and B be nonempty subsets of the metric space (X, d). Then (A, B) is said to satisfies*

(i) *the property UC, if it can be concluded from relation*

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n) = d(A, B),$$

for sequences $\{x_n\}$ and $\{x'_n\}$ in A and $\{y_n\}$ in B, that $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

(ii) *the property WUC, if for any $\{x_n\} \subseteq A$ such that for every $\epsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \le d(A, B) + \epsilon$ for $n \ge n_0$, then it can be conclude that $\{x_n\}$ is Cauchy.*

If A and B be nonempty subsets of a metric space (X, d) such that $d(A, B) = 0$, then (A, B) satisfies the property UC [16]. In 2011, Espínola and Fernández-León [7], proved that if A and B are nonempty subsets of the metric space (X, d) such that A is complete and the pair (A, B) has the property UC, then (A, B) has the property WUC.

Definition 2.2. *[11] Let $A_1, A_2, \cdots, A_k$ be nonempty subsets of the metric space (X, d) and let T be a cyclic mapping on $\bigcup_{i=1}^k A_i$, that is*

$$T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, \cdots, T(A_k) \subseteq A_1.$$

If there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tu, Tv) \le \varphi(d(u, v)),$$

for all $u \in A_i$ and $v \in A_{i+1}$, where $1 \le i \le k$ and $A_{k+1} := A_1$, then T is a cyclic φ-contraction on $\bigcup_{i=1}^k A_i$.

Definition 2.3. [9] *The modulus of convexity of a Banach space X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by*

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{u+v}{2}\| : \|u\| \le 1, \|v\| \le 1, \|u - v\| \ge \epsilon\}.$$

The norm is called uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$. The space $(X, \|.\|)$ is called uniformly convex Banach space.

Suzuki et al. [16] proved that if A and B are nonempty subsets of a uniformly convex Banach space X such that A is convex, then (A, B) has the property UC.

Remark 2.1. [9] *Let $(X, \|.\|)$ be a uniformly convex Banach space. Then for all $u, v, p \in X$, $R > 0$ and $r \in [0, 2R]$ satisfying that $\|u - p\| \leq R$, $\|v - p\| \leq R$ and $\|u - v\| \geq r$; the following implication holds:*

$$\|\frac{u+v}{2} - p\| \leq \left(1 - \delta_X(\frac{r}{R})\right)R. \tag{2.1}$$

If $(X, \|.\|)$ is a uniformly convex Banach space, then $\delta_X(\epsilon)$ is strictly increasing function, so there exists the inverse function δ_X^{-1} of the modulus of convexity. If the inequality $\delta_X(\epsilon) \geq C\epsilon^q$ holds for some $C > 0$ and $q > 0$ and every $\epsilon \in (0, 2]$, we say that the modulus of convexity is of power type p. For example, in [10] the authors proved that if $p \geq 2$ the modulus of convexity with respect to the canonical norm $\|.\|_p$ in l_p or L_p, is of power type p and if $p \in (1, 2)$ is of power type 2. For more details about the geometry of Banach spaces, see references [1,5,8].

3. STRONGLY CYCLIC SEHGAL TYPE φ-CONTRACTIONS ON $A \cup B$ OF TYPE ONE

We start this section with the following lemma.

Lemma 3.1. *Let I be an identity function defined on $\mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function, then we have*

(i) $\varphi(s) < s$, *for all* $s > 0$;
(ii) $\varphi(0) = 0$;

in addition, if $I - \varphi$ is a strictly increasing function, then we have

(iii) *φ is continuous.*

Proof. (i) If $s > 0$ and $\varphi(s) \geq s$ then $\varphi^n(s) \geq \varphi^{n-1}(s) \geq \cdots \geq \varphi^2(s) \geq \varphi(s) \geq s > 0$, for every $n \in \mathbb{N}$. So $\{\varphi^n(s)\} \nrightarrow 0$ that is a contradiction. (ii) If $\varphi(0) = s > 0$ then $0 < s = \varphi(0) \leq \varphi(\frac{s}{2}) < \frac{s}{2}$ which is impossible. To prove (iii), let $0 \leq s_1 < s_2$. Since $I - \varphi$ is strictly increasing, we get $s_1 - \varphi(s_1) < s_2 - \varphi(s_2)$ and so $\varphi(s_2) - \varphi(s_1) < s_2 - s_1$. Hence φ is continuous. □

There are several conditions for the comparison function φ that have been considered in the studied articles; see for instance [3,11] and references therein. In order to obtain some information about the convergence of the Picard iteration $\{x_n\}$ in this paper, according to the previous lemma, we only mention the condition $I - \varphi$ is a strictly increasing function.

Definition 3.4. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a conditional comparison function if it is a comparison function and $I - \varphi$ is a strictly increasing function.*

Given nonempty subsets A and B of a metric space (X, d), we set $d^*(u, v) := d(u, v) - d(A, B)$ for every $u, v \in X$. It is immediately that

$$d^*(u, v) \leq d(u, z) + d^*(z, v)$$

and

$$d^*(u, v) - d(A, B) \leq d^*(u, z) + d^*(z, v),$$

for all $u, v, z \in X$. Now, to establish the main results of this section, adapted from the contractions introduced by Sehgal [15], we introduce the following class of cyclic contraction type mappings.

Definition 3.5. *Let A and B be nonempty subsets of the metric space (X,d) and let T be a cyclic mapping on $A\cup B$. If there exists a conditional comparison function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$d^*(Tu,Tv)\leq\varphi\left(\max\left\{d^*(u,v),d^*(u,Tu),d^*(v,Tv)\right\}\right), \tag{3.2}$$

for all $u\in A$ and $v\in B$, then T is said to be a strongly cyclic Sehgal type φ-contraction on $A\cup B$ of type one.

Remark 3.2. *If conditional comparison function φ is subadditive and*

$$d(Tu,Tv)\leq\varphi\left(\max\left\{d(u,v),d(u,Tu),d(v,Tv)\right\}\right)+(I-\varphi)(d(A,B)), \tag{3.3}$$

for all $u\in A$ and $v\in B$, then T is a strongly cyclic Sehgal type φ-contraction on $A\cup B$ of type one.
Note that for every $a\in A$ and $b\in B$ we have

$$\varphi(d(a,b))=\varphi(d^*(a,b)+d(A,B))\leq\varphi(d^*(a,b))+\varphi(d(A,B)),$$

so

$$\varphi(d(a,b))-\varphi(d(A,B))\leq\varphi(d^*(a,b))\quad\forall\, a\in A,\ b\in B. \tag{3.4}$$

Hence from (3.3) and (3.4), we have

$$\begin{aligned}
d^*(Tu,Tv)&\leq\varphi\left(\max\left\{d(u,v),d(u,Tu),d(v,Tv)\right\}\right)-\varphi(d(A,B))\\
&=\max\left\{\varphi(d(u,v)),\varphi(d(u,Tu)),\varphi(d(v,Tv))\right\}-\varphi(d(A,B))\\
&=\max\left\{\varphi(d(u,v))-\varphi(d(A,B)),\varphi(d(u,Tu))\right.\\
&\qquad\left.-\varphi(d(A,B)),\varphi(d(v,Tv))-\varphi(d(A,B))\right\}\\
&\leq\max\left\{\varphi(d^*(u,v)),\varphi(d^*(u,Tu)),\varphi(d^*(v,Tv))\right\}\\
&=\varphi\left(\max\left\{d^*(u,v),d^*(u,Tu),d^*(v,Tv)\right\}\right).
\end{aligned}$$

Example 3.1. *Let a cyclic φ-contraction map on $A\cup B$, it is easy to get $d(A,B)=0$. If $I-\varphi$ is a strictly increasing function, we have a strongly cyclic Sehgal type φ-contraction on $A\cup B$ of type one.*

Example 3.2. *A cyclic contraction map on $A\cup B$ in the sense of Suzuki et al. in [16], is a strongly cyclic Sehgal type φ-contraction on $A\cup B$ of type one with $\varphi(t)=\lambda t$ for $t\geq 0$, that $\lambda\in[0,1)$.*

We begin with the following lemma which will be used later.

Lemma 3.2. *Let A and B be nonempty subsets of the metric space (X,d) with $d(A,B)>0$ and let T be a strongly cyclic Sehgal type φ-contraction on $A\cup B$ of type one. For given $x_0\in A$, define Picard iteration $\{x_n\}$ by $x_{n+1}:=Tx_n$ for each $n\geq 0$. Then*

(i) *for every $m,n\in\mathbb{N}$ with $n\geq m\geq 0$, we have*

$$0\leq d^*(x_{2n},x_{2m+1})\leq\varphi^{2m}\left(\mathcal{M}_{x_0}\right),$$

where

$$\mathcal{M}_{x_0}=(I-\varphi)^{-1}(d(x_0,T^2x_0));$$

(ii) $\forall\,\epsilon>0,\quad\exists\, m\in\mathbb{N}:\quad d(x_{2n},x_{2m+1})\leq d(A,B)+\epsilon,$ *for* $n\geq m$.

Proof. For every $x\in A\cup B$ and each $n\in\mathbb{N}$, let $\mathcal{O}_T(x,n):=\{x,Tx,\cdots,T^nx\}$ and

$$\delta^*[\mathcal{O}_T(x,n)]:=\max\{d^*(T^ix,T^jx):\quad 0\leq i\leq n \text{ is even and } 0\leq j\leq n\text{ is odd}\}.$$

First, we show that for each $n\in\mathbb{N}$ and $x_0\in A$, we have

$$\delta^*[\mathcal{O}_T(x_0,n)]=d^*(x_0,T^jx_0),\ \text{for some odd } j \text{ that } 1\leq j\leq n. \tag{3.5}$$

We may assume that $\delta^*[\mathcal{O}_T(x_0,n)] = d^*(T^i x_0, T^j x_0)$, where $1 \le i \le n$ is even and $1 \le j \le n$ is odd. Since T is a strongly cyclic Sehgal type φ-contraction of type one then from (3.2), we have

$$\begin{aligned} d^*(T^i x_0, T^j x_0) =& d^*(TT^{i-1}x_0, TT^{j-1}x_0) \\ \le& \varphi\big(\max\{d^*(T^{i-1}x_0, T^{j-1}x_0), d^*(T^{i-1}x_0, T^i x_0), d^*(T^{j-1}x_0, T^j x_0)\}\big) \\ \le& \varphi\left(\delta^*[\mathcal{O}_T(x_0,n)]\right). \end{aligned} \tag{3.6}$$

Thus, we get $(I-\varphi)(\delta^*[\mathcal{O}_T(x_0,n)]) \le 0$, which, along with Lemma 3.1(i), requires that $\delta^*[\mathcal{O}_T(x_0,n)] = 0$. So $\delta^*[\mathcal{O}_T(x_0,n)] = d^*(x_0, Tx_0)$, and hence (3.5), holds.

Now we show that for each $n \in \mathbb{N}$,

$$\delta^*[\mathcal{O}_T(x_0,n)] \le \mathcal{M}_{x_0}. \tag{3.7}$$

To prove the claim note that from (3.5) we have, $\delta^*[\mathcal{O}_T(x_0,n)] = d^*(x_0, T^j x_0)$, where $1 \le j \le n$ and j is odd. On the other hand, note that if $d^*(x_0, Tx_0) < d^*(Tx_0, T^2x_0)$ then from Lemma 3.1(i) and (3.2), we have

$$\begin{aligned} 0 < d^*(T^2x_0, Tx_0) &\le \varphi\left(\max\{d^*(x_0,Tx_0), d^*(Tx_0,T^2x_0)\}\right) \\ &= \varphi\left(d^*(Tx_0,T^2x_0)\right) \\ &< d^*(Tx_0,T^2x_0), \end{aligned}$$

a contradiction. So

$$d^*(Tx_0,T^2x_0) \le d^*(x_0,Tx_0).$$

Hence from (3.2)

$$d^*(Tx_0,T^2x_0) \le \varphi(d^*(x_0,Tx_0)). \tag{3.8}$$

If $j = 1$ (so $n \ge 1$), applying the triangle inequality and (3.8), we get

$$\begin{aligned} \delta^*[\mathcal{O}_T(x_0,n)] &= d^*(x_0,Tx_0) \\ &\le d(x_0,T^2x_0) + d^*(T^2x_0,Tx_0) \\ &\le d(x_0,T^2x_0) + \varphi(d^*(x_0,Tx_0)) \\ &\le d(x_0,T^2x_0) + \varphi\big(\delta^*[\mathcal{O}_T(x_0,n)]\big). \end{aligned}$$

If $j > 1$ (so $n \ge 3$), applying the triangle inequality, (3.2) and (3.6), we obtain

$$\begin{aligned} \delta^*[\mathcal{O}_T(x_0,n)] &= d^*(x_0,T^jx_0) \\ &\le d(x_0,T^2x_0) + d^*(T^2x_0,T^jx_0) \\ &\le d(x_0,T^2x_0) + \varphi\big(\delta^*[\mathcal{O}_T(x_0,n)]\big). \end{aligned}$$

So we have in both cases

$$(I-\varphi)\big(\delta^*[\mathcal{O}_T(x_0,n)]\big) \le d(x_0,T^2x_0).$$

On the other hand, from the strictly increasing $I-\varphi$ there exists its inverse function $(I-\varphi)^{-1}$, which is strictly increasing too, so

$$\delta^*[\mathcal{O}_T(x_0,n)] \le (I-\varphi)^{-1}\left(d(x_0,T^2x_0)\right) = \mathcal{M}_{x_0}.$$

From relation (3.2), it can be easily concluded that

$$\delta^*[\mathcal{O}_T(x_{n_0},n)] \le \varphi\big(\delta^*[\mathcal{O}_T(x_{n_0-1},n+1)]\big), \tag{3.9}$$

for $n_0 \geq 1$.

(i) Since T is a strongly cyclic Sehgal type φ-contraction of type one, for every $n \geq m \geq 0$, from (3.9), we have

$$\begin{aligned} d^*(x_{2m+1}, x_{2n}) \leq& \delta^*[\mathcal{O}_T(x_{2m}, 2n+1-2m)] \\ \leq& \varphi\left(\delta^*[\mathcal{O}_T(x_{2m-1}, 2n+2-2m)]\right) \\ \leq& \varphi^2\left(\delta^*[\mathcal{O}_T(x_{2m-2}, 2n+3-2m)]\right). \end{aligned}$$

By continuing this process and using (3.7), for every $n \geq m \geq 0$, we obtain

$$\begin{aligned} 0 &\leq d^*(x_{2m+1}, x_{2n}) \\ &\leq \varphi^{2m}\left(\delta^*[\mathcal{O}_T(x_0, 2n+1)]\right) \\ &\leq \varphi^{2m}\left(\mathcal{M}_{x_0}\right). \end{aligned} \tag{3.10}$$

Therefore (i) holds.

(ii) φ is comparison function, so $\{\varphi^k(\mathcal{M}_{x_0})\}$ is converges to 0. From (3.10), we get

$$\forall\, \epsilon > 0, \quad \exists\, m \in \mathbb{N}: \quad d^*(x_{2n}, x_{2m+1}) \leq \epsilon, \quad \text{for} \quad n \geq m,$$

and hence (ii). □

Lemma 3.3. *Let A and B be nonempty subsets of the metric space (X, d) with $d(A, B) = 0$ and let T be a strongly cyclic Sehgal type φ-contraction on $A \cup B$ of type one. For given $x_0 \in A$, define Picard iteration $\{x_n\}$ by $x_{n+1} := Tx_n$ for each $n \geq 0$. Then*

(i) *for every $m, n \in \mathbb{N}$ with $n \geq m$ that $n - m$ is odd, we have*

$$0 \leq d(x_n, x_m) \leq \varphi^m\left(\mathcal{M}_{x_0}\right),$$

where

$$\mathcal{M}_{x_0} = (I - \varphi)^{-1}(d(x_0, T^2 x_0));$$

(ii) $\forall\, \epsilon > 0, \quad \exists\, m \in \mathbb{N}: \quad d(x_n, x_m) \leq \epsilon, \quad$ *for* $\quad n \geq m.$

Proof. For every $x \in A \cup B$ and each $n \in \mathbb{N}$, let

$$\delta[\mathcal{O}_T(x, n)] := \max\{d(T^i x, T^j x) : \quad 0 \leq i \leq n \text{ is even and } 0 \leq j \leq n \text{ is odd}\}.$$

(i) Similar to proof Lemma 3.2(i), for every $n \geq m$ that $n - m$ is odd, it can be proved

$$\begin{aligned} d(x_m, x_n) \leq& \delta[\mathcal{O}_T(x_m, n-m)] \\ \leq& \varphi\left(\delta[\mathcal{O}_T(x_{m-1}, n-m+1)]\right) \\ \leq& \varphi^2\left(\delta[\mathcal{O}_T(x_{m-2}, n-m+2)]\right). \end{aligned}$$

By continuing this process and using (3.7), for every $n \geq m$ that $n - m$ is odd, we obtain

$$\begin{aligned} 0 &\leq d(x_m, x_n) \\ &\leq \varphi^m\left(\delta[\mathcal{O}_T(x_0, n)]\right) \\ &\leq \varphi^m\left(\mathcal{M}_{x_0}\right). \end{aligned} \tag{3.11}$$

(ii) φ is comparison function, so $\{\varphi^k(\mathcal{M}_{x_0})\}$ is converges to 0. Hence, from (3.11), we get

$$\forall\, \epsilon > 0, \quad \exists\, m \in \mathbb{N}: \quad d(x_n, x_m) \leq \epsilon, \quad \text{for} \quad n \geq m \quad \text{with} \quad n - m \quad \text{is odd},$$

so, it can be easily concluded that

$$\forall\, \epsilon > 0, \quad \exists\, m \in \mathbb{N}: \quad d(x_n, x_m) \leq \epsilon, \quad \text{for} \quad n \geq m.$$

□

Lemma 3.4. *Let A and B be nonempty subsets of the metric space (X, d) such that the pair (A, B) (resp. (B, A)) has the property WUC. Let T be a strongly cyclic Sehgal type φ-contraction on $A \cup B$ of type one. For $x_0 \in A$, define Picard iteration $\{x_n\}$ by $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $\{x_{2n}\}$ (resp. $\{x_{2n+1}\}$) is a Cauchy sequence.*

Proof. By using Lemma 3.2(ii), since the pair (A, B) (resp. (B, A)) has the property WUC, we get $\{x_{2n}\}$ (resp. $\{x_{2n+1}\}$) is a Cauchy sequence. □

Theorem 3.1. *Let A and B be nonempty subsets of the metric space (X, d) such that A is complete and (A, B) has the property WUC. Assume that T is a strongly cyclic Sehgal type φ-contraction on $A \cup B$ of type one. For $x_0 \in A$, define Picard iteration $\{x_n\}$ by $x_{n+1} := Tx_n$ for each $n \geq 0$. Then the following statements hold*

(i) *T has a unique best proximity point x^* in A;*
(ii) *x^* is a unique fixed point of T^2 in A;*
(iii) *$\{T^{2n}x\}$ converges to x^* for every $x \in A$;*
(iv) *Tx^* is a best proximity point of T in B that if (B, A) has the WUC property, it is unique.*
(v) *If (B, A) has the WUC property, then $\{T^{2n}y\}$ converges to Tx^* for every $y \in B$.*

Proof. (i) Let $x_0 \in A$. By Lemma 3.4, $\{x_{2n}\}$ is a Cauchy sequence in A. Since A is complete, $\{x_{2n}\}$ is converges to some $x^* \in A$ and we have

$$d^*(x_{2n}, Tx^*) \leq \varphi\left(\max\left\{d^*(x_{2n-1}, x^*), d^*(x_{2n-1}, x_{2n}), d^*(x^*, Tx^*)\right\}\right).$$

Hence

$$\begin{aligned} d^*(x^*, Tx^*) &= \overline{\lim}_{n\to\infty} d^*(x_{2n}, Tx^*) \\ &\leq \varphi\left(d^*(x^*, Tx^*)\right), \end{aligned}$$

so from Lemma 3.1(i), we obtain $d(x^*, Tx^*) = d(A, B)$. Also

$$\begin{aligned} d^*(Tx^*, T^2x^*) &\leq \varphi\left(\max\left\{d^*(x^*, Tx^*), d^*(Tx^*, T^2x^*)\right\}\right) \\ &= \varphi(d^*(Tx^*, T^2x^*)), \end{aligned}$$

hence from Lemma 3.1(i), we obtain $d(Tx^*, T^2x^*) = d(A, B)$. Since (A, B) has the WUC property, then $T^2x^* = x^*$. Suppose that z^* is other best proximity point of T in A, then z^* is a fixed point of T^2, too. Without loss of generality suppose that $d^*(x^*, Tz^*) \leq d^*(z^*, Tx^*)$ then we have

$$\begin{aligned} d^*(z^*, Tx^*) = d^*(T^2z^*, Tx^*) &\leq \varphi\left(\max\left\{d^*(Tz^*, x^*), d^*(Tz^*, T^2z^*), d^*(x^*, Tx^*)\right\}\right) \\ &= \varphi\left(d^*(Tz^*, x^*)\right) \\ &\leq d^*(Tz^*, x^*) \\ &= d^*(Tz^*, T^2x^*) \\ &\leq \varphi\left(\max\left\{d^*(z^*, Tx^*), d^*(z^*, Tz^*), d^*(Tx^*, T^2x^*)\right\}\right) \\ &= \varphi\left(d^*(z^*, Tx^*)\right). \end{aligned}$$

So

$$d^*(z^*, Tx^*) \leq \varphi\left(d^*(z^*, Tx^*)\right),$$

and hence from Lemma 3.1(i), we get $d(z^*, Tx^*) = d(A, B)$. Since $d(x^*, Tx^*) = d(A, B)$ and (A, B) has the property WUC, we get $z^* = x^*$.

(ii) Suppose that u^* is other fixed point of T^2 in A, then

$$\begin{aligned} d^*(u^*, Tu^*) &= d^*(T^2u^*, Tu^*) \\ &\leq \varphi\left(\max\left\{d^*(u^*, Tu^*), d^*(Tu^*, T^2u^*)\right\}\right) \\ &= \varphi(d^*(u^*, Tu^*)), \end{aligned}$$

hence from Lemma 3.1(i), we obtain $d(u^*, Tu^*) = d(A, B)$ that is u^* is a best proximity point of T, so from (i), we get $u^* = x^*$.

(iii) is naturally obtained from the proof of (i) and (ii).

(iv) From (i) and (ii), Tx^* is a best proximity point of T in B. To prove uniqueness, suppose that v^* is an another best proximity point of T in B, so it can be proved as before $d(Tv^*, T^2v^*) = d(A, B)$, from (i) $Tv^* = x^*$. Because (B, A) has the WUC property, $T^2v^* = v^*$. So $Tx^* = T^2v^* = v^*$.

(v) Since $\lim_{n\to\infty} d(x^*, T^{2n}y) = \lim_{n\to\infty} d(T^{2n-1}y, T^{2n}y) = d(A, B) = d(x^*, Tx^*)$ and (B, A) has the WUC property, then

$$\lim_{n\to\infty} d(Tx^*, T^{2n}y) = 0.$$

□

4. Priori and posteriori error estimates

Now, according to the results of the previous section, we have the next approximation theorem that is a direct consequence of Lemma 3.3 and Theorem 3.1.

Theorem 4.2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a conditional comparison function and A and B be nonempty subsets of the metric space (X, d) such that A is complete. Assume that T is a strongly cyclic Sehgal type φ-contraction on $A \cup B$ of type two, that is*

$$d(Tu, Tv) \leq \varphi\left(\max\left\{d(u, v), d(u, Tu), d(v, Tv)\right\}\right),$$

for all $u \in A$ and $v \in B$. Then T has a unique fixed point $x^* \in X$ such that $\{T^n x_0\}$ converges to x^* for every starting point $x_0 \in A$. Also, the following estimates

$$d(T^m x_0, x^*) \leq \varphi^m\left((I - \varphi)^{-1}(d(x_0, T^2x_0)\right)$$

and

$$\begin{aligned} d(T^m x_0, x^*) &\leq \varphi^2\left((I - \varphi)^{-1}(d(T^{m-2}x_0, T^m x_0)\right) \\ &\leq \varphi\left((I - \varphi)^{-1}(d(T^{m-2}x_0, T^m x_0))\right), \end{aligned}$$

hold, for all $x_0 \in A$.

Corollary 4.1. *Especially in the previous theorem in the case that $\varphi(t) = \lambda t$ for $t \geq 0$, that $\lambda \in [0, 1)$ is a constant, T has a unique fixed point $x^* \in X$ such that $\{T^n x_0\}$ converges to x^* for every starting point $x_0 \in X$. Also, the following estimates*

$$d(T^m x_0, x^*) \leq \frac{\lambda^m}{1 - \lambda} d(x_0, T^2x_0)$$

and

$$d(T^m x_0, x^*) \leq \frac{\lambda}{1 - \lambda} d(T^{m-2}x_0, T^m x_0),$$

hold, for all $x_0 \in X$.

The next theorem is proved exactly like Lemma 3.3 and Theorem 3.1, which we omit from bringing it. This is our answer to Zlatanov's question.

Theorem 4.3. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a conditional comparison function and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of the metric space (X, d). Assume that T is a cyclic φ-contraction on $\bigcup_{i=1}^k A_i$ such that $I - \varphi$ be a strictly increasing function. Then T has a unique fixed point $x^* \in \bigcap_{i=1}^k A_i$ such that $\{T^n x_0\}$ converges to x^* for every starting point $x_0 \in \bigcup_{i=1}^k A_i$. Also, the following estimates*

$$d(T^m x_0, x^*) \leq \varphi^m \left((I - \varphi)^{-1}(d(x_0, T^k x_0)\right)$$

and

$$d(T^m x_0, x^*) \leq \varphi^k \left((I - \varphi)^{-1}(d(T^{m-k} x_0, T^m x_0)\right),$$

hold, for all $x_0 \in A$.

It is important that even if we have a self-mapping on a complete metric space (X, d), we can obtain the following important theorem with the same methods of proving the results in the previous section. This theorem is also a generalization of Theorem 1 of [4], which adds to its importance.

Theorem 4.4. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a conditional comparison function. Assume that self map T is a strongly Ćirić type φ-contraction on a complete metric space (X, d), that is*

$$d(Tu, Tv) \leq \varphi\left(\max\left\{d(u, v), d(u, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\right\}\right), \tag{4.12}$$

for all $u, v \in X$. Then T has a unique fixed point $x^* \in X$ such that $\{T^n x_0\}$ converges to x^* for every starting point $x_0 \in X$. Also, the following estimates

$$d(T^m x_0, x^*) \leq \varphi^m \left((I - \varphi)^{-1}(d(x_0, T x_0)\right)$$

and

$$d(T^m x_0, x^*) \leq \varphi\left((I - \varphi)^{-1}(d(T^{m-1} x_0, T^m x_0)\right),$$

hold, for all $x_0 \in X$.

Let $\varphi' : \mathbb{R}_+^5 \to \mathbb{R}_+$ be a 5-dimensional comparison [3, Definition 2.4] function and

$$\psi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \psi(t) = \varphi(t, t, t, t, t), \quad t \in \mathbb{R}_+.$$

It is not difficult to see that a generalized φ'-contraction map [3, Definition 2.5], satisfies in (4.12) with $\varphi(t) := \varphi'(t, t, t, t, t)$ for $t \geq 0$. Therefore, Theorem 4.4 is another version of Theorem 2.10 [3].

Corollary 4.2. *[4, Theorem 1] Especially in the previous theorem, in the case that $\varphi(t) = \lambda t$ for $t \geq 0$, that $\lambda \in [0, 1)$ is a constant, T has a unique fixed point $x^* \in X$ such that $\{T^n x_0\}$ converges to x^* for every starting point $x_0 \in X$. Also, the following estimates*

$$d(T^m x_0, x^*) \leq \frac{\lambda^m}{1 - \lambda} d(x_0, T x_0)$$

and

$$d(T^m x_0, x^*) \leq \frac{\lambda}{1 - \lambda} d(T^{m-1} x_0, T^m x_0),$$

hold, for all $x_0 \in X$.

In the next result, we find a priori and a posteriori error estimates of the best proximity point for the Picard iteration associated to a strongly cyclic Sehgal type φ-contraction of type one, which is defined on a uniformly convex Banach space with modulus of convexity of power type.

Theorem 4.5. *Let $(X, \|.\|)$ be a uniformly convex Banach space such that $\delta_X(\epsilon) \geq C\epsilon^q$ for some $C > 0$, $q \geq 2$ and every $\epsilon \in (0, 2]$. Suppose that A and B be nonempty, closed and convex subsets of X with $d := d(A, B) > 0$. Let $T : A \cup B \to A \cup B$ be strongly cyclic Sehgal type φ-contraction on $A \cup B$ of type one, then*

- *(i) there exists a unique best proximity point x^* of T in A such that Tx^* is a unique best proximity point of T in B and $T^2x^* = x^*$;*
- *(ii) for every $x_0 \in A$, the sequence $\{T^{2n}x_0\}$ converges to x^* and $\{T^{2n+1}x_0\}$ converges to Tx^*;*
- *(iii) a priori error estimate holds in the following implication:*

$$\|x^* - T^{2m}x_0\| \leq \left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right) \sqrt[p]{\frac{\varphi^{2m}(\mathcal{M}_{x_0})}{Cd}};$$

- *(iv) a posteriori error estimate holds in the following implication:*

$$\|T^{2m}x_0 - x^*\| \leq \left(\mathcal{M}_{x_{2m}} + d\right) \sqrt[p]{\frac{\mathcal{M}_{x_{2m}}}{Cd}};$$

where

$$\mathcal{M}_{x_0} = (I - \varphi)^{-1}\left(d(x_0, T^2x_0)\right) \quad \text{and} \quad \mathcal{M}_{x_{2m}} = (I - \varphi)^{-1}\left(d(x_{2m}, T^2x_{2m})\right).$$

Proof. The proof of (i) and (ii) follows directly from Theorem 3.1.

(iii) For every $n \in \mathbb{N}$ let $x_n = T^n x_0$. From Lemma 3.2(i), for $n \geq m \geq 0$ we have the inequalities

$$\|x_{2n} - x_{2m+1}\| \leq \varphi^{2m}(\mathcal{M}_{x_0}) + d,$$

$$\|x_{2m} - x_{2m+1}\| \leq \varphi^{2m}(\mathcal{M}_{x_0}) + d$$

and

$$\|x_{2n} - x_{2m}\| \leq 2\left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right).$$

Now from (2.1), with $u = x_{2n}$, $v = x_{2m}$, $p = x_{2m+1}$, $r = \|x_{2n} - x_{2m}\|$, $R = \varphi^{2m}(\mathcal{M}_{x_0}) + d$ and using the convexity of the set A, we get

$$\begin{aligned} d &\leq \left\|\frac{x_{2n} + x_{2m}}{2} - x_{2m+1}\right\| \\ &\leq \left(1 - \delta\Big(\frac{\|x_{2n} - x_{2m}\|}{\varphi^{2m}(\mathcal{M}_{x_0}) + d}\Big)\right)\left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right), \end{aligned}$$

so, we obtain the inequality

$$\delta\Big(\frac{\|x_{2n} - x_{2m}\|}{\varphi^{2m}(\mathcal{M}_{x_0}) + d}\Big) \leq \frac{\varphi^{2m}(\mathcal{M}_{x_0})}{\varphi^{2m}(\mathcal{M}_{x_0}) + d}. \tag{4.13}$$

From the uniform convexity of X it follows that δ_X is strictly increasing and therefore there exists its inverse function δ_X^{-1}, which is strictly increasing. From (4.13), we get

$$\|x_{2n} - x_{2m}\| \leq \left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right)\delta_X^{-1}\left(\frac{\varphi^{2m}(\mathcal{M}_{x_0})}{\varphi^{2m}(\mathcal{M}_{x_0}) + d}\right). \tag{4.14}$$

It follows from the inequality $\delta_X(t) \geq Ct^q$ that $\delta_X^{-1}(t) \leq \left(\frac{t}{C}\right)^{\frac{1}{q}}$. Using (4.14), we obtain

$$\|x_{2n} - x_{2m}\| \leq \left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right) \sqrt[q]{\frac{\varphi^{2m}(\mathcal{M}_{x_0})}{C\left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right)}}$$

$$\leq \left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right) \sqrt[q]{\frac{\varphi^{2m}(\mathcal{M}_{x_0})}{Cd}}. \tag{4.15}$$

So, from (4.15), for $n \geq m \geq 0$ we obtain

$$\|x_{2n} - x_{2m}\| \leq \left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right) \sqrt[q]{\frac{\varphi^{2m}(\mathcal{M}_{x_0})}{Cd}}. \tag{4.16}$$

Letting $n \to \infty$ in (4.16), we obtain

$$\|x^* - T^{2m}x_0\| \leq \left(\varphi^{2m}(\mathcal{M}_{x_0}) + d\right) \sqrt[q]{\frac{\varphi^{2m}(\mathcal{M}_{x_0})}{Cd}}.$$

(iv) If we use relation (4.16) to get a posteriori error estimate, we have

$$\|x_{2m+2i} - x_{2m}\| \leq \left(\varphi^{0}(\mathcal{M}_{x_{2m}}) + d\right) \sqrt[q]{\frac{\varphi^{0}(\mathcal{M}_{x_{2m}})}{Cd}}. \tag{4.17}$$

After letting $i \to \infty$ in (4.17), we obtain the inequality

$$\|T^{2m}x_0 - x^*\| \leq \left(\mathcal{M}_{x_{2m}} + d\right) \sqrt[q]{\frac{\mathcal{M}_{x_{2m}}}{Cd}}. \tag{4.18}$$

□

If T is a weak cyclic Kannan contraction map [12] or a cyclic contraction map in the sense of Suzuki et al. in [16], we have the following corollary.

Corollary 4.3. *In the previous theorem, in the special case, if $\varphi(t) = \lambda t$ for every $t \geq 0$ and some $\lambda \in [0, 1)$, then we have*

(i) a priori error estimate holds

$$\|x^* - T^{2m}x_0\| \leq \left(\frac{\|x_0 - T^2x_0\|}{1-\lambda} + d\right) \sqrt[q]{\frac{\|x_0 - T^2x_0\|}{Cd(1-\lambda)}}\left(\sqrt[q]{\lambda}\right)^{2m};$$

(ii) a posteriori error estimate holds

$$\|T^{2m}x_0 - x^*\| \leq \left(\frac{\|T^{2m-2}x_0 - T^{2m}x_0\|}{(1-\lambda)} + d\right) \sqrt[q]{\frac{\|T^{2m-2}x_0 - T^{2m}x_0\|}{Cd(1-\lambda)}}\left(\sqrt[q]{\lambda}\right)^{2}.$$

Proof. (i) First note that, since $\varphi(t) = \lambda t$, then $\mathcal{M}_{x_0} = (I - \varphi)^{-1}(\|x_0 - T^2x_0\|) = \frac{1}{1-\lambda}\|x_0 - T^2x_0\|$, so from Theorem 4.5(iii), we have

$$\|x^* - T^{2m}x_0\| \leq \left(\frac{\lambda^{2m}}{1-\lambda}\|x_0 - T^2x_0\| + d\right) \sqrt[q]{\frac{\frac{\lambda^{2m}}{1-\lambda}\|x_0 - T^2x_0\|}{Cd}}$$

$$\leq \left(\frac{\|x_0 - T^2x_0\|}{1-\lambda} + d\right) \sqrt[q]{\frac{\|x_0 - T^2x_0\|}{Cd(1-\lambda)}}\left(\sqrt[q]{\lambda}\right)^{2m}.$$

(ii) Since $\mathcal{M}_{x_{2m-2}} = \frac{1}{1-\lambda}\|x_{2m-2} - T^2 x_{2m-2}\|$, so similar to (4.17), we obtain the inequality

$$\begin{aligned}\|T^{2m}x_0 - x^*\| &\le \left(\varphi^2\left(\mathcal{M}_{x_{2m-2}}\right) + d\right)\sqrt[q]{\frac{\varphi^2\left(\mathcal{M}_{x_{2m-2}}\right)}{Cd}} \\ &= \left(\frac{\lambda^2\|T^{2m-2}x_0 - T^{2m}x_0\|}{(1-\lambda)} + d\right)\sqrt[q]{\frac{\|T^{2m-2}x_0 - T^{2m}x_0\|}{Cd(1-\lambda)}}\sqrt[q]{\lambda^2} \\ &\le \left(\frac{\|T^{2m-2}x_0 - T^{2m}x_0\|}{(1-\lambda)} + d\right)\sqrt[q]{\frac{\|T^{2m-2}x_0 - T^{2m}x_0\|}{Cd(1-\lambda)}}(\sqrt[q]{\lambda})^2.\end{aligned}$$

□

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