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Third-Order Neutral Difference Equations: Improved Oscillation Criteria via Linearization Method

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ABSTRACT. This paper studies the oscillatory and asymptotic behavior of solutions to the third-order neutral delay difference equation

$$\Delta\left(\mu(\nu)[\Delta^2\alpha(\nu)]^a\right) + b(\nu)\beta^a(\nu-\tau) = 0,$$

where $\alpha(\nu) = \beta(\nu) + \varrho\beta(\nu-\sigma)$, using linearization method and then comparing with the second-order delay difference equations whose oscillatory properties are known. The obtained criteria are new, improve and extend some of the known results. This is verified by means of two specific examples.

1. INTRODUCTION

In this paper, we study the oscillatory and asymptotic behavior of solutions to the third-order neutral difference equation of the form

$$\text{(E)} \qquad \Delta(\mu(\nu)[\Delta^2\alpha(\nu)]^a) + b(\nu)\beta^a(\nu-\tau) = 0, \nu \geq \nu_0,$$

where ν_0 is a positive integer and $\alpha(\nu) = \beta(\nu) + \varrho\beta(\nu-\sigma)$.
Throughout, we assume that

- (H_1) $\{\mu(\nu)\}$ and $\{b(\nu)\}$ are real sequences such that $\mu(\nu) > 0, b(\nu) \geq 0$, and $b(\nu)$ is not identically zero for large ν;
- (H_2) $a \geq 1$ is a ratio of odd positive integers;
- (H_3) $\varrho \geq 0$ is a real number and $\varrho \neq 1$.

Let $\vartheta = \min\{\sigma, \tau\}$. By a solution of (E), we mean a sequence $\{\beta(\nu)\}$ defined for all $\nu \geq \nu_0 - \vartheta$ and satisfying (E) for all $\nu \geq \nu_0$. We consider only solutions of (E) that satisfy $\sup\{|\beta(\nu)| : \nu \geq \aleph\} > 0$ for all $\aleph \geq \nu_0$, and we tacitly assume that (E) possesses such solutions. A solution of (E) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise the solution is said to be *nonoscillatory*.

In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g.[9-12]. Determining oscillatory and asymptotic behavior of solutions of (E) or its special cases (including the continuous case) received great attention in recent years, see, for example [1-6, 8, 13-19,21,22] and the references cited therein. In [2-6,14-19,21], the authors established several criteria imply that all solutions of (E) are either oscillatory or tend to zero as $\upsilon \to \infty$.

In [13,15-18], the authors investigated the oscillatory behavior of solutions of (E) assuming the following conditions:

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(1.1) $$\sum_{\nu=\nu_0}^{\infty} \mu^{-1/a}(\nu) = \infty,$$

(1.2) $$0 \le \varrho < 1$$

and

(1.3) $$\textit{either } \Delta\mu(\nu) \ge 0, \ \textit{or } \Delta\mu(\nu) \le 0.$$

Finally in [19], the authors analyzed the behavior of solutions of (E) under the condition (1.3) and $0 \le \varrho < \infty$.

Recently in [5], the authors studied the behavior of solutions of (E) under the condition (1.1), and using comparison with first-order delay difference equations whose oscillatory behavior is well-known. But in this paper, the oscillation criteria are established first by applying linearization technique and then using comparison with second-order delay difference equations whose oscillatory properties are known instead of first-order delay difference equations. Therefore, the obtained criteria are new and different from [5, 15-19] and this is verified through examples.

2. Main result

From the form and the assumptions on the studied equation it is enough to consider only eventually positive solutions of (E), when we deal with nonoscillatory solutions. We start with the lemmas that are used to prove our main results.

Lemma 2.1. *Let* (1.1) *holds and assume* $\{\beta(\nu)\}$ *is an eventually positive solution of* (E). *Then, there exists an integer* $\nu_1 \ge \nu_0$ *such that for all* $\nu \ge \nu_1$, *either*

(I) $\alpha(\nu) > 0, \ \Delta\alpha(\nu) < 0, \ \Delta^2\alpha(\nu) > 0, \ \Delta(\mu(\nu)(\Delta^2\alpha(\nu))^a) \le 0,$

(II) $\alpha(\nu) > 0, \ \Delta\alpha(\nu) > 0, \ \Delta^2\alpha(\nu) > 0, \ \Delta(\mu(\nu)(\Delta^2\alpha(\nu))^a) \le 0.$

Proof. Assume that $\{\beta(\nu)\}$ is a positive solution of (E) for all $\nu \ge \nu_0$. It is easy to see that $\alpha(\nu) > \beta(\nu) > 0$ and

$$\Delta(\mu(\nu)[\Delta^2\alpha(\nu)]^a) = -b(\nu)\beta^a(\nu-\tau) < 0,$$

for $\nu \ge \nu_1 \ge \nu_0$. Thus $\mu(\nu)(\Delta^2\alpha(\nu))^a$ is nonincreasing and of one sign. Therefore $\Delta^2\alpha(\nu)$ is also of one sign and hence we have two possibilities: $\Delta^2\alpha(\nu) < 0$ or $\Delta^2\alpha(\nu) > 0$ for $\nu \ge \nu_1$. If we take $\Delta^2\alpha(\nu) < 0$, then there exists a constant $M > 0$ such that

$$\mu(\nu)(\Delta^2\alpha(\nu))^a \le -M < 0.$$

Summing up the last inequality from ν_1 to $\nu - 1$, we get

$$\Delta\alpha(\nu) \le \Delta\alpha(\nu_1) - M^{\frac{1}{a}} \sum_{s=\nu_1}^{\nu-1} \mu^{-\frac{1}{a}}(s).$$

Letting ν tends to ∞ and using (1.1), we get $\Delta\alpha(\nu)$ tends to $-\infty$. Thus, $\Delta\alpha(\nu) < 0$ eventually. But $\Delta^2\alpha(\nu) < 0$ and $\Delta\alpha(\nu) < 0$ eventually imply that $\alpha(\nu) < 0$ for $\nu \ge \nu_1$ which is a contradiction. This contradiction proves that $\Delta^2\alpha(\nu) > 0$ and we have only two cases (I) and (II) for $\alpha(\nu)$. This ends the proof. □

Lemma 2.2. *Assume that* $\{\theta(\nu)\}$ *and* $\{\chi(\nu)\}$ *are real sequences with* $\theta(\nu) = \chi(\nu) + \varrho\chi(\nu - \ell)$ *for* $\nu \ge \nu_0 + \max\{0, \ell\}$, *where* $\varrho \ne 1$, *is a constant and* ℓ *is an integer. Let that there exists a constant* $\delta \in R$ *such that* $\lim_{\nu\to\infty} \theta(\nu) = \delta$.

(I) *If* $\liminf_{\nu\to\infty} \chi(\nu) = \gamma \in R$, *then* $\gamma = \dfrac{\delta}{1+\varrho}$.

(II) *If* $\limsup_{\nu\to\infty} \chi(\nu) = \gamma^* \in R$, *then* $\gamma^* = \dfrac{\delta}{1+\varrho}$.

Proof. We shall prove (I). The proof of (II) is similar and will be omitted. From

$$\theta(\nu) = \chi(\nu) + \varrho\chi(\nu - \ell), \tag{2.4}$$

we see that

$$\theta(\nu + \ell) - \theta(\nu) = \chi(\nu + \ell) + (\varrho - 1)\chi(\nu) - \varrho\chi(\nu - \ell). \tag{2.5}$$

Let $\{\nu_j\}$ be a sequence of integers such that

$$\lim_{j\to\infty} \nu_j = \infty \text{ and } \lim_{j\to\infty} \chi(\nu_j) = \gamma. \tag{2.6}$$

Case 1. Assume $\varrho > 1$. By replacing ν by $\nu_j + \ell$ in (2.5) and then by taking limits, we see that

$$\lim_{j\to\infty} (\chi(\nu_j + 2\ell) + (\varrho - 1)\chi(\nu_j + \ell)) = \varrho\gamma. \tag{2.7}$$

Let

$$\gamma_1 = \lim_{j\to\infty} \inf \chi(\nu_j + 2\ell) \text{ and } \gamma_2 = \lim_{j\to\infty} \inf \chi(\nu_j + \ell). \tag{2.8}$$

Then, $\gamma_1 \geq \gamma,\ \gamma_2 \geq \gamma$ and (2.7) implies that

$$\gamma_1 + (\varrho - 1)\gamma_2 \leq \varrho\gamma. \tag{2.9}$$

We want to show that

$$\gamma_1 = \gamma_2 = \gamma. \tag{2.10}$$

If $\gamma_1 > \gamma$, then by (2.9), $\gamma + (\varrho - 1)\gamma_2 \leq \varrho\gamma$, so $(\varrho - 1)\gamma_2 \leq (\varrho - 1)$, or $\gamma_2 \leq \gamma$ which is a contradiction. Hence, $\gamma_1 = \gamma$.

If $\gamma_2 > \gamma$, then by (2.9), $\gamma_1 + (\varrho - 1)\gamma \leq \varrho\gamma$, so $\gamma_1 \leq \gamma$, which again is a contradiction. Hence, $\gamma_2 = \gamma$. Therefore, $\gamma_1 = \gamma_2 = \gamma$ which is what we wanted to show.

It follows from (2.10), (2.6) and (2.8) that there exists a subsequence $\{\nu_{j_m}\}$ of $\{\nu_j\}$ such that

$$\lim_{m\to\infty} \chi(\nu_{j_m} + 2\ell) = \lim_{m\to\infty} \chi(\nu_{j_m} + \ell) = \gamma.$$

Replacing ν with $\nu_{j_m} + 2\ell$ in (2.4) and by taking limits as $m \to \infty$, we find that

$$\delta = (1 + \varrho)\gamma,$$

which completes the proof for the case $\varrho > 1$.

Case 2. $0 \leq \varrho < 1$. Replacing ν by $\nu_j - \ell$ in (2.5), taking limits and applying (2.6) results in

$$\lim_{j\to\infty} [(1 - \varrho)\chi(\nu_j - \ell) + \varrho\chi(\nu_j - 2\ell)] = \gamma.$$

The conclusion follows by an argument similar to that used in Case 1.

Case 3. $\varrho < 0$. By replacing ν by ν_j in (2.5) and then taking limits, we obtain

$$\lim_{j\to\infty} [\chi(\nu_j + \ell) - \varrho\chi(\nu_j - \ell)] = (1 - \varrho)\gamma,$$

from which the conclusion follows as in Case 1. This complete the proof of the lemma. $\square$

Lemma 2.3. *Assume $\{\beta(\nu)\}$ is an eventually positive solution of* (E) *and let $\{\alpha(\nu)\}$ satisfy case (I) of Lemma 2.1. If either*

$$\sum_{\nu=\nu_0}^{\infty} b(\nu) = \infty \tag{2.8}$$

or

$$\sum_{\nu=\nu_0}^{\infty}\sum_{s=\nu}^{\infty}\left(\left(\frac{1}{\mu(s)}\sum_{t=s}^{\infty} b(t)\right)^{\frac{1}{a}}\right) = \infty, \tag{2.9}$$

then

$$\lim_{\nu\to\infty} \beta(\nu) = 0. \tag{2.10}$$

Proof. In view of $\alpha(\nu) > 0$ and $\Delta\alpha(\nu) < 0$, there exists a constant $C \geq 0$ such that $\lim_{\nu\to\infty} \alpha(\nu) = C$. We will prove $C = 0$. If not, then using Lemma 2.2, we have $\liminf_{\nu\to\infty} \beta(\nu) = \frac{C}{1+\varrho} > 0$. Hence there exists an integer ν_2 such that for $\nu \geq \nu_2 \geq \nu_1$,

$$\beta(\nu-\tau) > \frac{C}{2(1+\varrho)} > 0.$$

Using this in (E), we obtain

$$\Delta(\mu(\nu)(\Delta^2\alpha(\nu))^a) \leq -\left(\frac{C}{2(1+\varrho)}\right)^a b(\nu)$$

for $\nu \geq \nu_2$. Summing up the last inequality from ν_2 to ν, we get

$$\left(\frac{C}{2(1+\varrho)}\right)^a \sum_{s=\nu_2}^{\nu} b(s) \leq \mu(\nu_2)(\Delta^2\alpha(\nu_2))^a < \infty$$

which contradicts (2.8). Therefore $\lim_{\nu\to\infty} \alpha(\nu) = C = 0$, so (2.10) holds since $0 < \beta(\nu) \leq \alpha(\nu)$.

Next, if condition (2.9) is satisfied, then the proof is similar to that in Lemma 2.3 of [5] and so the details are omitted. This completes the proof. □

Lemma 2.4. *Assume (1.1) holds and let $\{\beta(\nu)\}$ be an eventually positive solution of* (E)*. If case (II) of Lemma 2.1 holds then there exists an integer ν_* such that for all $\nu \geq \nu_*$, we have*

- (i) $\Delta\alpha(\nu) \geq \Omega(\nu)\mu^{1/a}(\nu)\Delta^2\alpha(\nu)$;
- (ii) $\left\{\frac{\Delta\alpha(\nu)}{\Omega(\nu)}\right\}$ *is eventually decreasing;*
- (iii) $\alpha(\nu) \geq \frac{\Omega_1(\nu)}{\Omega(\nu)}\Delta\alpha(\nu)$;
- (iv) $\left\{\frac{\alpha(\nu)}{\Omega_1(\nu)}\right\}$ *is eventually decreasing,*

where $\Omega(\nu) = \sum_{s=\nu_*}^{\nu-1} \mu^{-1/a}(s)$ *and* $\Omega_1(\nu) = \sum_{s=\nu_*}^{\nu-1} \Omega(s)$.

Proof. Let $\{\beta(\nu)\}$ be an eventually positive solution of (E). Then there exists an integer $\nu_* \geq \nu_0$ such that $\beta(\nu-\tau) > 0$ and $\beta(\nu-\sigma) > 0$ for all $\nu \geq \nu_*$. Since $\alpha(\nu) \in Case\ (II)$, we have

$$\Delta\alpha(\nu) = \Delta\alpha(\nu_*) + \sum_{s=\nu_*}^{\nu-1} \mu^{1/a}(s)\frac{\Delta^2\alpha(s)}{\mu^{1/a}(s)} \geq \Omega(\nu)\mu^{1/a}(\nu)\Delta^2\alpha(\nu) \tag{2.11}$$

which proves (i).
Now (2.11) gives

$$\Delta\left(\frac{\Delta\alpha(\nu)}{\Omega(\nu)}\right) = \frac{\mu^{1/a}(\nu)\Omega(\nu)\Delta^2\alpha(\nu) - \Delta\alpha(\nu)}{\mu^{1/a}(\nu)\Omega(\nu)\Omega(\nu+1)} \leq 0,$$

which means that

$$\frac{\Delta\alpha(\nu)}{\Omega(\nu)} \text{ is decreasing.} \tag{2.12}$$

That is, (ii) is satisfied.
In view of (2.12), we obtain

$$\alpha(\nu) = \alpha(\nu_*) + \sum_{s=\nu_1}^{\nu-1} \frac{\Omega(s)\Delta\alpha(s)}{\Omega(s)} \geq \Omega_1(\nu)\frac{\Delta\alpha(\nu)}{\Omega(\nu)} \tag{2.13}$$

which proves (iii).
From (2.13), we get

$$\Delta\left(\frac{\alpha(\nu)}{\Omega_1(\nu)}\right) = \frac{\Omega_1(\nu)\Delta\alpha(\nu) - \Omega(\nu)\alpha(\nu)}{\Omega_1(\nu)\Omega_1(\nu+1)} < 0$$

which implies that $\dfrac{\alpha(\nu)}{\Omega_1(\nu)}$ is decreasing. The proof of the lemma is complete. □

Theorem 2.1. *Let (1.1) and (2.9) hold. If the second-order delay difference equation*

$$\Delta(\mu^{1/a}(\nu)\Delta Z(\nu)) + \frac{b(\nu)}{a(1+\varrho)^a}\frac{\Omega_1^a(\nu-\tau)}{\Omega(\nu-\tau)}Z(\nu-\tau) = 0 \tag{2.14}$$

is oscillatory, then every solution of (E) *is either oscillatory or* (2.10) *holds.*

Proof. Let $\{\beta(\nu)\}$ be an eventually positive solution of (E). From Lemma 2.1 there exists an integer $\nu_1 \geq \nu_0$ such that $\alpha(\nu) > 0$ for all $\nu \geq \nu_1$ and either case (I) or case (II) holds for all $\nu \geq \nu_1$. For case (I), it follows immediately from Lemma 2.3 that (2.10) holds, and so we consider only case (II). Since $\Delta\alpha(\nu) > 0,\ \Delta^2\alpha(\nu) > 0$, there exists a positive constant $l \leq \infty$ such that $\lim_{\nu\to\infty} \Delta\alpha(\nu) = l > 0$. Therefore, using Lemma 2.2, we see that $\liminf_{\nu\to\infty} \Delta\beta(\nu) = \dfrac{l}{1+\varrho} > 0$, and so we conclude that

$$\Delta\beta(\nu) > 0 \tag{2.15}$$

for all $\nu \geq \nu_2 \geq \nu_1$. In view of (2.15), we observe that $\alpha(\nu) = \beta(\nu)+\varrho\beta(\nu-\sigma) \leq (1+\varrho)\beta(\nu)$, that is,

$$\beta(\nu) \geq \frac{1}{(1+\varrho)}\alpha(\nu). \tag{2.16}$$

From the inequalities $\tau > 0$, (2.15), (2.16), we obtain

$$\beta(\nu-\tau) \geq \frac{1}{(1+\varrho)}\alpha(\nu-\tau). \tag{2.17}$$

Combining (2.17) with (E), we get

$$\Delta(\mu(\nu)(\Delta^2\alpha(\nu))^a) + \frac{b(\nu)}{(1+\varrho)^a}\alpha^a(\nu-\tau) \leq 0. \tag{2.18}$$

Now Δ-derivative yields

$$\Delta((\mu^{1/a}(\nu)\Delta^2\alpha(\nu))^a) \geq a(\mu^{1/a}(\nu)\Delta^2\alpha(\nu))^{a-1}\Delta(\mu^{1/a}(\nu)\Delta^2\alpha(\nu))$$

and from (2.18), we get

$$\Delta(\mu^{1/a}(\nu)\Delta^2\alpha(\nu)) + \frac{1}{a}(\mu^{1/a}(\nu)\Delta^2\alpha(\nu))^{1-a}\frac{b(\nu)}{(1+\varrho)^a}\alpha^a(\nu-\tau) \le 0. \tag{2.19}$$

Using (2.11) and (2.13), we get

$$\alpha(\nu-\tau) \ge \Omega_1(\nu-\tau)\mu^{1/a}(\nu-\tau)\Delta^2\alpha(\nu-\tau) \ge \Omega_1(\nu-\tau)\mu^{1/a}(\nu)\Delta^2\alpha(\nu), \tag{2.20}$$

for $\nu \ge \nu_2$. Since $a \ge 1$ and so combining (2.19) with (2.20), we have

$$\Delta(\mu^{1/a}(\nu)\Delta^2\alpha(\nu)) + \frac{1}{a}\Omega_1^{a-1}(\nu-\tau)\frac{b(\nu)}{(1+\varrho)^a}\alpha(\nu-\tau) \le 0, \nu \ge \nu_2. \tag{2.21}$$

Let $Z(\nu) = \Delta\alpha(\nu)$. Using (2.13) in (2.21), we see that $\{Z(\nu)\}$ is a positive solution of the inequality

$$\Delta(\mu^{1/a}(\nu)\Delta Z(\nu)) + \frac{b(\nu)}{a(1+p)^a}\frac{\Omega_1^a(\nu-\tau)}{\Omega(\nu-\tau)}Z(\nu-\tau) \le 0.$$

But by Lemma 1 of [15], the corresponding equation (2.14) has a positive solution. This contradiction completes the proof. □

In view of Theorem 2.5, we immediately obtain the following explicit criteria for the oscillation of (E).

Corollary 2.1. *Let (1.1) and (2.9) hold. If*

$$\liminf_{\nu\to\infty}\Omega(\nu)\sum_{s=\nu}^{\infty}\frac{\Omega_1^a(s-\tau)b(s)}{\Omega(s)} > \frac{a(1+\varrho)^a}{4}, \tag{2.22}$$

then the conclusion of Theorem 2.5 holds.

Proof. Assume the contrary that $\{\beta(\nu)\}$ is an eventually positive solution of (E). Then proceeding as in the proof of Theorem 2.5, we see that condition (2.9) implies that (2.10) holds. Also from Theorem 2.5, we are led to (2.14), that is,

$$\Delta(\mu^{1/a}(\nu)\Delta Z(\nu)) + \frac{b(\nu)\Omega_1^a(\nu-\tau)}{a(1+\varrho)^a\Omega(\nu-\tau)}Z(\nu-\tau) \le 0, \nu \ge \nu_1. \tag{2.23}$$

From Lemma 2.4 (ii), we see that $\left\{\frac{Z(\nu)}{\Omega(\nu)}\right\}$ is decreasing and using this in (2.23), we have

$$\Delta(\mu^{1/a}(\nu)\Delta Z(\nu)) + \frac{b(\nu)\Omega_1^a(\nu-\tau)}{a(1+\varrho)^a\Omega(\nu)}Z(\nu) \le 0. \tag{2.24}$$

Define

$$\omega(\nu) = \frac{\mu^{1/a}(\nu)\Delta Z(\nu)}{Z(\nu)} > 0.$$

Thus, in view of (2.24), we get

$$\Delta\omega(\nu) \le -\frac{b(\nu)\Omega_1^a(\nu-\tau)}{a(1+\varrho)^a\Omega(\nu)} - \frac{\omega(\nu)\omega(\nu+1)}{\mu^{1/a}(\nu)}.$$

Summing up the last inequality from ν to ∞, we obtain

$$\omega(\nu) \ge \frac{1}{a(1+\varrho)^a}\sum_{s=\nu}^{\infty}\frac{b(s)\Omega_1^a(s-\tau)}{\Omega(s)} + \sum_{s=\nu}^{\infty}\frac{\omega(s)\omega(s+1)}{\mu^{1/a}(s)}$$

and so

$$\Omega(\nu)\omega(\nu) \ge \frac{\Omega(\nu)}{a(1+\varrho)^a}\sum_{s=\nu}^{\infty}\frac{b(s)\Omega_1^a(s-\tau)}{\Omega(s)} + \Omega(\nu)\sum_{s=\nu}^{\infty}\frac{\omega(s)\omega(s+1)}{\mu^{1/a}(s)}. \tag{2.25}$$

Letting $\liminf_{\nu\to\infty} \Omega(\nu)\omega(\nu) = M_1 > 0$, then from (2.25), we get

$$M_1 > \frac{1}{4} + M_1^2 \tag{2.26}$$

since $\Omega(\nu) \sum_{s=\nu}^{\infty} \frac{1}{\mu^{1/a}(s)\Omega(s)\Omega(s+1)} = 1$. For $M_1 > 0$, the relation (2.26) is not possible and this completes the proof. □

Next, by applying Theorem 3.5 in [7] to equation (2.14), we have the following result.

Corollary 2.2. *Let (1.1) and (2.9) hold. If there exists a nondecreasing positive sequence $\{\phi(\nu)\}$ such that for any $\nu \geq \nu_0$*

$$\limsup_{\nu\to\infty} \sum_{s=\nu_0}^{n} \left[\phi(s)Q(s) - \frac{a(1+\varrho)^a \mu^{1/a}(s)(\Delta\phi(s))^2}{4\phi(s)}\right] = \infty, \tag{2.27}$$

where $Q(\nu) = \dfrac{b(\nu)\Omega_1^a(\nu-\tau)}{\Omega(\nu+1)}$, then the conclusion of Theorem 2.5 holds.

Proof. Proceeding as in the proof of Corollary 2.6, we get (2.23). Using the fact that $\left\{\frac{Z(\nu)}{\Omega(\nu)}\right\}$ is decreasing in (2.23), we have

$$\Delta(\mu^{1/a}(\nu)\Delta Z(\nu)) + \frac{b(\nu)\Omega_1(\nu-\tau)}{a(1+\varrho)^a\Omega(\nu+1)} Z(\nu+1) \leq 0.$$

Now an application of Theorem 3.5 in [11] completes the proof. □

Theorem 2.2. *Let (1.1) and (2.9) hold. If*

$$\limsup_{\nu\to\infty} \left\{ \frac{1}{\Omega(\nu-\tau)} \sum_{s=\nu_1}^{\nu-1-\tau} \Omega(s+1)\Omega_1^a(s-\tau)b(s) + \sum_{s=\nu-\tau}^{\nu-1} b(s)\Omega_1^a(s-\tau) + \Omega(\nu-\tau)\sum_{s=\nu}^{\infty} \frac{b(s)\Omega_1^a(s-\tau)}{\Omega(s-\tau)} \right\} > a(1+\varrho)^a, \tag{2.28}$$

then the conclusion of Theorem 2.5 holds.

Proof. Let $\{\beta(\nu)\}$ be an eventually positive solution of (E). From Lemma 2.1, there exists an integer $\nu \geq \nu_0$ such that $\alpha(\nu) > 0$ and either case (I) or case (II) holds for all $\nu \geq \nu_1$. For the case (I), it follows from Lemma 2.3 that conclusion (2.10) holds. So, we consider the case (II). Proceeding as in the proof of Theorem 2.5, we are led to (2.14), that is,

$$\Delta(\mu^{1/a}(\nu)\Delta Z(\nu)) + \frac{b(\nu)\Omega_1^a(\nu-\tau)}{a(1+\varrho)^a\Omega(\nu-\tau)} Z(\nu-\tau) \leq 0, \nu \geq \nu_1.$$

Summing up the last inequality from ν to ∞, we get

$$\Delta Z(\nu) \geq \frac{1}{\mu^{1/a}(\nu)} \sum_{s=\nu}^{\infty} \frac{b(s)\Omega_1^a(s-\tau)}{a(1+\varrho)^a\Omega(s-\tau)} Z(s-\tau).$$

Again summing up this from $\nu_1 \geq \nu_0$ to $\nu-1$, and then using summation by parts formula, we obtain

$$a(1+\varrho)^a Z(\nu) \geq \sum_{s=\nu_1}^{\nu-1} \frac{\Omega(s+1)\Omega_1^a(s-\tau)b(s)}{\Omega(s-\tau)} Z(s-\tau) + \Omega(\nu)\sum_{s=\nu}^{\infty} \frac{b(s)\Omega_1^a(s-\tau)}{\Omega(s-\tau)} Z(s-\tau).$$

So, we have

$$a(1+\varrho)^a Z(\nu-\tau) \geq \sum_{s=\nu_1}^{\nu-\tau-1} \frac{\Omega(s+1)\Omega_1^a(s-\tau)b(s)}{\Omega(s-\tau)} Z(s-\tau)$$
$$+\Omega(\nu-\tau)\sum_{s=\nu-\tau}^{\nu-1} \frac{\Omega_1^a(s-\tau)b(s)}{\Omega(s-\tau)} Z(s-\tau)$$
$$+\Omega(\nu-\tau)\sum_{s=\nu}^{\infty} \frac{\Omega_1^a(s-\tau)b(s)}{\Omega(s-\tau)} Z(s-\tau).$$

By Lemma 2.4 (ii), we see that $\{Z(\nu)\}$ is increasing and $\left\{\frac{Z(\nu)}{\Omega(\nu)}\right\}$ is decreasing and using this in the above inequality, we get

$$a(1+\varrho)^a Z(\nu-\tau) \geq \frac{Z(\nu-\tau)}{\Omega(\nu-\tau)} \sum_{s=\nu_1}^{\nu-\tau-1} b(s)\Omega(s+1)\Omega_1^a(s-\tau)$$
$$+Z(\nu-\tau)\sum_{s=\nu-\tau}^{\nu-1} b(s)\Omega_1^a(s-\tau)$$
$$+Z(\nu-\tau)\Omega(\nu-\tau)\sum_{s=\nu}^{\infty} \frac{b(s)\Omega_1^a(s-\tau)}{\Omega(s-\tau)}.$$

So,

$$a(1+\varrho)^a \geq \left\{ \frac{1}{\Omega(\nu-\tau)} \sum_{s=\nu_1}^{\nu-\tau-1} b(s)\Omega(s+1)\Omega_1^a(s-\tau) \right.$$
$$\left. + \sum_{s=\nu-\tau}^{\nu-1} b(s)\Omega_1^a(s-\tau) + \Omega(\nu-\tau)\sum_{s=\nu}^{\infty} \frac{b(s)\Omega_1^a(s-\tau)}{\Omega(s-\tau)} \right\}.$$

This contradicts (2.28) and the proof of the theorem is complete. □

3. Examples

In this section, we present two examples to illustrate our results which are progressive, comparing to the already known ones.

Example 3.1. *Consider the third-order neutral delay difference equation*

$$\Delta(\nu(\Delta^2(\beta(\nu)+\varrho\beta(\nu-1))^3)) + \frac{\lambda}{\nu^6}\beta^3(\nu-2) = 0, \nu \geq 1, \tag{3.1}$$

where $\lambda > 0$ and $\varrho \neq 1$ are positive constants.

Comparing with (E), we see that $b(\nu) = \dfrac{\lambda}{\nu^6}$, $\mu(\nu) = \nu$, $\sigma = 1$, $\tau = 2$, and $a = 3$. A simple computation shows that $\Omega(\nu) \simeq \dfrac{3}{2}\nu^{2/3}$ and $\Omega_1(\nu) \simeq \dfrac{9}{10}\nu^{5/3}$. The condition (1.1) is evidently satisfied. The condition (2.9) becomes

$$\sum_{\nu=1}^{\infty}\sum_{s=1}^{\infty}\left(\frac{1}{s}\sum_{t=s}^{\infty}\frac{\lambda}{t^6}\right)^{1/3} \geq (\lambda/5)^{1/3}\sum_{\nu=1}^{\infty}\frac{1}{\nu} = \infty,$$

that is, condition (2.9) is satisfied. The condition (2.22) becomes

$$\begin{aligned}\liminf_{\nu\to\infty}\frac{3}{2}\nu^{2/3}\sum_{s=\nu}^{\infty}\frac{\lambda}{s^6}\left(\frac{2}{3}\right)\left(\frac{9}{10}\right)^3\frac{(s-2)^5}{s^{2/3}} &\simeq \liminf_{\nu\to\infty}\lambda(9/10)^3\nu^{2/3}\sum_{s=\nu}^{\infty}\frac{1}{s^{5/3}}\\ &= \lambda\left(\frac{3}{2}\right)\left(\frac{9}{10}\right)^3 > \frac{3(1+\varrho)^3}{4}\end{aligned}$$

that is, condition (2.22) is satisfied if $\lambda > \frac{500}{729}(1+\varrho)^3$. Hence all conditions of Corollary 2.6 are satisfied, and therefore every solution of (3.1) is either oscillatory or satisfies (2.10) if $\lambda > \frac{500}{729}(1+\varrho)^3$,

Next, we see that condition (2.20) is satisfied if $\lambda > \frac{1000}{729}(1+\varrho)^3$. Thus, Corollary 2.2 improves Theorem 2.4.

Note that equation (3.1) is considered in [15] and using Theorem 2.1 in [15], the authors obtained the same conclusion if $\lambda > \frac{625}{64}(1+\varrho)^3$. For $\varrho = 2$, we see that Corollary 2.6 gives $\lambda > 18.51581$ and Theorem 2.8 gives $\lambda > 37.037037$ but Theorem 2.1 in [15] gives $\lambda > 87.890625$. So our results are significantly better than Theorem 2.1 of [15].

Example 3.2. *Consider the third-order neutral difference equation*

$$\Delta^3\left(\beta(\nu)+\frac{1}{2}\beta(\nu-3)\right)+\frac{2\lambda(\nu+1)}{(\nu-1)^2(\nu-2)^2}\beta(\nu-1)=0, \nu\geq 3, \tag{3.2}$$

where $\lambda > 0$ *is a constant.*

Here $\mu(\nu) = 1$, $a = 1$, $\varrho = \frac{1}{2}$, $\sigma = 3$, $\tau = 1$ and $b(\nu) = \frac{2\lambda(\nu+1)}{(\nu-1)^2(\nu-2)^2}$. By a simple calculation we see that $\Omega(\nu) \simeq \nu$ and $\Omega_1(\nu) \simeq \frac{\nu^2}{2}$. The condition (1.1) clearly holds. The condition (2.9) becomes

$$\sum_{\nu=3}^{\infty}\sum_{s=\nu}^{\infty}\sum_{t=s}^{\infty}\frac{\lambda(t+1)}{(t-1)^2(t-2)^2} \geq \lambda\sum_{\nu=3}^{\infty}\frac{1}{\nu} = \infty,$$

that is, condition (2.9) is satisfied. The condition (2.22) becomes

$$\begin{aligned}\liminf_{\nu\to\infty}\nu\sum_{s=\nu}^{\infty}\frac{\lambda(s+1)}{(s-1)^2(s-2)^2}\left(\frac{(s-1)^2}{s}\right) &\geq \liminf_{n\to\infty}\lambda\nu\sum_{s=\nu}^{\infty}\frac{1}{s(s+1)}\\ &= \lambda > \frac{3}{8},\end{aligned}$$

that is, condition (2.22) is satisfied if $\lambda > \frac{3}{8}$.

Next, by taking $\phi(\nu) = \nu$, we see that the condition (2.27) is also satisfied if $\lambda > \frac{3}{8}$. Therefore, by Corollary 2.6 or by Corollary 2.7, every solution of (3.2) is either oscillatory or satisfies (2.10) if $\lambda > \frac{3}{8}$. The same equation is considered in Example 2 of [14] and the same conclusion is obtained if $\lambda > 8$. Therefore our results give better condition than that in [14].

4. Conclusions

The results presented in this paper are new and improve some of the known ones. The main technique here is to reduce the oscillation of the studied third-order delay difference equation to that of linear second-order delay difference equations whose oscillatory behavior is known in the literature. Two examples are presented to point out the progress of our results over the known ones.

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