CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 253-272

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.01

A two-step inertial algorithm for solving equilibrium problems on Hadamard manifolds

H. A. ABASS[1], A. ADAMU[2,3] AND M. APHANE[1]

ABSTRACT. In the context of Hadamard manifolds, we develop a two-step inertial subgradient extragradient method for approximating solutions of equilibrium problems involving a pseudomonotone operator. To avoid dependency of the step-size on the Lipschitz constant of the underlying operator, we propose an iterative method with a self-adaptive step size that can increase with each iteration. Furthermore, we prove a strong convergence result concerning the sequence generated by our two-step inertial subgradient extragradient method in the setting of Hadamard manifolds. To illustrate the performance of our iterative method relative to other methods of a similar nature, we provide some numerical examples. The results presented in this article are new in this space and extend many related findings in the literature.

1. INTRODUCTION

Many nonlinear analysis problems are extended from Euclidean spaces to Hadamard manifolds, including problems in variational inequality, convex analysis, fixed point theory, and optimization problems. Indeed, many optimization-related problems on Hadamard manifolds are revised by endowing the space with a Riemannian metric. It is important to note that Riemannian manifolds-particularly Hadamard manifolds-hold the topological structure, making them the ideal framework for developing ideas and techniques from Euclidean spaces to nonlinear forms. Hence, the fundamental notion behind the aforementioned issues is to apply ideas and methods that work in Euclidean spaces to Riemannian manifolds (see [6, 34, 49] and other references).

The equilibrium problem, also referred to as Ky Fan's inequality because of his substantial contribution (see [21]), is a general problem in that it includes a variety of mathematical models, including complementarity problems, variational inequality problems (also known as VIP), saddle points, vector minimization problems, Nash equilibrium of noncooperative games, and the Kirszbraun problem (see [10, 14, 24, 21, 50, 53]). Let $\mathcal{K}$ be a nonempty, closed geodesic convex subset of a Hadamard manifold $\mathbb{P}$, $T_x\mathbb{P}$ be the tangent space of $\mathbb{P}$ at $x \in \mathbb{P}$ and $T\mathbb{P}$ be the tangent bundle of $\mathbb{P}$. Let $h : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be a bifunction satisfying $h(x, x) = 0$, for all $x \in \mathbb{P}$. The Equilibrium Problem (in short, EP) is to find $x \in \mathbb{P}$ such that

$$h(x, y) \geq 0, \ \forall\, y \in \mathcal{K}. \tag{1.1}$$

We denote the solution set of (1.1) by Ω. In nonlinear analysis and optimization theory, another fascinating area of study is the creation of an efficient algorithm for approximating

Received: 30.04.2024. In revised form: 28.08.2024. Accepted: 22.09.2024
2010 *Mathematics Subject Classification.* 47H09, 49J27, 65K10, 90C25.
Key words and phrases. *Equilibrium problem, double inertial method, Hadamard manifold, pseudomonotone operator, Riemannian manifold.*
Corresponding author: A. Adamu; abubakar.adamu@neu.edu.tr

solutions to optimization problems. To approximate solutions of EP in linear and nonlinear space settings, various iterative methods have been used (see [4, 13, 45, 52, 15]).

The Proximal Point Method (in short, PPM) was proposed by Martinet [32] to solve VIP and later Moudafi [33] proposed it to solve monotone equilibrium problems. In order to overcome this limitation, Flam *et al.* [23] and Tran *et al.* [45] successively introduced a PPM which is called the extragradient method (in short, EGM) due to Korpelevich's contributions to saddle point problems in [28]. From the point of view, the EGM can be easier to compute numerically than the PPM in solving optimization problems. In the case when the feasible set has a simple structure, then the projection can be easily computed. However, if the feasible set is any closed convex set, the computation of projections, in general, is difficult to compute which can affect the efficiency of the EGM. Spurred by the EGM, Censor *et al.* [12] introduced the subgradient extragradient method (SEGM) for solving VIP in Hilbert spaces. In this method, the second projection in the EGM is replaced by a projection onto a constructed half-space and allows a clear computation. The projection onto a half-space is inherently explicit, thus the SEGM can be considered as an improvement of EGM over each computational step. The SEGM have been employed for solving equilibrium problem (see [45, 12, 37, 54, 55]). Extension of concepts and techniques from linear spaces to Riemannian manifolds has some important advantages (see [29, 42, 22]). For instance, some optimization problems with nonconvex objective functions become convex from the Riemannian geometry point of view, and some constrained optimization problems can be regarded as unconstrained ones with an appropriate Riemannian metric. In addition, the study of convex minimization problems and equilibrium problems in nonlinear spaces have proved to be very useful in computing medians and means of trees, which are very important in computational phylogenetics, diffusion tensor imaging, consensus algorithms and modeling of airway systems in human lungs and blood vessels (see [7, 8, 9]). Thus, nonlinear spaces are more suitable frameworks for the study of optimization problems from linear to Riemannian manifolds.

Iterative methods for solving pseudomonotone EP on Hadamard manifolds have received a lot of attention (see [17, 30, 38]). Recently, Cruz Neto *et al.* [35] proposed the following EGM for solving equilibrium problem on Hadamard manifolds as follows:

$$
\begin{cases} w_k = arg\min_{y\in\mathbb{P}} \left\{h(x_k,y) + \frac{1}{2\lambda_k}d^2(x_k,y)\right\} \\ x_{k+1} = arg\min_{y\in\mathbb{P}} \left\{h(w_k,y) + \frac{1}{2\lambda_k}d^2(x_k,y)\right\}, \end{cases} \tag{1.2}
$$

where $0 < \lambda_k < \beta < \min\{\alpha_1^{-1}, \alpha_2^{-1}\}, \alpha_1, \alpha_2$ are constants related to Lipschitz-type constants are unknown in general and they are difficult to approximate. Very recently, Ali Akbar [4] introduced the following subgradient extragradient method for solving EP as follows:

Initialization: Choose an inertial $x_0 \in \mathcal{K}$ and a parameter λ satisfying $0 < \lambda < \min\{\frac{1}{2\alpha_1}, \frac{1}{2\alpha_2}\}$.

Iterative steps; Assume that $x_k \in \mathcal{K}$ and we calculate $x_{k+1} \in \mathcal{K}$ as follows:

Step 1: Compute

$$y_k = arg\min\{\lambda h(x_k,y) + \frac{1}{2}d^2(x_k,y) : y \in \mathcal{K}\}.$$

If $x_k = y_k$, then stop and x_k is a solution to EP. Otherwise,
Step 2: Compute

$$x_{k+1} = arg\min\{\lambda h(y_k,y) + \frac{1}{2}d^2(x_k,y) : y \in T_k\},$$

where $T_k := \{p \in \mathbb{P} : \langle \exp_{y_k}^{-1} x_k - \lambda v_k, \exp_{y_k}^{-1} p \rangle \le 0\}$ and $v_k \in \partial_2 h(x_k, y_k)$.
They proved that their iterative algorithm converges to a solution of EP (1.1).

However, Polyak [39] was the first to propose an inertial-type algorithm as an acceleration process for solving a smooth convex minimization problem. Using the previous two iterates to define the next iterate is the process of the inertial-type algorithm, which is a two-step iterative method. It is common knowledge that adding an inertial term to another algorithm quickens the algorithm's sequence's rate of convergence. Many authors have recently focused a great deal of attention on inertial techniques intended for solving equilibrium problems in linear and nonlinear spaces (see [1, 16, 37, 20, 5, 44]). In 2003, Moudafi and Oliny [33] introduced the following inertial proximal point method for finding the zero of the sum of two monotone operators:

$$\begin{cases} w_k = q_k + \theta_k(q_k - q_{k-1}), \\ q_{k+1} = (I + r_k\Psi)^{-1}(w_k - r_k\Phi q_k), \; k \ge 1, \end{cases} \tag{1.3}$$

where $r > 0$, $\Psi : \mathbb{M} \to 2^{\mathbb{M}}$ is a set-valued operator, $\Phi : \mathbb{M} \to \mathbb{M}$ is an operator and $\mathbb{M}$ is a real Hilbert space. They obtained a weak convergence theorem provided that $r_k < \frac{2}{L}$ with L being the Lipschitz constant of Φ and $\sum_{k=1}^{\infty} \theta_k \|q_k - q_{k-1}\| < \infty$ holds. In [38], Polyak explained that the numerical iteration method with multistep inertial extrapolation steps could improve the rate of convergence of such methods for solving optimization problems even though neither the convergence analysis nor the rate of convergence result of such multi-step inertial method is established in [38]. In view of this, some authors have proposed multi-step inertial methods for solving optimization problems in linear spaces, (see [44, 51, 18]).

We highlight our contributions as follows:

- (i) Two extragradient techniques for solving EP in a Hadamard manifold were presented by the authors in [35]. It is observed that the iterative algorithm in [35] necessitates the estimation of the bifunction's Lipschitz-like constants beforehand, which can be highly challenging to determine. When it comes to numerical computation, the subgradient extragradient method is preferable over the extragradient method. The self-adaptive used in this article is simple to compute.
- (ii) In addition, the authors of [4] suggested a subgradient extragradient approach to solving EP in the Hadamard manifold. We point out that their algorithm's convergence depends on a prior estimate of the Lipschitz-like constants.
- (iii) Aside from the inertial extrapolation technique we included in our iterative process, which is known to accelerate iterative methods' rate of convergence. In our paper, we also introduced a parameter λ, which further accelerates our algorithm's rate of convergence.
- (iv) Our result clearly extends the results of [3, 19, 54, 55] from real Hilbert spaces to the Hadamard manifold.

In the context of a Hadamard manifold, we propose a double step inertial subgradient extragradient method for solving the pseudomontone equilibrium problem. The previously mentioned findings on inertial extrapolation techniques, subgradient extragradient techniques, and extragradient methods served as inspiration for this approach. We show that the sequence generated by our iterative approach converges to a solution of a pseudomonotone equilibrium problem under some mild conditions. We provide some numerical examples to compare the performance of our iterative method with several relevant ones in the literature. To the best of our knowledge, the result presented here is new in the context of a Hadamard manifold and generalizes other similar results found in the literature.

2. Preliminaries

Let $\mathbb{P}$ be an m-dimensional manifold, let $x \in \mathbb{P}$ and let $T_x\mathbb{P}$ be the tangent space of $\mathbb{P}$ at $x \in \mathbb{P}$. We denote by $T\mathbb{P} = \bigcup_{x\in\mathbb{P}} T_x\mathbb{P}$ the tangent bundle of $\mathbb{P}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{P}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{P}\times T_x\mathbb{P} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{P}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{P}$ is denoted by $\|\cdot\|_x$. We will drop the subscript x and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{P}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{P}$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of γ by $l(\gamma) := \int_a^b \|\gamma'(t)\|dt$. The Riemannian distance $d(x, y)$ is the minimal length over the set of all such curves joining x to y. The metric topology induced by d coincides with the original topology on $\mathbb{P}$. We denote by ∇ the Levi-Civita connection associated with the Riemannian metric [42]. Let γ be a smooth curve in $\mathbb{P}$. A vector field X along γ is said to be parallel if $\nabla_{\gamma'}X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' itself is parallel along γ, then we say that γ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\| = 1$, then the geodesic γ is said to be normalized. A geodesic joining x to y in $\mathbb{P}$ is called a minimizing geodesic if its length equals $d(x, y)$. A Riemannian manifold $\mathbb{P}$ equipped with a Riemannian distance d is a metric space $(\mathbb{P}, d)$. A Riemannian manifold $\mathbb{P}$ is said to be complete if for all $x \in \mathbb{P}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. The Hopf-Rinow theorem [42], posits that if $\mathbb{P}$ is complete, then any pair of points in $\mathbb{P}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{P}, d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{P}$ is compact. If $\mathbb{P}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{P} \to \mathbb{P}$ at $x \in \mathbb{P}$ is defined by

$$\exp_x v := \gamma_v(1, x)\ \forall\, v \in T_x\mathbb{P},$$

where $\gamma_v(\cdot, x)$ is the geodesic starting from x with velocity v (that is, $\gamma_v(0, x) = x$ and $\gamma_v'(0, x) = v$). Then, for any t, we have $\exp_x tv = \gamma_v(t, x)$ and $\exp_x \mathbf{0} = \gamma_v(0, x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{P}$ for every $x \in \mathbb{P}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{P} \to T_x\mathbb{P}$. For any $x, y \in \mathbb{P}$, we have $d(x, y) = \|\exp_y^{-1} x\| = \|\exp_x^{-1} y\|$ (see [42] for more details). The parallel transport $\Gamma_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{P} \to T_{\gamma(b)}\mathbb{P}$ on the tangent bundle $T\mathbb{P}$ along $\gamma : [a, b] \to \mathbb{R}$ with respect to ∇ is defined by

$$\Gamma_{\gamma,\gamma(b),\gamma(a)}v = F(\gamma(b)),\ \forall\, a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{P},$$

where F is the unique vector field such that $\nabla_{\gamma'(t)}v = \mathbf{0}$ for all $t \in [a, b]$ and $F(\gamma(a)) = v$. If γ is a minimizing geodesic joining x to y, then we write $\Gamma_{y,x}$ instead of $\Gamma_{\gamma,y,x}$. Note that for every $a, b, r, s \in \mathbb{R}$, we have

$$\Gamma_{\gamma(s),\gamma(r)} \circ \Gamma_{\gamma(r),\gamma(a)} = \Gamma_{\gamma(s),\gamma(a)} \text{ and } \Gamma^{-1}_{\gamma(b),\gamma(a)} = \Gamma_{\gamma(a),\gamma(b)}.$$

Also, $\Gamma_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{P}$ to $T_{\gamma(b)}\mathbb{P}$, that is, the parallel transport preserves the inner product

$$\langle\Gamma_{\gamma(b),\gamma(a)}(u), \Gamma_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)} = \langle u, v\rangle_{\gamma(a)},\ \forall\, u, v \in T_{\gamma(a)}\mathbb{P}. \tag{2.4}$$

Below there are three (3) examples of a Hadamard manifold (see [40]).

Example 2.1. *The Hyperbolic plane (without boundary) is defined as*

$$\mathcal{H}^2 = \{(x, y) \in \mathbb{R}^2, y > 0\},$$

with Riemannian metric defined as

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

is a Hadamard manifold with sectional curvature -1.

Example 2.2. *Let $n \geq 1$ be a natural number. The Hyperbolic n-space $\mathbb{H}^n$ is defined:*

$$\mathbb{H}^n = \{(x_1, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.$$

Recall that the metric $\mathbb{H}^n$ is induced from Lorentz metric $< .,. >_1$ which is defined on $\mathbb{R}^{n+1}$ as follows:

$$< (x_1, \cdots, x_{n+1}), (y_1, \cdots, y_{n+1}) >_1 = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

So $\mathbb{H}^n$ is a Hadamard manifold with sectional curvature -1 (see [11]).

Example 2.3. *Let $\mathbb{R}^m_{++}$ be the product space $\mathbb{R}^m_{++} := \{(x_1, x_2, \cdots, x_m) : x_i \in \mathbb{R}_{++},\ i = 1, 2, \cdots, m\}$. Let $\mathbb{P} = ((R)_{++}, \langle \cdot, \cdot \rangle)$ be the m-dimensional Hadamard manifold with the Riemannian metric $\langle p, q \rangle = p^T q$ and the distance $d(x, y) = |\ln \frac{x}{y}| = |\ln \sum_{i=1}^{m} \frac{x_i}{y_i}|$, where $x, y \in \mathbb{P}$ with $x = \{x_i\}_{i=1}^m$ and $y = \{y_i\}_{i=1}^m$.*

A subset $\mathcal{K} \subset \mathbb{P}$ is said to be convex if for any two points $x, y \in \mathcal{K}$, the geodesic γ joining x to y is contained in $\mathcal{K}$. That is, if $\gamma : [a, b] \to \mathbb{P}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in \mathcal{K}$ for all $t \in [0, 1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called an Hadamard manifold. We denote by $\mathbb{P}$ a finite dimensional Hadamard manifold. Henceforth, unless otherwise stated, we represent by $\mathcal{K}$ a nonempty, closed and convex subset of $\mathbb{P}$.

The subdifferential of a function $f : \mathbb{P} \to \mathbb{R}$ at a point $x \in \mathbb{P}$ is given

$$\partial f(x) := \{z \in T_x\mathbb{P} : f(y) \geq f(x) + \langle z, \exp_x^{-1} y \rangle \leq 0,\ \forall\, y \in \mathcal{K}\},$$

and its elements are called subgradient of f at x. The convex function f is called subdifferential at a point $x \in \mathbb{P}$ if the set $\partial f(x)$ is nonempty. The set $\partial f(x)$ is closed and convex, and it is known to be nonempty if f is convex on $\mathbb{P}$. We denote by $\partial_2 h$ the partial derivative of h at the second argument, that is $\partial_2 h(x, .)$, for all $x \in \mathbb{P}$. The normal cone, denoted $N_{\mathcal{K}}$, is defined at a point $x \in \mathbb{P}$ by

$$N_{\mathcal{K}}(x) := \{z \in T_x\mathbb{P} : \langle z, \exp_x^{-1} y \rangle \leq 0, \forall\, y \in \mathcal{K}\}.$$

We state some results and definitions which are needed in the next section.

Definition 2.1. *[37] Let $\mathcal{K}$ be a nonempty, closed and convex subset of $\mathbb{P}$. A bifunction $h : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ is said to be*

(i) monotone on $\mathcal{K}$, if

$$h(u, v) + h(v, u) \leq 0,\ \forall\, u, v \in \mathcal{K};$$

(ii) pseudomonotone on $\mathcal{K}$, if

$$h(u, v) \geq 0 \Rightarrow g(v, u) \leq 0,\ \forall\, u, v \in \mathcal{K};$$

(iii) Lipschitz-type continuous if there exists $\alpha_1 > 0$ and $\alpha_2 > 0$, such that

$$h(u, v) + h(v, w) \geq h(u, w) - \alpha_1 d^2(u, v) - \alpha_2 d^2(v, w),\ \forall\, u, v, w \in \mathcal{K}.$$

Definition 2.2. *[22] Let $\mathcal{K}$ be a nonempty, closed and subset of $\mathbb{P}$ and $\{x_n\}$ be a sequence in $\mathbb{P}$. Then $\{x_n\}$ is said to be Fejèr convergent with respect to $\mathcal{K}$ if for all $p \in \mathcal{K}$ and $n \in \mathbb{N}$,*

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Lemma 2.1. *[22] Let $\mathcal{K}$ be a nonempty, closed and closed subset of $\mathbb{P}$ and $\{x_n\} \subset \mathbb{P}$ be a sequence such that $\{x_n\}$ be a Fejér convergent with respect to $\mathcal{K}$. Then the following hold:*

(i) For every $p \in \mathcal{K}$, $d(x_n, p)$ converges,

(ii) $\{x_n\}$ is bounded,

(iii) Assume that every cluster point of $\{x_n\}$ belongs to $\mathcal{K}$, then $\{x_n\}$ converges to a point in $\mathcal{K}$.

Proposition 2.1. *[42]. Let $x \in \mathbb{P}$. The exponential mapping $\exp_x : T_x\mathbb{P} \to \mathbb{P}$ is a diffeomorphism. For any two points $x, y \in \mathbb{P}$, there exists a unique normalized geodesic joining x to y, which is given by*

$$\gamma(t) = \exp_x t \exp_x^{-1} y \;\forall\, t \in [0,1].$$

A geodesic triangle $\Delta(p,q,r)$ of a Riemannian manifold $\mathbb{P}$ is a set containing three points p, q, r and three minimizing geodesics joining these points.

Proposition 2.2. *[42]. Let $\Delta(p,q,r)$ be a geodesic triangle in $\mathbb{P}$. Then*

$$d^2(p,q) + d^2(q,r) - 2\langle \exp_q^{-1} p, \exp_q^{-1} r\rangle \leq d^2(r,q) \tag{2.5}$$

and

$$d^2(p,q) \leq \langle \exp_p^{-1} r, \exp_p^{-1} q\rangle + \langle \exp_q^{-1} r, \exp_q^{-1} p\rangle. \tag{2.6}$$

Moreover, if θ is the angle at p, then we have

$$\langle \exp_p^{-1} q, \exp_p^{-1} r\rangle = d(q,p)d(p,r)\cos\theta. \tag{2.7}$$

Also,

$$\|\exp_p^{-1} q\|^2 = \langle \exp_p^{-1} q, \exp_p^{-1} q\rangle = d^2(p,q). \tag{2.8}$$

Remark 2.1. *[31] If $x, y \in \mathbb{P}$ and $v \in T_y\mathbb{P}$, then*

$$\langle v, -\exp_y^{-1} x\rangle = \langle v, \Gamma_{y,x} \exp_x^{-1} y\rangle = \langle \Gamma_{x,y} v, \exp_x^{-1} y\rangle. \tag{2.9}$$

Remark 2.2. *From (2.6) and Remark 2.1, let $v \in T_p\mathbb{P}$, we have*

$$\langle v, \exp_p^{-1} q\rangle \leq \langle v, \exp_p^{-1} r\rangle + \langle v, \Gamma_{p,r} \exp_r^{-1} q\rangle. \tag{2.10}$$

Lemma 2.2. *[31] Let $x_0 \in \mathbb{P}$ and $\{x_n\} \subset \mathbb{P}$ with $x_n \to x_0$. Then the following assertions hold:*

(i) For any $y \in \mathbb{P}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} x_n$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$,
(ii) If $v_n \in T_{x_n}\mathbb{P}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathbb{P}$,
(iii) Given $u_n, v_n \in T_{x_n}\mathbb{P}$ and $u_0, v_0 \in T_{x_0}\mathbb{P}$, if $u_n \to u_0$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$,
(iv) For any $u \in T_{x_0}\mathbb{P}$, the function $F : \mathbb{P} \to T\mathbb{P}$, defined by $F(x) = \Gamma_{x,x_0} u$ for each $x \in \mathbb{P}$ is continuous on $\mathbb{P}$.

The next lemma presents the relationship between triangles in $\mathbb{R}^2$ and geodesic triangles in Riemannian manifolds (see [11]).

Lemma 2.3. *[11]. Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in $\mathbb{P}$. Then there exists a triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ corresponding to $\Delta(x_1, x_2, x_3)$ such that $d(x_i, x_{i+1}) = \|\bar{x}_i - \bar{x}_{i+1}\|$ with the indices taken modulo 3. This triangle is unique up to isometries of $\mathbb{R}^2$.*

The triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Lemma 2.3 is said to be the comparison triangle for $\Delta(x_1, x_2, x_3) \subset \mathbb{P}$. The points $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are called comparison points to the points x_1, x_2 and x_3 in $\mathbb{P}$.
A function $h : \mathbb{P} \to \mathbb{R}$ is said to be geodesic if for any geodesic $\gamma \in \mathbb{P}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leq \lambda h \circ \gamma(u) + (1-\lambda) h \circ \gamma(v), \; u, v \in \mathbb{R}, \; \lambda \in [0,1].$$

Lemma 2.4. *[31] Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold $\mathbb{P}$ and $\Delta(p^{'}, q^{'}, r^{'})$ be its comparison triangle.*

(i) Let α, β, γ (resp. $\alpha^{'}, \beta^{'}, \gamma^{'}$) be the angles of $\Delta(p, q, r)$ (resp. $\Delta(p^{'}, q^{'}, r^{'})$) at the vertices p,q,r (resp. $p^{'}, q^{'}, r^{'}$). Then, the following inequalities hold:

$$\alpha^{'} \geq \alpha, \; \beta^{'} \geq \beta, \; \gamma^{'} \geq \gamma,$$

(ii) Let z be a point in the geodesic joining p to q and $z^{'}$ its comparison point in the interval $[p^{'}, q^{'}]$. Suppose that $d(z, p) = \|z^{'} - p^{'}\|$ and $d(z^{'}, q^{'}) = \|z^{'} - q^{'}\|$. Then the following inequality holds:

$$d(z, r) \leq \|z^{'} - r^{'}\|.$$

Lemma 2.5. *[31] Let $x_0 \in \mathbb{P}$ and $\{x_n\} \subset \mathbb{P}$ be such that $x_n \to x_0$. Then, for any $y \in \mathbb{P}$, we have $\exp^{-1}_{x_n} y \to \exp^{-1}_{x_0} y$ and $\exp^{-1}_{y} x_n \to \exp^{-1}_{y} x_0$;*

The following propositions (see [22, 31]) are very useful in our convergence analysis:

Proposition 2.3. *Let $\mathcal{K}$ be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$ and $h : \mathcal{K} \to \mathbb{R}$ be a convex subdifferential and lower semicontinuous function on $\mathcal{K}$. Then, x^* is a solution to the following convex problem*

$$\min\{h(x) : x \in \mathcal{K}\},$$

if and only if $0 \in \partial g(x^*) + N_{\mathcal{K}}(x^*)$.

Proposition 2.4. *Let $\mathbb{P}$ be an Hadamard manifold and $d : \mathbb{P} \times \mathbb{P} :\to \mathbb{R}$ be the distance function. Then the function d is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics $\gamma_1 : [0,1] \to \mathbb{P}$ and $\gamma_2 : [0,1] \to \mathbb{P}$, then for all $t \in [0,1]$, we have*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in \mathbb{P}$, the function $d(\cdot, y) : \mathbb{P} \to \mathbb{R}$ is a convex function.

Proposition 2.5. *Let $\mathbb{P}$ be a Hadamard manifold and $x \in \mathbb{P}$. The map $\Phi_x = d^2(x, y)$ satisfying the following:*

(1) Φ_x is convex. Indeed, for any geodesic $\gamma : [0,1] \to \mathbb{P}$, the following inequality holds for all $t \in [0,1]$:

$$d^2(x, \gamma(t)) \leq (1-t) d^2(x, \gamma(0)) + t d^2(x, \gamma(1)) - t(1-t) d^2(\gamma(0), \gamma(1)).$$

(2) Φ_x is smooth. Moreover, $\partial \Phi_x(y) = -2 \exp^{-1}_y x$.

Lemma 2.6. *[25] Let $\{v_n\}$ and $\{\delta_n\}$ be nonnegative sequences which satisfy*

$$v_{n+1} = (1 + \delta_n) v_n + \delta_n v_{n-1}, \; n \geq 1.$$

Then,

$$v_{n+1} \leq M \cdot \prod_{j=1}^{n} (1 + 2\delta_j), \; \text{where } M = \max\{v_1, v_2\}.$$

Moreover, if $\sum_{n=1}^{\infty} \delta_n < +\infty$, then $\{v_n\}$ is bounded.

Lemma 2.7. *[36] Let $\{a_n\}, \{\varphi_n\}$ and $\{\beta_n\}$ be nonnegative sequences which satisfy*

$$a_{n+1} = (1+\beta_n)a_n + \varphi_n, \ n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < +\infty$ and $\sum_{n=1}^{\infty} \varphi_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main result

In this section, we present a two steps subgradient extragradient method for solving equilibrium problem in the settings of Hadamard manifolds. For solving EP (1.1), we state the following assumptions:

Assumption 3.6. *(1) (A1) h is pseudomonotone on $\mathcal{K}$ and $h(u,u) = 0$ for all $u \in \mathbb{P}$;*
(2) (A2) h(.,v) is upper semicontinuous for all $y \in \mathbb{P}$;
(3) (A3) $h(u,.)$ is convex and subdifferential for all fixed $u \in \mathbb{P}$;
(4) (A4) h satisfies a Lipschitz-type condition on $\mathbb{P}$.

Assumption 3.7. *(L1) The solution set Ω is nonempty.*

(L2) $\{\tau_k\}$ is a nonnegative sequence of real numbers satisfying $\sum_{k=1}^{\infty} \tau_k < +\infty$.

Algorithm 3.8. *Two steps inertial subgradient extragradient method for EP.*

Initialization: *Choose $\rho_1 > 0$, $\delta \in (0,1), \lambda \in (0,1]$, $\{\theta_k\}, \{\beta_k\}$ are real positive sequences. Let $t_0, t_1 \in \mathbb{P}$ be arbitrary.*

Iterative steps: *Given the current iterate t_k, t_{k+1} calculate ρ_{k+1} as follows:*

Step 1: *Evaluate*

$$\begin{cases} u_k = \exp_{t_k}(-\theta_k \exp^{-1}_{t_k} t_{k-1}) \\ v_k = \exp_{u_k}(-\beta_k \exp^{-1}_{u_k} t_{k-1}), \end{cases} \tag{3.11}$$

and

$$w_k = arg \min_{y\in\mathbb{P}} \{\rho_k h(v_k, y) + \frac{1}{2} d^2(v_k, y)\} \tag{3.12}$$

If $w_k = v_k$, then stop. Otherwise, proceed to the next step.

Step 2: *Choose $z_k \in \partial_2 h(v_k, w_k)$ and compute*

$$t_{k+1} = arg \min_{y\in T_k} \{\lambda\rho_k h(w_k, y) + \frac{1}{2} d^2(v_k, y)\}, \tag{3.13}$$

where $T_k := \{x \in \mathbb{P} : \langle \exp^{-1}_{w_k} v_k - \rho_k z_k, \exp^{-1}_{w_k} x\rangle \leq 0\}$.
and

$$\rho_{k+1} = \begin{cases} \min\left\{\rho_k + \tau_k \ , \ \dfrac{\delta\left[d^2(w_k, v_k) + d^2(t_{k+1}, w_k)\right]}{2\left[h(v_k, t_{k+1}) - h(v_k, w_k) - h(w_k, t_{k+1})\right]}\right\}, \\ \quad \text{if } h(v_k, t_{k+1}) - h(v_k, w_k) - h(w_k, t_{k+1}) > 0, \\ \rho_k + \tau_k, \qquad\qquad \text{otherwise.} \end{cases} \tag{3.14}$$

Update *Set $k := k+1$ and return to* **Iterative step 1**.

We start by establishing a technical lemma useful to our analysis.

Lemma 3.8. *[2, 37] Let $\{t_k\}$ be a sequence generated by Algorithm 3.8 and $\{\rho_k\}$ be generated by (3.14)*

$$\min\left\{\frac{\delta}{2\max\{c_1, c_2\}}, \rho_1\right\} \leq \rho \leq \rho_1 + \tau,$$

where $\tau = \sum_{k=0}^{\infty} \tau_k$.

Theorem 3.9. *Let* $\{t_k\}$ *be a sequence generated by Algorithm 3.8. If* $\sum_{k=1}^{\infty} \theta_k < +\infty$ *and* $\sum_{k=1}^{\infty} \beta_k < +\infty$, *then*

(i) $d(t_{k+1}, q) \leq M \cdot \prod_{j=1}^{k} (1 + 2(\theta_j + \beta_j(1 + \theta_j)))$, *where* $M := \max\{d(t_1, q), d(t_2, q)\}$ *and* $q \in \Omega$.

(ii) The sequence $\{q_k\}$ *converges to an element in* Ω.

(i) *Proof.* Let $q \in \Omega$, then by applying Proposition 2.3 and (3.13) in Algorithm 3.8, we deduce that

$$0 \in \partial_2 \big[\lambda \rho_k h(w_k, y) + \frac{1}{2} d^2(v_k, y)\big](t_{k+1}) + N_{T_k}(t_{k+1}), \ \forall\, y \in T_k. \tag{3.15}$$

It follows that there exists $d_k \in \partial_2 h(w_k, t_{k+1})$ and $r_k \in N_{T_k}(t_{k+1})$ such that

$$\lambda \rho_k d_k - \exp^{-1}_{t_{k+1}} v_k + r_k = 0.$$

Thus, for any $y \in T_k$, we obtain that

$$\langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} y \rangle = \lambda \rho_k \langle d_k, \exp^{-1}_{t_{k+1}} y \rangle + \langle r_k, \exp^{-1}_{t_{k+1}} y \rangle.$$

Since $r_k \in N_{T_k}(t_{k+1})$, we obtain $\langle r_k, \exp^{-1}_{t_{k+1}} y \rangle \leq 0$. Then

$$\lambda \rho_k \langle d_k, \exp^{-1}_{t_{k+1}} y \rangle \geq \langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} y \rangle, \ \forall\, y \in T_k. \tag{3.16}$$

By the definition of subdifferential and $d_k \in \partial_2 h(w_k, t_{k+1})$, we get

$$h(w_k, y) - h(w_k, t_{k+1}) \geq \langle d_k, \exp^{-1}_{t_{k+1}} y \rangle, \ \forall\, y \in T_k. \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\lambda \rho_k \big(h(w_k, y) - h(w_k, t_{k+1})\big) \geq \langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} y \rangle, \ \forall\, y \in T_k. \tag{3.18}$$

Let $y = q \in \Omega \subset \mathcal{K} \subset T_k$ in (3.18), then

$$\lambda \rho_k \big(h(w_k, q) - h(w_k, t_{k+1})\big) \geq \langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} q \rangle. \tag{3.19}$$

Since $q \in \Omega$, we have $h(q, w_k) \geq 0$. By the pseudomonotonicity of h, we deduce that $h(w_k, q) \leq 0$. Thus (3.19) can be transformed into

$$\langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} q \rangle \geq \lambda \rho_k h(w_k, t_{k+1}). \tag{3.20}$$

Similarly, since $z_k \in \partial_2 h(v_k, w_k)$, we obtain that

$$h(v_k, z) - h(v_k, w_k) \geq \langle z_k, \exp^{-1}_{w_k} z \rangle, \ \forall\, z \in \mathbb{P}.$$

Let $z := t_{k+1}$, then

$$h(v_k, t_{k+1}) - h(v_k, w_k) \geq \langle z_k, \exp^{-1}_{w_k} t_{k+1} \rangle. \tag{3.21}$$

By definition of T_k and $t_{k+1} \in T_k$, we have $\langle \exp^{-1}_{w_k} v_k - \rho_k z_k, \exp^{-1}_{w_k} t_{k+1} \rangle \leq 0$. This implies that

$$\rho_k \langle z_k, \exp^{-1}_{w_k} t_{k+1} \rangle \geq \langle \exp^{-1}_{w_k} v_k, \exp^{-1}_{w_k} t_{k+1} \rangle. \tag{3.22}$$

Combining (3.21) and (3.22), we get

$$\rho_k \big(h(v_k, t_{k+1}) - h(v_k, w_k)\big) \geq \langle \exp^{-1}_{w_k} v_k, \exp^{-1}_{w_k} t_{k+1} \rangle. \tag{3.23}$$

From (3.14), we obtain

$$\rho_{k+1}\big(h(v_k,t_{k+1})-h(v_k,w_k)-h(w_k,t_{k+1})\big)\leq \frac{\delta}{2}\big(d^2(v_k,w_k)+d^2(t_{k+1},w_k)\big),$$

or equivalently

$$\rho_k\big(h(v_k,t_{k+1})-h(v_k,w_k)-h(w_k,t_{k+1})\big)\leq \frac{\rho_k}{\rho_{k+1}}\frac{\delta}{2}\big(d^2(v_k,w_k)+d^2(t_{k+1},w_k)\big). \tag{3.24}$$

On substituting (3.24) into (3.23), it yields

$$\langle \exp^{-1}_{w_k} v_k, \exp^{-1}_{w_k} t_{k+1}\rangle \leq \rho_k h(w_k,t_{k+1})+\frac{\rho_k}{\rho_{k+1}}\frac{\delta}{2}\big(d^2(v_k,w_k)+d^2(t_{k+1},w_k)\big). \tag{3.25}$$

Adding (3.20) and (3.25), we get

$$\langle \exp^{-1}_{w_k} v_k, \exp^{-1}_{w_k} t_{k+1}\rangle \leq \frac{1}{\lambda}\langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} q\rangle+\frac{\rho_k}{\rho_{k+1}}\frac{\delta}{2}\big(d^2(v_k,w_k)+d^2(t_{k+1},w_k)\big). \tag{3.26}$$

By applying Proposition 2.2, we obtain

$$d^2(v_k,t_{k+1})+d^2(t_{k+1},q)-d^2(v_k,q)\leq 2\langle \exp^{-1}_{t_{k+1}} v_k, \exp^{-1}_{t_{k+1}} q\rangle \tag{3.27}$$

and

$$-2\langle \exp^{-1}_{w_k} v_k, \exp^{-1}_{w_k} t_{k+1}\rangle \leq d^2(v_k,t_{k+1})-d^2(v_k,w_k)-d^2(t_{k+1},w_k). \tag{3.28}$$

On substituting (3.27) and (3.28) into (3.26), we get

$$\begin{aligned} d^2(t_{k+1},q)\leq{}& d^2(v_k,q)-(1-\lambda)d^2(v_k,t_{k+1})\\ &-\lambda\Big(1-\delta\frac{\rho_k}{\rho_{k+1}}\Big)\big(d^2(v_k,w_k)+d^2(t_{k+1},w_k)\big). \end{aligned} \tag{3.29}$$

By utilizing the geodesic triangles $\triangle(u_k,t_k,q)\subset\mathbb{P}$ and $\triangle(t_k,t_{k-1},q)\subset\mathbb{P}$ with their respective comparison triangles $\triangle(u'_k,t'_k,q')\subseteq\mathbb{R}^2$. Then by Lemma 2.4 (ii), we have $d(u_k,t_k)=\|u'_k-t'_k\|$, $d(u_k,q)=\|u'_k-q'\|$ and $d(t_k,t_{k-1})=\|t'_k-t'_{k-1}\|$. Similarly, using the geodesic triangles $\triangle(v_k,u_k,q)\subset\mathbb{P}$ and $\triangle(t_k,t_{k-1},q)\subset\mathbb{P}$ with their respective comparison triangle $\triangle(v'_k,u'_k,q')\subseteq\mathbb{R}^2$. Then by Lemma 2.4 (ii), we have $d(v_k,u_k)=\|v'_k-u'_k\|$, $d(v_k,t_k)=\|v'_k-t'_k\|$ and $d(v_k,q)=\|v'_k-q'\|$. From (3.11) of Algorithm 3.8, we have that $u'_k=t'_k+\theta_k(t'_k-t'_{k-1})$ and $v'_k=u'_k+\beta_k(u'_k-t'_{k-1})$, thus

$$\begin{aligned} d(u_k,q)&=\|u'_k-q'\|\\ &=\|t'_k+\theta_k(t'_k-t'_{k-1})-q'\|\\ &\leq\|t'_k-q'\|+\theta_k\|t'_k-t'_{k-1}\|\\ &=d(t_k,q)+\theta_k d(t_k,t_{k-1}). \end{aligned} \tag{3.30}$$

Also,

$$\begin{aligned} d(u_k,t_{k-1})&=\|u'_k-t'_{k-1}\|\\ &=\|t'_k+\theta_k(t'_k-t'_{k-1})-t'_{k-1}\|\\ &\leq\|t'_k-t'_{k-1}\|+\theta_k\|t'_k-t'_{k-1}\|\\ &=d(t_k,t_{k-1})+\theta_k d(t_k,t_{k-1})\\ &=(1+\theta_k)d(t_k,t_{k-1}). \end{aligned} \tag{3.31}$$

By definition of v_k, (3.30) and (3.31), we get

$$
\begin{aligned}
d(v_k, q) &= \|v'_k - q'\| \\
&= \|u'_k + \beta_k(u'_k - t'_{k-1}) - q'\| \\
&\leq \|u'_k - q'\| + \beta_k\|u'_k - t'_{k-1}\| \\
&= d(u_k, q) + \beta_k d(u_k, t_{k-1}) \\
&\leq d(t_k, q) + \theta_k d(t_k, t_{k-1}) + \beta_k(1+\theta_k)d(t_k, t_{k-1}) \\
&= d(t_k, p) + (\theta_k + \beta_k(1+\theta_k))d(t_k, t_{k-1}).
\end{aligned} \tag{3.32}
$$

Since $\lambda \in (0, 1]$ and $\delta \in (0, 1)$, we obtain

$$
\lim_{k\to\infty} (1 - \delta\frac{\rho_k}{\rho_{k+1}}) = 1 - \delta > 0 \text{ and } 1 - \lambda \geq 0.
$$

Thus, (3.29) and (3.32) becomes

$$
\begin{aligned}
d(t_{k+1}, q) &\leq d(v_k, q) \\
&\leq d(t_k, q) + (\theta_k + \beta_k(1+\theta_k))d(t_k, t_{k-1}) \\
&\leq d(t_k, q) + (\theta_k + \beta_k(1+\theta_k))\big(d(t_k, q) + d(t_{k-1}, q)\big) \\
&= (1 + \theta_k + \beta_k(1+\theta_k))d(t_k, q) + (\theta_k + \beta_k(1+\theta_k))d(t_{k-1}, q).
\end{aligned} \tag{3.33}
$$

By applying Lemma 2.6, we obtain that

$$
d(t_{k+1}, q) \leq M \cdot \prod_{j=1}^{k}(1 + 2(\theta_j + \beta_j(1+\theta_j))), \tag{3.34}
$$

where $M = \max\{d(t_1, q), d(t_2, q)\}$. Hence, the proof completes. □

(ii) *Proof.* Next, we establish that $\{t_k\}$ converges to a point in Ω. Since $\sum\limits_{k=1}^{\infty} \theta_k < +\infty$ and $\sum\limits_{k=1}^{\infty} \beta_k < +\infty$, then by Lemma 2.6 and (3.34). the sequence $\{t_k\}$ is bounded. This also implies that $\sum\limits_{k=1}^{\infty} \theta_k d(t_k, t_{k-1}) < +\infty$ and $\sum\limits_{k=1}^{\infty} \beta_k d(t_k, t_{k-1}) < +\infty$. By applying Lemma 2.7 in (3.33), we can claim that $\lim\limits_{k\to\infty} d(t_k, q)$ exists. From Lemma 2.4 (ii) and Proposition 2.5, we see that

$$
\begin{aligned}
d^2(u_k, q) &= \|u'_k - q'\|^2 \\
&= \|t'_k + \theta_k(t'_k - t'_{k-1}) - q'\|^2 \\
&= \|(1+\theta_k)(t'_k - q') - \theta_k(t'_{k-1} - q')\|^2 \\
&= (1+\theta_k)d^2(t_k, q) - \theta_k d^2(t_{k-1}, q) + \theta_k(1+\theta_k)d^2(t_k, t_{k-1}).
\end{aligned} \tag{3.35}
$$

We also consider

$$
\begin{aligned}
d^2(u_k, t_{k-1}) &= \|u'_k - t'_{k-1}\|^2 \\
&= \|t'_k + \theta_k(t'_k - t'_{k-1}) - t'_{k-1}\|^2 \\
&= \|t'_k - t'_{k-1}\|^2 + 2\langle t'_k - t'_{k-1}, \theta_k(t'_k - t'_{k-1})\rangle \\
&\quad + \theta_k^2\|t'_k - t'_{k-1}\|^2.
\end{aligned} \tag{3.36}
$$

But from (2.8), we have

$$\begin{aligned}\langle t'_k - t'_{k-1}, t'_k - t'_{k-1}\rangle &\leq \langle \exp^{-1}_{t_{k-1}} t_k, \exp^{-1}_{t_{k-1}} t_k\rangle \\ &= \| \exp^{-1}_{t_{k-1}} t_k\|^2 \\ &= d^2(t_k, t_{k-1}). \end{aligned} \tag{3.37}$$

On substituting (3.37) into (3.36), we get

$$\begin{aligned} d^2(u_k, t_{k-1}) &\leq d^2(t_k, t_{k-1}) + 2\theta_k d^2(t_k, t_{k-1}) + \theta_k^2 d^2(t_k, t_{k-1}) \\ &= (1+\theta_k)^2 d^2(t_k, t_{k-1}). \end{aligned} \tag{3.38}$$

We deduce from Lemma 2.4, (3.35) and (3.38) that

$$\begin{aligned} d^2(v_k, q) &= \|v'_k - q'\|^2 \\ &= \|u'_k + \beta_k(u'_k - t'_{k-1}) - q'\|^2 \\ &= \|(1+\beta_k)(u'_k - t'_{k-1}) - \beta_k(t'_{k-1} - q')\|^2 \\ &= (1+\beta_k)d^2(u_k, t_{k-1}) - \beta_k d^2(t_{k-1}, q) + \beta_k(1+\beta_k)d^2(u_k, t_{k-1}) \\ &= (1+\beta_k)\big((1+\theta_k)d^2(t_k, q) - \theta_k d^2(t_{k-1}, q) + \theta_k(1+\theta_k)d^2(t_k, t_{k-1})\big) \\ &\quad - \beta_k d^2(t_{k-1}, q) + \beta_k(1+\beta_k)d^2(u_k, t_{k-1}) \\ &\leq (1+\beta_k)\Big(d^2(t_k, q) + \theta_k\big(d^2(t_k, q) - d^2(t_{k-1}, q)\big) + \theta_k(1+\theta_k)d^2(t_k, t_{k-1})\Big) \\ &\quad - \beta_k d^2(t_{k-1}, q) + \beta_k(1+\beta_k)(1+\theta_k)^2 d^2(t_k, t_{k-1}) \\ &= d^2(t_k, q) + \big(\beta_k + (1+\beta_k)\theta_k\big)\big(d^2(t_k, q) - d^2(t_{k-1}, q)\big) \\ &\quad + \theta_k(1+\theta_k)(1+\beta_k)d^2(t_k, t_{k-1}) + \beta_k(1+\beta_k)(1+\theta_k)^2 d^2(t_k, t_{k-1}). \end{aligned} \tag{3.39}$$

On substituting (3.39) into (3.29), we obtain

$$\begin{aligned} d^2(t_{k+1}, q) &\leq d^2(t_k, q) + (\beta_k + (1+\beta_k)\theta_k)\big(d^2(t_k, q) - d^2(t_{k-1}, q)\big) \\ &\quad + \theta_k(1+\theta_k)(1+\beta_k)d^2(t_k, t_{k-1}) + \beta_k(1+\beta_k)(1+\theta_k)^2 d^2(t_k, t_{k-1}) \\ &\quad - (1-\lambda)d^2(v_k, t_{k+1}) - \lambda\left(1 - \delta\frac{\rho_k}{\rho_{k+1}}\right)\big(d^2(v_k, w_k) + d^2(t_{k+1}, w_k)\big), \end{aligned} \tag{3.40}$$

which implies that

$$\begin{aligned} &(1-\lambda)d^2(v_k, t_{k+1}) + \lambda\left(1 - \delta\frac{\rho_k}{\rho_{k+1}}\right)\big(d^2(v_k, w_k) + d^2(t_{k+1}, w_k)\big) \\ &\leq d^2(t_k, q) - d^2(t_{k+1}, q) + (\beta_k + (1+\beta_k)\theta_k)\big(d^2(t_k, q) - d^2(t_{k-1}, q)\big) \\ &+ \theta_k(1+\theta_k)(1+\beta_k)d^2(t_k, t_{k-1}) + \beta_k(1+\beta_k)(1+\theta_k)^2 d^2(t_k, t_{k-1}). \end{aligned} \tag{3.41}$$

Since $\lim_{k\to\infty} d(t_k, q)$ exists, $\sum_{k=1}^{\infty} \theta_k < +\infty$ and $\sum_{k=1}^{\infty} \beta_k < +\infty$. It follows from (3.41) that

$$\begin{cases} \lim_{k\to\infty} d(v_k, t_{k+1}) = 0, \\ \lim_{k\to\infty} d(v_k, w_k) = 0, \\ \lim_{k\to\infty} d(t_{k+1}, w_k) = 0. \end{cases} \tag{3.42}$$

Observe that

$$\begin{aligned} d(u_k, t_k) &= \|u'_k - t'_k\| \\ &= \|t'_k + \theta_k(t'_k - t'_{k-1}) - t'_k\| \\ &= \theta_k d(t_k, t_{k-1}) \to 0,\ k \to \infty. \end{aligned} \tag{3.43}$$

From (3.43), we get

$$\begin{aligned} d(v_k, t_k) &= \|v'_k - t'_k\| \\ &= \|u'_k + \beta_k(u'_k - t'_{k-1}) - t'_k\| \\ &\le \|u'_k - t'_k\| + \beta_k\|t'_k + \theta_k(t'_k - t'_{k-1}) - t'_{k-1}\| \\ &\le d(u_k, t_k) + \beta_k d(t_k, t_{k-1}) + \beta_k\theta_k d(t_k, t_{k-1}) \to 0,\ k \to \infty. \end{aligned} \tag{3.44}$$

From (3.42) and (3.44), we have

$$\begin{cases} \lim\limits_{k\to\infty} d(w_k, t_k) = 0, \\ \lim\limits_{k\to\infty} d(t_{k+1}, t_k) = 0. \end{cases} \tag{3.45}$$

Since $\{t_k\}$ is bounded, there exists a subsequence $\{t_{k_l}\}$ which converges to a cluster point x^*. Also from (3.45), there exists a subsequence $\{w_{k_l}\}$ of $\{w_k\}$ which converges weakly to $x^* \in \mathbb{P}$. From (3.18) and (3.25), we deduce that

$$\begin{aligned} \lambda\rho_{k_l} h(w_{k_l}, y) &\ge \lambda\rho_{k_l} h(w_{k_l}, t_{k_{l+1}}) + \langle \exp^{-1}_{t_{k_l+1}} v_{k_l}, \exp^{-1}_{t_{k_l+1}} y\rangle \\ &\ge \lambda\Big[\langle \exp^{-1}_{w_{k_l}} v_{k_l}, \exp^{-1}_{w_{k_l}} t_{k_{l+1}}\rangle - \frac{\rho_{k_l}}{\rho_{k_{l+1}}}\frac{\delta}{2}\big(d^2(v_{k_l}, w_{k_l}) + d^2(t_{k_{l+1}}, w_{k_l})\big)\Big] \\ &+ \langle \exp^{-1}_{t_{k_{l+1}}} v_{k_l}, \exp^{-1}_{t_{k_{l+1}}} y\rangle. \end{aligned}$$

Since $\lambda > 0,\ \lim\limits_{k\to\infty} \rho_{k_l} = \rho > 0$, we have that

$$0 \le \limsup_{l\to\infty} h(w_{k_l}, y) = h(x^*, y),\ \forall\, y \in \mathcal{K}.$$

Thus, $x^* \in \Omega$. Lastly, by Lemma 2.1, we obtain that $\{q_k\}$ converges to a point in Ω. □

Remark 3.3. *We present some consequences of our result as follows:*

(i) The result presented in this article generalizes the results of [3, 41, 55] from real Hilbert spaces to a Hadamard manifold.

(ii) We introduce a two step inertial method to fasten the rate of convergence of our iterative method. We also compare our result with some related results in the literature.

(iii) We emphasize that our step size is selected self-adaptively and varies from each iteration to the other which allows our iterative method to be computed easily without the prior knowledge of the Lipschitz constants.

4. Numerical Examples

In this section, we provide some numerical examples to illustrate the computational efficiency of the proposed algorithms compared with some iterative algorithms in the literature. First, we will perform a sensitivity analysis on the parameter λ in our proposed algorithm and use the best value of lambda to compare the performance of our proposed algorithm with recent inertial and non-inertial algorithms in the literature.

Example 4.4. *Let $\mathbb{P}$ be the space in Example 2.3. Assume that the operator F is defined by*

$$F(x) := \begin{pmatrix} (x_1^2 + (x_2-1)^2)(1+x_2) \\ -x_1 - x_1(x_2-1)^2 \end{pmatrix}.$$

Set $h(x,y) = \langle F(x), \exp_x^{-1} y\rangle$, $\forall\, x,y \in \mathcal{K}$, where $\mathcal{K} := \{x \in \mathbb{R}^2 : -5 \le x_i \le 5, i = 1,2\}$. The problem (1.1) has a unique solution $x^* = (0,-1)^T$. However, using the Monte Carlo approach (see [48]), it can be shown that F is pseudomonotone on $\mathcal{K}$ and thus the bifunction h is also pseudomonotone. Furthermore, observe that for h defined this way, the $\arg\min$ defined in w_k and t_{k+1} is the metric projection (interested readers may see, e.g., [19] for how this claim is proved). For this example, in our proposed algorithm 3.8 we let $\rho_1 = 0.01, \delta = 0.9, \theta_k = \frac{1}{(k+1)^{1.1}}, \tau_k = \frac{1}{(k+1)^4}$ and $\beta_k = \frac{1}{(5k+2)^2}$. We set maximum number of iterations to be $3,000$, $D_1 = 10$ and terminate the process when $D_k = \|x_{k+1} - x_k\| < 10^{-6}$ is not satisfied. To determine the optimal choice of λ, we study the behaviour of our proposed algorithm for the values of $\lambda = \{0.7, 0.8, 0.9, 1\}$. We consider two set of initial points: First Initial Points (FIP): $x_0 = \begin{pmatrix} 1 \\ 0.25 \end{pmatrix}$, $x_1 = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}$. Second Initial Points (SIP): $x_0 = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$, $x_1 = \begin{pmatrix} 1 \\ -0.8 \end{pmatrix}$. The results of the numerical simulations are presented in Figure 1 below.

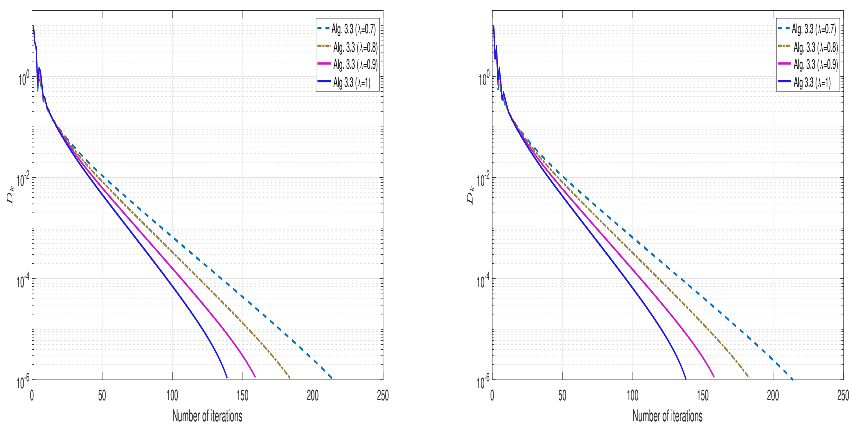

FIGURE 1. Numerical results for Example 4.4 for various values of λ

Remark 4.4. *We observe that the choice of $\lambda = 1$ in our proposed algorithm gave the best approximation. In comparison with other methods, since the choice of the parameters are very sensitive, we will first consider the parameters used by the authors in the papers we are comparing our algorithm with. However, if using the parameters in their papers does not give a better approximation, we might change it slightly looking at the parameters we choose for our proposed algorithm.*

Next, using this choice of $\lambda = 1$, we shall compare the performance of our proposed algorithm with Algorithm 3 of Xie et al. [55] (XCT Alg), Algorithm 3.1 of Hieu [26] (Hieu Alg), Algorithm 1 of Vinh and Muu [47] (VM Alg), Algorithm 1 of Oyewole and Reich [37] (OR Alg) and Algorithm 3.1 of Yang and Liu [54] (YL Alg). In XCT Alg, we choose $\mu = 0.1,\ \tau = 0.01,\ \kappa = 0.7,\ \theta_k = 0.5,\ \alpha_k = \frac{1}{1000k+1}$ and $\beta_k = 0.5(1-\alpha_k)$. In Hieu Alg, we choose $\lambda_k = \frac{1}{(k+1)^3}$ and $\theta = 0.3$. In VM Alg, we choose $\lambda = 0.1,\ \theta = 0.1,\ \epsilon_k = \frac{1}{(k+1)^2}$. In OR Alg, we choose $\mu = 0.5,\ \lambda_1 = 0.1,\ \theta = 0.3,\ \theta_k = \bar{\theta}_k,\ \delta_k = \frac{1}{(k+1)^4},\ \epsilon_k = \frac{1}{(k+1)^{1.1}}$ and $\beta_k = \frac{1}{(k+1)}$. In YL Alg, we choose

$\lambda_0 = 0.1,\ \mu = 0.5,\ \alpha_k = \frac{1}{k+1}$ *and* $\beta_k = 0.6$. *We will use the same initial points and stopping criteria as mentioned above. In the experiment, all the methods satisfy the stopping criteria in less than 0.2 seconds. Hence, we will not report CPU time in this example. The result of the numerical simulations are presented in Figure 2.*

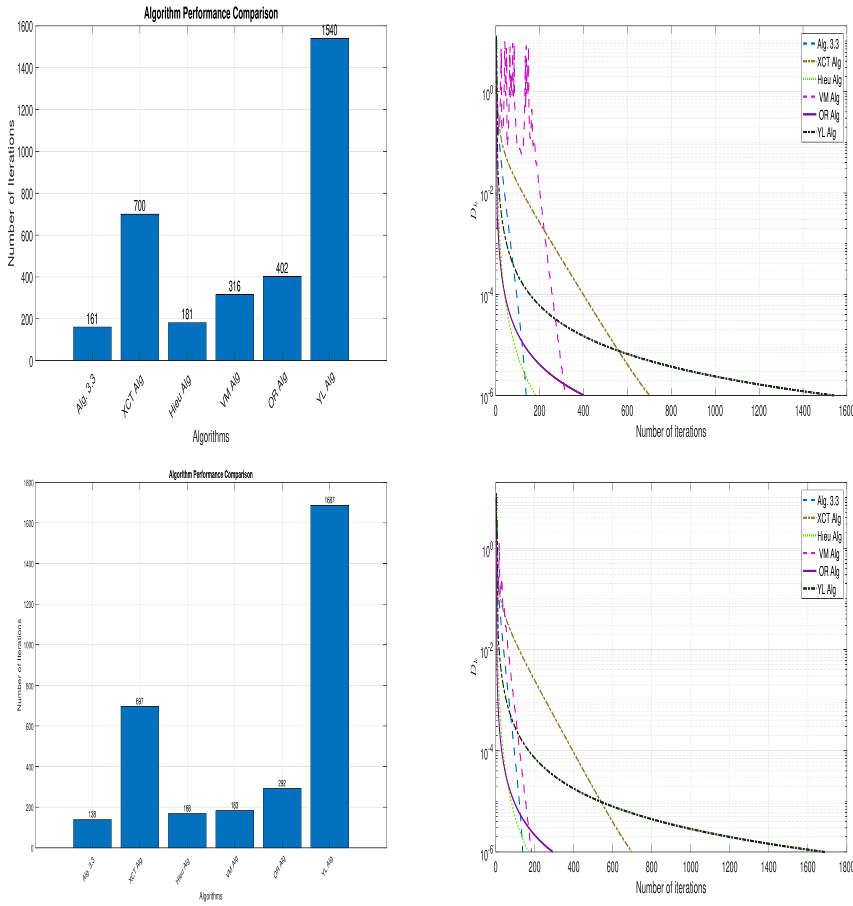

FIGURE 2. Numerical results for Example 4.4, Top FIP; Bottom SIP

Example 4.5. *Let* $\mathbb{P}$ *be space 3 defined in section 2 of our manuscript. Let* $h : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ *be given by*

$$h(x, y) = \langle Mx + Ny + c, y - x\rangle,\ \forall\, x, y \in \mathcal{K},$$

where the feasible set $\mathcal{K}$ *is defined by* $\mathcal{K} = \{x \in \mathbb{R}^m : 1 \leq x_i \leq 100, i = 1, 2, \cdots, m\}$, $c \in \mathbb{R}^m$ *and* $M, N \in \mathbb{R}^{m\times m}$. *The Matrix* M *is symmetric positive semi-definite and the matrix* $(N - M)$ *is symmetric negative. It is known that* h *is pseudomonotone and satisfies (A2) with Lipschitz constants* $\alpha_1 = \alpha_2 = \frac{\|M-N\|}{2}$. *Assumption (A3) and (A4) are also satisfied (see [27, 45]). Furthermore, observe that for this example the* $\arg\min$ *defined in* w_k *and* t_{k+1} *is no longer the metric projection. Thus, we use the matlab built-in function "*$fmincon$*" to compute these terms. For this example, we use the same parameters used in Example 4.4 for all the algorithms. We consider two cases for the dimension: Case I:* $m = 10$. *Case II:* $m = 20$. *We choose the initial* *points $x_0 = x_1 = randn(m, 1)$. The stopping criteria is same as used in Example 4.4. The results of the numerical simulations are presented in Figure 3 below:*

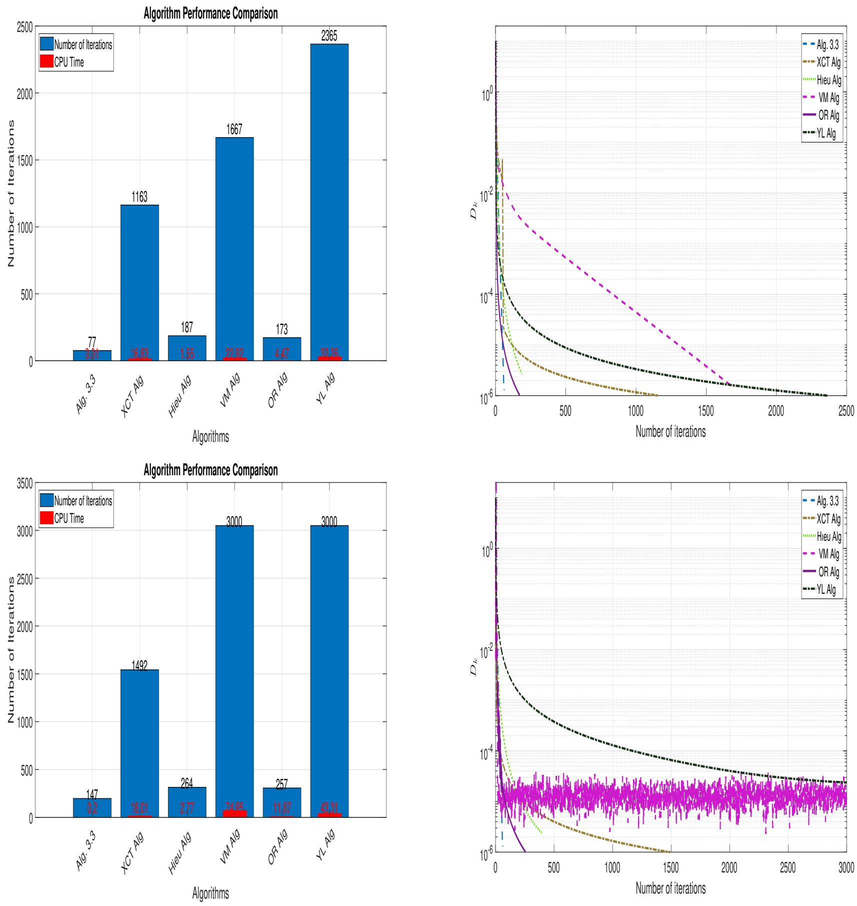

FIGURE 3. Numerical results for Example, Top Case I; Bottom Case II

Remark 4.5. *Using different values of λ and varying the key parameters in Example 4.4 and 4.5, we compare our results with the results of Xie et al. [55] (XCT Alg), Algorithm 3.1 of Hieu [26] (Hieu Alg), Algorithm 1 of Vinh and Muu [47] (VM Alg), Algorithm 1 of Oyewole and Reich [37] (OR Alg) and Algorithm 3.1 of Yang and Liu [54] (YL Alg). We obtained the numerical results displayed in Figures 1, 2 and 3 respectively. From our displayed results, it can be inferred that our algorithm performs better in both number of iterations and computational time taken to satisfy the stopping criterion.*

5. Conclusions

In the context of a Hadamard manifold, we propose a double step inertial subgradient extragradient method for solving the pseudomontone equilibrium problem. The previously mentioned findings on inertial extrapolation techniques, subgradient extragradient techniques, and extragradient methods served as inspiration for this approach. We show that the sequence generated by our iterative approach converges to a solution of a pseudomonotone equilibrium problem under some mild conditions. We provide some numerical examples to compare the performance of our iterative method with several relevant ones in the literature. To the best of our knowledge, the result presented here is new in the context of a Hadamard manifold and generalizes other similar results found in the literature.

6. Acknowledgment

The authors appreciate the support provided by their institutions. The author would like to thank the reviewers for their esteemed comments and suggestions.

References

[1] Abass, H. A.; Izuchukwu, C.; Mewomo, O. T.; Dong, Q. L. Strong convergence of an inertial forward-backward splitting method for accretive operators in real Banach spaces, *Fixed Point Theory*, **21** (2020), no. 2, 397-412.

[2] Abass, H. A.; Oyewole, O. K.; Jolaoso, L. O.; Aremu, K. O. Modified inertial Tseng method for solving variational inclusion and fixed point problems on Hadamard manifolds. *Appl. Anal.* (2024), 1604-1627.

[3] Abass, H. A.; Mewomo, O. T. Accelerated extragradient algorithm for equilibrium and fixed point problems for countable family of certain multi-valued mappings, *Bol. Soc. Paran. Mat.*, **41**(2023), 1-14.

[4] Ali-Akbari, M. A subgradient extragradient method for equilibrium problems on Hadamard manifolds, *Int. J. Nonlinear Anal. Appl.,* **13**(2022), 75-84.

[5] Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.,* **9**(2001), 3-11.

[6] Azagra, D.; Ferrera, J.; López-Mesas, F. Nonsmooth analysis and Hamilton Jacobi on Riemannian manifolds. *J. Funct. Anal.*, **220**(2005), 304-361.

[7] Baćak, M. Old and new challenges in Hadamard spaces, Arxiv 1807.01355v2 $[Math.FA]$, (2018).

[8] Baćak, M. Computing medians and means in Hadamard spaces, *SIAM J. Optim.*, **24**(2014), 1542-1566.

[9] Baćak, M. The proximal point algorithm in metric spaces, *Israel J. Math.***194**(2013), 689-701.

[10] Blum, E.; Oettli, W. From optimization and variational inequalities to equilibrium problems, *Math. Stud.,* **63(1–4)**(1994), 123-145.

[11] Bridson, M. R.; Haefliger, A. *Metric Spaces of Non-positive Curvature. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences)* vol. 319. Springer, Berlin (1999).

[12] Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert spaces, *J. Optim. Theory Appl.,* **148**(2011), 318-335.

[13] Chen, J.; Liu, S. Extragradient-like method for pseudomonotone equilibrium problems on Hadamard manifolds, *J. Ineq. Appl.,* (2020) 2020:205.

[14] Chidume, C.E.; Adam, A.A.; Adamu, A. An iterative method involving a class of quasi-phi-nonexpansive mappings for solving split equality fixed point problems, *Creat. Math. Inform.,* **32**(2023), 29-40.

[15] Chidume, C.E.; Adamu, A. On split generalized mixed equality equilibrium and split equality fixed point problems, *Appl. Set-Valued Anal. Optim.,* **2**(2020), 273-283.

[16] Chidume, C.E.; Adamu, A.; Nnakwe, M.O. An inertial algorithm for solving Hammerstein equations, *Symmetry*, **13**(2021), 376.

[17] Colao, V.; López, G.; Marino, G.; Martín-Márquez, V. Equilibrium problems in Hadamard manifolds, *J. Math. Anal. Appl.*,**388 (1)**(2012), 61-77.

[18] Q. L. Dong, Y. J. Cho and T. M. Rassias, General inertial Mann algorithms and their convergence analysis for nonexpansive mappings, *Appl. Nonlinear Anal.* (2018), 175-191.

[19] Dadashi, V.; Iyiola O.S.; Shehu, Y. The subgradient extragradient method for pseudomonotone equilibrium problems, *Optimization,* (2020), 901-923.

[20] Ofem, A.E.; Abuchu, J.A.; Ugwunnadi, G.C.; Nabwey, H.A.; Adamu, A.; Narain, O.K. Double inertial steps extragadient-type methods for solving optimal control and image restoration problems, *AIMS Math.,* **9**(2024), 12870-12905.

[21] Fan, K. *A minimax inequality and its application,* in: O. Shisha (Ed.), Inequalities, **3** (1972), 103-113.

[22] Ferreira, O.P.; Oliveira, P.R. Proximal Point Algorithm on Riemannian Manifolds, *Optimization*, **51 (2)** (2002), 257-270.

[23] Flam, S. D.; Antipin, A. S. Equilibrium programming using proximal-like algorithms, *Math. Program*, **78**(1997), 29-41.

[24] Giannessi, F.; Maugeri, A. *Variational inequalities and Network equilibrium problems*, Springer, New York, (1995).

[25] Hanjing, A.; Suantai, S. A fast image restoration algorithm based on a fixed point and optimization method, *Mathematics*, **8**(2020), 378.

[26] Hieu, D.V. New inertial algorithm for a class of equilibrium problems, *Numer. Algor.* (2019), 1413-1436.

[27] Khammahawong, K.; Kumam, P.; Chaipunya, P.; Yao, J.C.; Wen, C.F.; Jirakitpuwapat, W.; An extragradient algorithm for strongly pseudomonotone equilibrium problems on Hadamard manifolds. *Thai J. Math.,* **18** (2020), 350-371.

[28] Korpelevich, G.M. An extragradient method for finding saddle points and for other problems,*Ekon. Mat. Metody.,* **12** (1976), 747-756.

[29] Li, C.; López, G.; Màrquez, V. M.; Resolvent of set-valued monotone vector fields in Hadamard manifolds, *J. Set-Valued Anal.*, **19**(2011), 361-383.

[30] Li, X. B.; Zhon, L. W.; Huang, N. J. Gap functions and descent method for equilibrium problems on Hadamard manifolds, *J. Nonlinear Convex Anal.*, **17**(2016), 807-826.

[31] Li, C.; López, G.; Martín-Márquez, V. Monotone vector fields and the proximal point algorithm on Hadamard manifolds, *J. Lond. Math. Soc.,* **79** (2009), 663-683.

[32] Marinet, B. Régularisation d'inéquations variationnelles par approximations successives. Rev Francaise In format Recherche Opérationnelle, (1970), 154-158.

[33] Moudafi, A.; Oliny, M. Convergence of a splitting inertial proximal method for monotone operators, *J. Comput. Appl. Math.*, **155** (2003), 447-454.

[34] Németh, S. Z. Variational inequalities on Hadamard manifolds, *Nonlinear Anal.*, **52** (2003), 1491-1498.

[35] Neto, J. C.; Santos, P.; Soares, P. An extragradient method for equilibrium problems on Hadamard manifolds, *Optim. Lett.*, **10**, (2016), 1327-1336.

[36] Osilike, M. O.; Aniagbosor, S. C. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, *Math. Comput. Model.*, **32**(2000), 1181-1191.

[37] Oyewole, O. K.; Reich, S. An inertial subgradient extragradient method for approximating solution to equilibrium problems in Hadamard manifold, *Axioms,* **12**, (2023), 256.

[38] Polyak, B. T. *Introduction to optimization.* New York: Optim. Softw., Publications Division, 1987.

[39] Polyak, B.T. Some methods of speeding up the convergence of iterative methods *Zh. Vychisl. Mat. Mat. Fiz.*, **4**(1964), 1-17.

[40] Rahimi, R.; Farajzadeh, A. P.; Vaezpour, S. M. Study of equilibrium problem in Hadamard manifolds by extending some concepts of nonlinear analysis, *RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.*, **112**(2018), 1521-1537.

[41] Rehman, H.; Kumam, P.; Sitthithakerngkiet, K. Viscosity-type method for solving pseudomonotone equilibrium problems in a real Hilbert space with applications, *AIMS Math.,* **6** (2021), 1538-1560.

[42] Sakai, T. *Riemannian geometry. Vol. 149, Translations of mathematical monographs,* Providence (RI): Amer. Math. Soc., 1996. Translated from the 1992 Japanese original by the author.

[43] Shehu, Y.; Yao, J. C. Rate of convergence for inertial iterative method for countable family of certain quasi-nonexpansive mappings, *J. Nonlinear Convex Anal.*, **21** (2020), 533-541.

[44] Suantai, S.; Inkrong, P.; Cholamjiak, P. Forward-backward-forward algorithms involving two inertial terms for monotone inclusions, *Comput. Appl. Math.*, **42** (2023), 255.

[45] Tran, D.; Dung, M.; Nguyen, V. Extragradient algorithms extended to equilibrium problems, *Optimization*, **57** (2008), 749-776.

[46] Thong, D. V.; Cholamjiak, P. Strong convergence of a forward-backward splitting method with a new step size for solving monotone inclusions, *Comput. Appl. Math.*, **38** (2019), 1-16.

[47] Vinh, N.; Muu, L.D. Inertial extragradient algorithms for solving equilibrium problems. Acta Math. Vietnam., (2019), 639–663.

[48] Hu, X.; Wang, J. Solving pseudo-monotone variational inequalities and pseudoconvex optimization problems using the projection neural network. IEEE Trans Neural Netw. (2006) 1487–1499.

[49] Wang, J.H; López, G.; Martín-Márquez, V.; Li, C. Monotone and accretive vector fields on Riemannian manifolds. J. Optim. Theory Appl., **146** (2010), 691-708.

[50] Wang, Z.; Sunthrayuth, P.; Adamu, A.; Cholamjiak, P. Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities, *Optimization,* **73** (2024), 2053-2087

[51] Yao, Y.; Iyiola, O. S.; Shehu, Y. Subgradient extragradient method with double inertial steps for variational inequalities, *J. Sci. Comput.*, **90** (2022), 1-29.

[52] Yao, Y.; Adamu, A.; Shehu, Y.; Yao, J.C. Simple proximal-type algorithm for equilibrium problems, *J. Global Optim.*, **89** (2024), 1069–1098.

[53] Yao, Y.; Adamu, A.; Shehu, Y. Strongly convergent inertial forward-backward-forward algorithm without on-line rule for variational inequalities, *Acta Math. Sci.*, **44**, (2024), 551-566.

[54] Yang, J.; Liu, H. The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space, *Optim. Lett.*, **14** (2020), 1803-1816.

[55] Zhongbing, X.; Cai, G.; Tan, B. Inertial subgradient extragradient method for solving pseudomonotone equilibrium problems in Hilbert spaces, *Optimization* (2022), 1-26.

[1]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Science University, P.O. Box 94, Pretoria 0204, South Africa

[2] Operational Research Center in Healthcare, Near East University, TRNC Mersin 10, Nicosia 99138, Turkey

[3] Charles Chidume Mathematics Institute, African University of Science and Technology, Abuja 900107, Nigeria

Email address: hammed.abass@smu.ac.za, abubakar.adamu@neu.edu.tr, maggie.aphane@smu.ac.za