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Existence and uniqueness of solution of a tripled system of fractional Langevin differential equations with cyclic boundary conditions

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ABSTRACT. The present work examines the solvability of a tripled system of fractional Langevin differential equations with cyclic antiperiodic boundary conditions. The Krasnoselskii fixed point theorem, the Banach contraction mapping theorem, and specific properties of the Mittag-Leffler functions are employed to establish sufficient conditions for the existence and uniqueness of solutions. The feasibility of the primary findings is illustrated through the discussion of several numerical examples.

1. INTRODUCTION

The subject of fractional differential equations (FDEs) has become highly popular and is regarded as a major area of research mostly because of its demonstrated progress in both theoretical and practical aspects in recent decades. FDEs are highly valuable in many domains such as fluid flow, physics, dynamical processes in self-similar and porous structures, blood flow phenomena, electrodynamics of complex media, capacitor theory, electrical circuits, biology, control theory of dynamical systems, and fitting of experimental data. For more details, we refer the reader to the monographs [3, 13, 16, 18, 20, 21, 22] and the papers [4, 6, 7, 8, 17].

The examination of fractional differential equation systems is deemed to be of great importance and practicality due to their prevalence in a diverse array of problems. The results indicate that fractional differential systems are more appropriate for describing physical phenomena that exhibit genetic characteristics and memory. In particular, the existence and uniqueness of systems that are supplemented with boundary conditions have become one of the primary areas of interest in mathematical analysis; see [1, 2, 19, 23, 24, 25].

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In an effort to heighten awareness of this matter, we shall examine several pertinent findings. Recently in [14], the authors considered the existence and uniqueness of solutions for the following boundary value problem of Langevin differential equation with two different fractional orders

$$\begin{cases}\mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega(t)=\mathfrak{h}(t,\Omega(t)), \quad 0<t<1, \quad 0<\nu\leq 1, \quad 1<\zeta\leq 2,\\ \Omega(0)+\Omega(1)=0, \mathfrak{D}^{\nu}\Omega(0)+\mathfrak{D}^{\nu}\Omega(1)=0, \mathcal{D}^{2\nu}\Omega(0)+\mathcal{D}^{2\nu}\Omega(1)=0,\end{cases} \tag{1.1}$$

where $\mathfrak{D}^{\sigma}$ is a fractional derivative in Caputo sense of order $\sigma \in \{\nu, \zeta\}$, $\mathfrak{h} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ω is a real number and $\mathcal{D}^{m\sigma}(m = 1, 2)$ denotes the sequential fractional derivatives with the features $\mathcal{D}^{\sigma}\Omega = \mathfrak{D}^{\sigma}\Omega$ and $\mathcal{D}^{k\sigma}\Omega = \mathcal{D}^{\sigma}\mathcal{D}^{(k-1)\sigma}\Omega, k = 2, 3, \dots$.

In light of the findings of [14], the authors in [9] studied a coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions of the form

$$\begin{cases}\mathfrak{D}^{\zeta_1}(\mathfrak{D}^{\nu_1}+\omega_1)\Omega_1(t)=\Psi_1(t,\Omega_1(t),\Omega_2(t)), \quad 0<t<1, \quad 0<\nu_1\leq 1, \quad 1<\zeta_1\leq 2,\\ \mathfrak{D}^{\zeta_2}(\mathfrak{D}^{\nu_2}+\omega_2)\Omega_2(t)=\Psi_2(t,\Omega_1(t),\Omega_2(t)), \quad 0<t<1, \quad 0<\nu_2\leq 1, \quad 1<\zeta_2\leq 2,\\ \Omega_1(0)+\Omega_1(1)=0, \mathcal{D}^{\nu_1}\Omega_1(0)+\mathcal{D}^{\nu_1}\Omega_1(1)=0, \mathcal{D}^{2\nu_1}\Omega_1(0)+\mathcal{D}^{2\nu_1}\Omega_1(1)=0,\\ \Omega_2(0)+\Omega_2(1)=0, \mathcal{D}^{\nu_2}\Omega_2(0)+\mathcal{D}^{\nu_2}\Omega_2(1)=0, \mathcal{D}^{2\nu_2}\Omega_2(0)+\mathcal{D}^{2\nu_2}\Omega_2(1)=0,\end{cases} \tag{1.2}$$

where $\mathfrak{D}^{\nu}$ is a fractional derivative in Caputo sense of order $\nu \in \{\nu_1, \zeta_1, \nu_2, \zeta_2\}$, $\mathcal{D}^{m\nu_i}(m, i = 1, 2)$ are the sequential fractional derivatives, $\Psi_1, \Psi_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\omega_1, \omega_2 \in \mathbb{R}$.

Very recently in [26], Zhang and Ni introduced the tripled systems of fractional differential equations with cyclic conditions in the following form:

$$\begin{cases}\mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega_1(t)=\Psi_1(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0<t<1,\\ \mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega_2(t)=\Psi_2(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0<t<1,\\ \mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega_3(t)=\Psi_3(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0<t<1,\\ \Omega_1(0)+\Omega_2(1)=0, \mathcal{D}^{\nu}\Omega_1(0)+\mathcal{D}^{\nu}\Omega_2(1)=0,\\ \Omega_2(0)+\Omega_3(1)=0, \mathcal{D}^{\nu}\Omega_2(0)+\mathcal{D}^{\nu}\Omega_3(1)=0,\\ \Omega_3(0)+\Omega_1(1)=0, \mathcal{D}^{\nu}\Omega_3(0)+\mathcal{D}^{\nu}\Omega_1(1)=0,\end{cases} \tag{1.3}$$

where $\Psi_i : [0, 1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}, i = 1, 2, 3$ are given continuous functions, $\omega \in \mathbb{R}^+$, $0 < \nu, \zeta < 1$ and $1 < \nu + \zeta < 2$. By means of fixed point theorems and some analytical skills, sufficient conditions for the existence, uniqueness and stability of the proposed problem are obtained.

Motivated by the above mentioned work, the objective of this paper is to study a tripled system of Langevin differential equations of fractional order and associated to cyclic antiperiodic boundary conditions of the form

$$
\begin{cases}
\mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega_1(t)=\Psi_1(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0<t<1,\\
\mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega_2(t)=\Psi_2(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0<t<1,\\
\mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu}+\omega)\Omega_3(t)=\Psi_3(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0<t<1,\\
\Omega_1(0)+\Omega_2(1)=0, \mathcal{D}^{\nu}\Omega_1(0)+\mathcal{D}^{\nu}\Omega_2(1)=0, \mathcal{D}^{2\nu}\Omega_1(0)+\mathcal{D}^{2\nu}\Omega_2(1)=0,\\
\Omega_2(0)+\Omega_3(1)=0, \mathcal{D}^{\nu}\Omega_2(0)+\mathcal{D}^{\nu}\Omega_3(1)=0, \mathcal{D}^{2\nu}\Omega_2(0)+\mathcal{D}^{2\nu}\Omega_3(1)=0,\\
\Omega_3(0)+\Omega_1(1)=0, \mathcal{D}^{\nu}\Omega_3(0)+\mathcal{D}^{\nu}\Omega_1(1)=0, \mathcal{D}^{2\nu}\Omega_3(0)+\mathcal{D}^{2\nu}\Omega_1(1)=0,
\end{cases}
\tag{1.4}
$$

where $\Psi_i : [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $i=1,2,3$ are given continuous functions, $\omega\in\mathbb{R}^+$, $0<\nu\le 1$ and $1<\zeta\le 2$. We prove the existence and uniqueness of solutions for system (1.4) via the Krasnoselskii fixed point theorem and the Banach contraction mapping theorem. In view of the results of [26], it is observed that the results are closely related to the friction coefficient ω. Nevertheless, these theorems are not valid for large values.

The key novelties that are presented in this study are as follows:

- The existence and uniqueness of solutions for equation (1.4) is established independently of the friction coefficient ω, hence highlighting the principal contribution of the paper.
- The primary findings hold true for high friction coefficient values of ω, as evidenced by alternative methods of demonstration.
- System (1.4) incorporates Langevin differential equations of fractional orders $0<\nu\le 1$ and $1<\zeta\le 2$, which are higher compared to those addressed in system (1.3). This necessitates more complex integral interpretations, broader use of generalized operators, and the inclusion of additional boundary conditions, leading to several challenging steps in the proof. Consequently, our study represents a significant generalization and a natural progression in the development of this theory.

The paper is outlined as follows: Section 2 will serve as prerequisites prior to the main results. We assemble some preparations of fractional calculus and state some essential theorems which are important to the existence of solution of the considered problem. The main theorems are reported in Section 3. In the section, by using linear algebra and inverse matrix tools, we first introduce the general solution of the system (1.4), then we conclude fixed points of the operator, which is acting as a solution of the system (1.4). In Section 4, the validity of the main results are illustrated by two particular examples.

2. Essential preliminaries

We begin by revisiting the fundamental definitions and lemmas that will be required in the following sections. For more detailed information on the properties of fractional calculus, readers can consult the monographs [18, 22].

Definition 2.1. *The Riemann-Liouville integral of order $v > 0$ for a function $\Theta : [0, \infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}^{v}\Theta(u) = \int_0^u \frac{(u-\ell)^{v-1}}{\Gamma(v)}\Theta(\ell)d\ell, \qquad 0 \le u \le 1, \tag{2.5}$$

where Γ is the Gamma function, provided the right side integral exists and is finite.

Definition 2.2. *The Riemann-Liouville fractional derivative of order ν for $\Theta : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{R}\mathfrak{D}_a^{\nu}\Theta(u) = \frac{d^k}{du^k}\mathfrak{I}_a^{k-\nu}\Theta(u) = \frac{1}{\Gamma(k-\nu)}\frac{d^k}{du^k}\int_a^u (u-\kappa)^{(k-\nu-1)}\Theta(\kappa)d\kappa,\ k-1 < \nu < k. \tag{2.6}$$

Definition 2.3. *The Caputo fractional derivative of order ν for $\Theta : [0, \infty) \to \mathbb{R}$ can be written as*

$$\mathfrak{D}_a^{\nu}\Theta(u) = {}^{R}\mathfrak{D}_a^{\nu}\Big[\Theta(u)-\Theta(a)-\Theta'(a)\frac{(u-a)}{1!}-\Theta''(a)\frac{(u-a)^2}{2!}-\cdots-\Theta^{(k-1)}(a)\frac{(u-a)^{k-1}}{(k-1)!}\Big]. \tag{2.7}$$

where $n-1 < v < n$, $n \in \mathbb{N}$, provided the right-hand-side integral exists and is finite.

For the relationship between (2.6) and (2.7) and for further properties of these concepts, we refer the reader to [18, 22].

Lemma 2.4 ([18])**.** *The general solution to $\mathfrak{D}^{v}\Theta = 0$, where $k \in \mathbb{N}^{+}$, $k-1 < v < k$, is given by*

$$\Theta(u) = \mathfrak{l}_0 + \mathfrak{l}_1 u + \mathfrak{l}_2 u^2 + \cdots + \mathfrak{l}_{k-1}u^{k-1},$$

where $\mathfrak{l}_n (n = 0, 1, \ldots, k-1)$ are real numbers.

Definition 2.5 ([22])**.** *The Laplace transform of the Caputo fractional derivative of order $v \in (q-1, q]$, $q \in \mathbb{N}$, is given by*

$$\ell(\mathfrak{D}^{v}\Theta)(s) = s^{v}U(s) - \sum_{i=0}^{q-1} s^{v-i-1}u^{(i)}(0),$$

where $U(s)$ represents the Laplace transform of the function Θ.

Definition 2.6 ([22])**.** *Let v and w be positive real numbers, and let r be a real number. The generalized Mittag-Leffler function is given by*

$$E_{v,w}^{r}(x) = \sum_{m=0}^{\infty} \frac{\Gamma(r+m)}{\Gamma(r)\Gamma(vm+w)} \cdot \frac{x^m}{m!}. \tag{2.8}$$

Remark 2.7. *In relation (2.8):*

- *If* $r = w = 1$*, we obtain*
$$E^1_{v,1}(x) = E_v(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(mv+1)},$$
and is referred to as Mittag-Leffler function of order v*.*
- *If* $r = 1$*, we have*
$$E^1_{v,w}(x) = E_{v,w}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(mv+w)},$$
and is referred to as the two-parameter Mittag-Leffler function with parameters v *and* w*.*
- *It is important to note that* $E_{v,w}(-x)$ *is completely monotonic function when* $0 < v \leq 1$ *and* $w \geq v$*. This means that* $E_{v,w}(-x)$ *has derivatives* $\frac{d^m}{dx^m}(E_{v,w}(-x))$ *for all* $m = 0, 1, 2, \ldots$ *and* $(-1)^m \frac{d^m}{dx^m}(E_{v,w}(-x)) \geq 0$ *for all* $x > 0$*. This property is particularly useful in proving the subsequent lemmas.*

We now present several key lemmas that are central to our discussion.

Lemma 2.8 ([5])**.** *For* $\alpha \geq \beta > 0$*,* $\lambda \in \mathbb{R}$*,* $l \in \{0, 1, 2, \cdots\}$ *and* $\boldsymbol{R}(s) > 0$*, where* $\boldsymbol{R}(s)$ *represents the real part of the complex number* s*, the Laplace inverse transform of the mapping* $F(s) = \frac{1}{(s^\alpha - \lambda s^\beta)^{l+1}}$ *is the following*
$$\ell^{-1}\{F(s)\}(t) = t^{(l+1)\alpha-1} E^{l+1}_{\alpha-\beta,(l+1)\alpha}(\lambda t^{\alpha-\beta}).$$

Lemma 2.9 ([10])**.** *For every* $0 < v \leq 1$ *and* $\theta > 0$*, it holds that*
$$|\frac{1 - E_v(-\theta)}{\theta}| \leq \frac{1}{\Gamma(v+1)}.$$

Lemma 2.10. *For any* $0 < v \leq 1$ *and* $\theta > 0$*, the following inequality holds:* $|\frac{1-E_{v,2}(-\theta)}{\theta}| \leq \frac{1}{\Gamma(v+1)}$*.*

Proof. Let $v \in (0, 1]$ and $\theta > 0$ be fixed real numbers, and define $f(t) = E_{v,2}(-t)$ on the interval $[0, \theta]$. It is clear that the function f is both continuous and differentiable. By the Lagrange mean value theorem, there exists a point $c_\theta \in (0, \theta)$ such that
$$\frac{1 - E_{v,2}(-\theta)}{\theta} = -f'(c_\theta).$$
Given that $(-1)^n \frac{d^n}{dx^n}(E_{v,2}(-x)) \geq 0$ for each $n = 0, 1, 2, \cdots$, and $x > 0$, it follows that $f'(t) = E'_{v,2}(-t)$ is an increasing function on $(0, \theta)$. This implies $-f'(c_\theta) \leq -f'(0) = \frac{1}{\Gamma(v+2)}$. Additionally, since $0 \leq 1 - E_{v,2}(-\theta) < 1$, we have $|\frac{1-E_{v,2}(-\Theta)}{\theta}| \leq \frac{1}{\Gamma(v+2)} \leq \frac{1}{\Gamma(v+1)}$. □

Lemma 2.11 ([12])**.** *Let* $v \in (0, 1)$ *and* θ *be a positive real number. Then, for every* $t \in [0, 1]$*, the following relations hold:*

I.: $t^{v+1}E_{v,v+2}(-\theta t^v) = \frac{t}{\theta}(1 - E_{v,2}(-\theta t^v))$.
II.: $t^v E_{v,v+1}(-\theta t^v) = \frac{1-E_v(-\theta t^v)}{\theta}$.

Proof. As we know, for each $v, w > 0$, the Mittag-Leffler function $E_{v,w}(x)$ is given by the second item in Remark 2.7. Then,

I.: For every $t \in [0,1]$, $v \in (0,1)$ and $\theta > 0$, we have

$$\begin{aligned} t^{v+1}E_{v,v+2}(-\theta t^v) &= t^{v+1}\sum_{n=0}^{\infty}\frac{(-1)^n\theta^n t^{nv}}{\Gamma(nv+v+2)} \\ &= t\sum_{n=0}^{\infty}\frac{(-1)^n\theta^n t^{(n+1)v}}{\Gamma((n+1)v+2)} = \frac{-t}{\theta}\sum_{m=1}^{\infty}\frac{(-1)^m\theta^m t^{mv}}{\Gamma(mv+2)} \\ &= \frac{-t}{\theta}\left(\sum_{m=0}^{\infty}\frac{(-1)^m\theta^m t^{mv}}{\Gamma(mv+2)} - 1\right) = \frac{t}{\theta}\left(1 - E_{v,2}(-\theta t^v)\right). \end{aligned}$$

II.: Also, for every $t \in [0,1]$, $v \in (0,1)$ and $\theta > 0$, we have

$$\begin{aligned} t^v E_{v,v+1}(-\theta t^v) &= t^v\sum_{n=0}^{\infty}\frac{(-1)^n\theta^n t^{nv}}{\Gamma(nv+v+1)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\theta^n t^{(n+1)v}}{\Gamma((n+1)v+1)} = \frac{-1}{\theta}\sum_{m=1}^{\infty}\frac{(-1)^m\theta^m t^{mv}}{\Gamma(mv+1)} \\ &= \frac{1}{\theta}\left(1 - \sum_{m=0}^{\infty}\frac{(-1)^m\theta^m t^{mv}}{\Gamma(mv+1)}\right) = \frac{1 - E_v(-\theta t^v)}{\theta}. \end{aligned}$$

The proof is completed. □

Lemma 2.12 ([11]). *For $v \in (0,1]$, $\theta > 0$, and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, the following inequality holds:*

$$|E_v(-\theta t_2^v) - E_v(-\theta t_1^v)| \le \frac{\theta(t_2^v - t_1^v)}{\Gamma(v+1)}.$$

Lemma 2.13. *For $v \in (0,1]$, $\theta > 0$, and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, the following inequality holds:*

$$|t_2 E_{v,2}(-\theta t_2^v) - t_1 E_{v,2}(-\theta t_1^v)| \le (t_2 - t_1).$$

Proof. Let $h(t) = tE_{v,2}(-\theta t^v)$ be defined on $[0,1]$. It is clear that the function h is continuous, and its derivative is given by $h'(t) = E_v(-\theta t^v)$. By the Lagrange mean value theorem, there exists $c_\theta \in (t_1, t_2)$ such that

$$\frac{t_2 E_{v,2}(-\theta t_2^v) - t_1 E_{v,2}(-\theta t_1^v)}{t_2 - t_1} = \frac{h(t_2) - h(t_1)}{t_2 - t_1} = h'(c_\theta) = E_v(-\theta c_\theta^v).$$

Hence,

$$\frac{|t_2 E_{v,2}(-\theta t_2^v) - t_1 E_{v,2}(-\theta t_1^v)|}{|t_2 - t_1|} \le E_v(-\theta c_\theta^v) \le 1.$$

Therefore,

$$|t_2 E_{v,2}(-\theta t_2^v) - t_1 E_{v,2}(-\theta t_1^v)| \le (t_2 - t_1).$$

□

Lemma 2.14 ([11]). *Let $v \in (0,1)$, $w \in (1,2]$, and ω be positive real numbers. Define the function $z(t) = t^{v+w-1}E_{v,v+w}(-\omega t^v)$ on the interval $[0,1]$. Then, z is an increasing function.*

Theorem 2.15 (The Krasnoselskii fixed point theorem [15]). *Let D be a closed, convex, bounded and nonempty subset of a Banach space X. Let $\mathbb{A}$ and $\mathbb{B}$ be two operators such that*

(I). $\mathbb{A}x + \mathbb{B}y \in D$ *for all* $x, y \in D$;
(II). $\mathbb{A}$ *is a completely continuous operator;*
(III). $\mathbb{B}$ *is a contraction mapping.*

Then there exists $z \in D$ such that $z = \mathbb{A}z + \mathbb{B}z$.

3. Main results

In the following lemma, which is the most important in our paper, we obtain the general solution of the problem (1.4).

Lemma 3.1. *Let $\mathfrak{H}_i \in \mathcal{C}([0,1],\mathbb{R}); i = 1,2,3$. Then, the general solution $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ on the interval $[0,1]$ of the tripled system of fractional Langevin equations*

(3.9)
$$\mathfrak{D}^{\zeta}(\mathfrak{D}^{\nu} + \omega)\Omega_i(t) = \mathfrak{H}_i(t), \quad 0 < t < 1, 0 < \nu \leq 1, 1 < \zeta \leq 2, i = 1,2,3,$$

with the cyclic boundary conditions

(3.10)
$$\begin{aligned}
&\Omega_1(0) + \Omega_2(1) = 0, \mathfrak{D}^{\nu}\Omega_1(0) + \mathfrak{D}^{\nu}\Omega_2(1) = 0, \mathcal{D}^{2\nu}\Omega_1(0) + \mathcal{D}^{2\nu}\Omega_2(1) = 0,\\
&\Omega_2(0) + \Omega_3(1) = 0, \mathfrak{D}^{\nu}\Omega_2(0) + \mathfrak{D}^{\nu}\Omega_3(1) = 0, \mathcal{D}^{2\nu}\Omega_2(0) + \mathcal{D}^{2\nu}\Omega_3(1) = 0,\\
&\Omega_3(0) + \Omega_1(1) = 0, \mathfrak{D}^{\nu}\Omega_3(0) + \mathfrak{D}^{\nu}\Omega_1(1) = 0, \mathcal{D}^{2\nu}\Omega_3(0) + \mathcal{D}^{2\nu}\Omega_1(1) = 0,
\end{aligned}$$

is given by

(3.11)

$$\Omega_1(t) = t^{\nu} E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}(\mathfrak{H}_1(\ell)+\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell))d\ell +$$
$$\frac{1}{2\Gamma(\zeta)} \int_0^1 (1-\ell)^{\zeta-1}(\mathfrak{H}_3(\ell)-\mathfrak{H}_2(\ell)-\mathfrak{H}_1(\ell))d\ell\Big)$$
$$- t^{\nu+1} E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_1(\ell)d\ell\Big)$$
$$+ \int_0^t (t-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{H}_1(\ell)d\ell$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}$$
$$\times \Big((ac^2-2bc^2-ac-a)\mathfrak{H}_1(\ell)+(ac^2+ac+a-2b)\mathfrak{H}_2(\ell)-(ac^2+ac-2bc-a)\mathfrak{H}_3(\ell)\Big)d\ell\Big)$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)} \int_0^1 (1-\ell)^{\zeta-1}$$
$$\times \Big(a(c^2-c-1)\mathfrak{H}_1(\ell)+a(c^2+c+1)\mathfrak{H}_2(\ell)-a(c^2+c-1)\mathfrak{H}_3(\ell)\Big)d\ell\Big)$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1} \int_0^1 (1-\ell)^{\zeta+\nu-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})(-c^2\mathfrak{H}_1(\ell)-\mathfrak{H}_2(\ell)+c\mathfrak{H}_3(\ell))d\ell\Big),$$

(3.12)

$$\Omega_2(t) = t^{\nu} E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}(\mathfrak{H}_2(\ell)+\mathfrak{H}_3(\ell)-\mathfrak{H}_1(\ell))d\ell +$$
$$\frac{1}{2\Gamma(\zeta)} \int_0^1 (1-\ell)^{\zeta-1}(\mathfrak{H}_1(\ell)-\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell))d\ell\Big)$$
$$- t^{\nu+1} E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_2(\ell)d\ell\Big)$$
$$+ \int_0^t (t-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{H}_2(\ell)d\ell$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}$$
$$\times \Big(-(ac^2+ac-2bc-a)\mathfrak{H}_1(\ell)+(ac^2-2bc^2-ac-a)\mathfrak{H}_2(\ell)+(ac^2+ac+a-2b)\mathfrak{H}_3(\ell)\Big)d\ell\Big)$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)} \int_0^1 (1-\ell)^{\zeta-1}$$
$$\times \Big(-a(c^2+c-1)\mathfrak{H}_1(\ell)+a(c^2-c-1)\mathfrak{H}_2(\ell)+a(c^2+c+1)\mathfrak{H}_3(\ell)\Big)d\ell\Big)$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1} \int_0^1 (1-\ell)^{\zeta+\nu-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})(c\mathfrak{H}_1(\ell)-c^2\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell))d\ell\Big),$$

(3.13)

$$\Omega_3(t) = t^{\nu} E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}(\mathfrak{H}_3(\ell)+\mathfrak{H}_1(\ell)-\mathfrak{H}_2(\ell))d\ell +$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}(\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell)-\mathfrak{H}_1(\ell))d\ell\Big)$$
$$- t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_3(\ell)d\ell\Big)$$
$$+ \int_0^t (t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{H}_3(\ell)d\ell$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}$$
$$\times\Big((ac^2+ac+a-2b)\mathfrak{H}_1(\ell) - (ac^2+ac-2bc-a)\mathfrak{H}_2(\ell) + (ac^2-2bc^2-ac-a)\mathfrak{H}_3(\ell)\Big)d\ell\Big)$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1 (1-\ell)^{\zeta-1}$$
$$\times\Big(-a(c^2+c-1)\mathfrak{H}_2(\ell) + a(c^2-c-1)\mathfrak{H}_3(\ell) + a(c^2+c+1)\mathfrak{H}_1(\ell)\Big)d\ell\Big)$$
$$+ E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})(-\mathfrak{H}_1(\ell)+c\mathfrak{H}_2(\ell)-c^2\mathfrak{H}_3(\ell))d\ell\Big),$$

where $a := E_{\nu,\nu+1}(-\omega)$, $b := E_{\nu,\nu+2}(-\omega)$ *and* $c := E_{\nu}(-\omega)$.

Proof. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be a solution. The relationship between the Caputo derivative and the Riemann-Liouville operator allows us to conclude that

$$\mathfrak{D}^{\zeta}[(\mathfrak{D}^{\nu}+\omega)\Omega_i(t) - \mathfrak{I}^{\zeta}\mathfrak{H}_i(t)] = 0, i = 1,2,3.$$

Now by applying Lemma 2.4, we deduce

$$(\mathfrak{D}^{\nu}+\omega)\Omega_i(t) - \mathfrak{I}^{\zeta}\mathfrak{H}_i(t) = c_0^i + c_1^i t, i = 1,2,3. \tag{3.14}$$

Applying the Laplace transform to both sides of (3.14) for the Caputo derivative yields:

$$s^{\nu}U_i(s) - s^{\nu-1}\Omega_i(0) + \omega U_i(s) = \frac{c_0^i}{s} + \frac{c_1^i}{s^2} + \frac{H_i(s)}{s^{\zeta}}, \tag{3.15}$$

where $U_i(s)$ and $H_i(s)$ denote the Laplace transforms of the functions Ω_i and $\mathfrak{H}_i$, $i = 1, 2, 3$.

We can then express the above formula in the following explicit form:

$$(s^{\nu}+\omega)U_i(s) = \frac{c_0^i}{s} + \frac{c_1^i}{s^2} + \frac{H_i(s)}{s^{\zeta}} + \frac{\Omega_i(0)}{s^{1-\nu}}, i = 1,2,3. \tag{3.16}$$

We now solve (3.16) for $U(s)$,

(3.17)

$$U_i(s) = \frac{c_0^i}{s^{\nu+1}+\omega s} + \frac{c_1^i}{s^{\nu+2}+\omega s^2} + \frac{H_i(s)}{s^{\nu+\zeta}+\omega s^{\zeta}} + \frac{\Omega_i(0)}{s+\omega s^{1-\nu}}, i = 1,2,3.$$

Taking the inverse Laplace transform of (3.17) and applying Lemma 2.8, we obtain an explicit representation of the solution to (3.9):

$$\begin{aligned}\Omega_i(t) &= c_0^i t^\nu E_{\nu,\nu+1}(-\omega t^\nu) + c_1^i t^{\nu+1} E_{\nu,\nu+2}(-\omega t^\nu) + \Omega_i(0) E_\nu(-\omega t^\nu) \\ &+ \int_0^t (t-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(t-\ell)^v)\mathfrak{H}_i(\ell)d\ell, i = 1,2,3.\end{aligned} \tag{3.18}$$

Hence, by using the boundary conditions (3.10), we have

$$c_0^1 E_{\nu,\nu+1}(-\omega) + c_1^1 E_{\nu,\nu+2}(-\omega) + \Omega_1(0)E_\nu(-\omega) + \Omega_3(0) = -\int_0^1 (1-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_1(\ell)d\ell,$$

$$c_0^2 E_{\nu,\nu+1}(-\omega) + c_1^2 E_{\nu,\nu+2}(-\omega) + \Omega_2(0)E_\nu(-\omega) + \Omega_1(0) = -\int_0^1 (1-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_2(\ell)d\ell,$$

$$c_0^3 E_{\nu,\nu+1}(-\omega) + c_1^3 E_{\nu,\nu+2}(-\omega) + \Omega_3(0)E_\nu(-\omega) + \Omega_2(0) = -\int_0^1 (1-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_3(\ell)d\ell,$$

$$\begin{aligned} c_0^1 + c_0^2 + c_1^2 &= -\mathfrak{I}^\zeta \mathfrak{H}_2(t)_{|t=1}, \\ c_0^2 + c_0^3 + c_1^3 &= -\mathfrak{I}^\zeta \mathfrak{H}_3(t)_{|t=1}, \\ c_0^3 + c_0^1 + c_1^1 &= -\mathfrak{I}^\zeta \mathfrak{H}_1(t)_{|t=1}, \end{aligned}$$

$$c_1^p = -\Gamma(2-\nu)\mathfrak{I}^{\zeta-\nu}\mathfrak{H}_p(t)_{|t=1}, p = 1,2,3.$$

The matrix representation of the above linear equations is as follows and to obtain the values $c_0^i, c_1^i, \Omega_i(0)$, $i = 1,2,3$, we must solve the system

$$AX = B, \tag{3.19}$$

where

$$A = \begin{pmatrix} a & 0 & 0 & b & 0 & 0 & c & 0 & 1 \\ 0 & a & 0 & 0 & b & 0 & 1 & c & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 1 & c \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} c_0^1 \\ c_0^2 \\ c_0^3 \\ c_1^1 \\ c_1^2 \\ c_1^3 \\ \Omega_1(0) \\ \Omega_2(0) \\ \Omega_3(0) \end{pmatrix}, B = \begin{pmatrix} -\int_0^1 (1-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_1(\ell)d\ell \\ -\int_0^1 (1-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_2(\ell)d\ell \\ -\int_0^1 (1-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_3(\ell)d\ell \\ -\mathfrak{I}^\zeta \mathfrak{H}_2(t)_{|t=1} \\ -\mathfrak{I}^\zeta \mathfrak{H}_3(t)_{|t=1} \\ -\mathfrak{I}^\zeta \mathfrak{H}_1(t)_{|t=1} \\ -\Gamma(2-\nu)\mathfrak{I}^{\zeta-\nu}\mathfrak{H}_1(t)_{|t=1} \\ -\Gamma(2-\nu)\mathfrak{I}^{\zeta-\nu}\mathfrak{H}_2(t)_{|t=1} \\ -\Gamma(2-\nu)\mathfrak{I}^{\zeta-\nu}\mathfrak{H}_3(t)_{|t=1} \end{pmatrix}.$$

It is not difficult to verify that A is an invertible matrix. So, the system of (3.19) has the unique solution

$$c_0^1 = \frac{1}{2\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}(\mathfrak{H}_3(\ell)-\mathfrak{H}_1(\ell)-\mathfrak{H}_2(\ell))d\ell + \frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}(\mathfrak{H}_1(\ell)+\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell))d\ell,$$

$$c_0^2 = \frac{1}{2\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}(\mathfrak{H}_1(\ell)-\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell))d\ell + \frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}(\mathfrak{H}_2(\ell)+\mathfrak{H}_3(\ell)-\mathfrak{H}_1(\ell))d\ell,$$

$$c_0^3 = \frac{1}{2\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}(\mathfrak{H}_2(\ell)-\mathfrak{H}_3(\ell)-\mathfrak{H}_1(\ell))d\ell + \frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}(\mathfrak{H}_3(\ell)+\mathfrak{H}_1(\ell)-\mathfrak{H}_2(\ell))d\ell,$$

$$c_1^p = -\Gamma(2-\nu)\mathfrak{I}^{\zeta-\nu}\mathfrak{H}_p(t)_{|t=1}, p=1,2,3,$$

$$\begin{aligned}
\Omega_1(0) = &-\frac{c^2}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_1(\ell)d\ell \\
&- \frac{1}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_2(\ell)d\ell \\
&+ \frac{c}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_3(\ell)d\ell \\
&+ \frac{a(c^2+c+1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_2(\ell)d\ell \\
&- \frac{a(c^2+c-1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_3(\ell)d\ell + \frac{a(c^2-c-1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_1(\ell)d\ell \\
&- (\frac{ac^2-2bc^2-ac-a}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_1(\ell)d\ell \\
&- (\frac{ac^2+ac+a-2b}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_2(\ell)d\ell \\
&+ (\frac{ac^2-2bc+ac-a}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_3(\ell)d\ell,
\end{aligned}$$

$$\begin{aligned}
\Omega_2(0) = &\frac{c}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_1(\ell)d\ell \\
&- \frac{c^2}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_2(\ell)d\ell \\
&- \frac{1}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_3(\ell)d\ell \\
&+ \frac{a(c^2-c-1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_2(\ell)d\ell \\
&+ \frac{a(c^2+c+1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_3(\ell)d\ell - \frac{a(c^2+c-1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_1(\ell)d\ell \\
&+ (\frac{ac^2-2bc+ac-a}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_1(\ell)d\ell \\
&- (\frac{ac^2-2bc^2-ac-a}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_2(\ell)d\ell \\
&- (\frac{ac^2+ac+a-2b}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_3(\ell)d\ell,
\end{aligned}$$

$$
\begin{aligned}
\Omega_3(0) = & \frac{-1}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_1(\ell)d\ell \\
& + \frac{c}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_2(\ell)d\ell \\
& - \frac{c^2}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)\mathfrak{H}_3(\ell)d\ell \\
& - \frac{a(c^2+c-1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_2(\ell)d\ell \\
& + \frac{a(c^2-c-1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_3(\ell)d\ell + \frac{a(c^2+c+1)}{2(c^3+1)\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}\mathfrak{H}_1(\ell)d\ell \\
& - (\frac{ac^2+ac+a-2b}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_1(\ell)d\ell \\
& + (\frac{ac^2+ac-2bc-a}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_2(\ell)d\ell \\
& - (\frac{ac^2-ac-a-2bc^2}{2(c^3+1)})\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}\mathfrak{H}_3(\ell)d\ell.
\end{aligned}
$$

Substituting the values of $c_0^i, c_1^i, \Omega_i(0)$, $i = 1,2,3$, in (3.18), we obtain the desired solution (3.11-3.13). Conversely, it is not difficult to verify that $(\Omega_1,\Omega_2,\Omega_3)$ given by (3.11-3.13) satisfies the system (3.9) and the boundary conditions (3.10). The proof is finished. □

Here, we define the operator $\psi : \mathcal{C}[0,1]\times\mathcal{C}[0,1]\times\mathcal{C}[0,1] \to \mathcal{C}[0,1]\times\mathcal{C}[0,1]\times\mathcal{C}[0,1]$ as

$$\psi(\Omega_1,\Omega_2,\Omega_3) = (\psi_1(\Omega_1,\Omega_2,\Omega_3), \psi_2(\Omega_1,\Omega_2,\Omega_3), \psi_3(\Omega_1,\Omega_2,\Omega_3)),$$

where $\psi_1,\psi_2,\psi_3 : \mathcal{C}[0,1]\times\mathcal{C}[0,1]\times\mathcal{C}[0,1] \to \mathcal{C}[0,1]$ define

$$\psi_1(\Omega_1,\Omega_2,\Omega_3)(t)$$
$$= t^{\nu}E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(\mathfrak{f}_{1,\Omega}(\ell)+\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell))d\ell+$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(\mathfrak{f}_{3,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{1,\Omega}(\ell))d\ell\Big)$$
$$- t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\mathfrak{f}_{1,\Omega}(\ell)d\ell\Big)$$
$$+\int_0^t(t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{f}_{1,\Omega}(\ell)d\ell$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}$$
$$\times\Big((ac^2-2bc^2-ac-a)\mathfrak{f}_{1,\Omega}(\ell)+(ac^2+ac+a-2b)\mathfrak{f}_{2,\Omega}(\ell)-(ac^2+ac-2bc-a)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}$$
$$\times\Big(a(c^2-c-1)\mathfrak{f}_{1,\Omega}(\ell)+a(c^2+c+1)\mathfrak{f}_{2,\Omega}(\ell)-a(c^2+c-1)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})$$
$$\times(-c^2\mathfrak{f}_{1,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell)+c\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big), t\in[0,1],$$

$$\psi_2(\Omega_1,\Omega_2,\Omega_3)(t)$$
$$= t^{\nu}E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(\mathfrak{f}_{2,\Omega}(\ell)+\mathfrak{f}_{3,\Omega}(\ell)-\mathfrak{f}_{1,\Omega}(\ell))d\ell+$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(\mathfrak{f}_{1,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big)$$
$$- t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\mathfrak{f}_{2,\Omega}(\ell)d\ell\Big)$$
$$+\int_0^t(t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{f}_{2,\Omega}(\ell)d\ell$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}$$
$$\times\Big(-(ac^2+ac-2bc-a)\mathfrak{f}_{1,\Omega}(\ell)+(ac^2-2bc^2-ac-a)\mathfrak{f}_{2,\Omega}(\ell)+(ac^2+ac+a-2b)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}$$
$$\times\Big(-a(c^2+c-1)\mathfrak{f}_{1,\Omega}(\ell)+a(c^2-c-1)\mathfrak{f}_{2,\Omega}(\ell)+a(c^2+c+1)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})$$
$$\times(c\mathfrak{f}_{1,\Omega}(\ell)-c^2\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big), t\in[0,1],$$

$$\psi_3(\Omega_1,\Omega_2,\Omega_3)(t)$$
$$= t^{\nu}E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(\mathfrak{f}_{3,\Omega}(\ell)+\mathfrak{f}_{1,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell))d\ell+$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell)-\mathfrak{f}_{1,\Omega}(\ell))d\ell\Big)$$
$$-t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\mathfrak{f}_{3,\Omega}(\ell)d\ell\Big)$$
$$+\int_0^t(t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{f}_{3,\Omega}(\ell)d\ell$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}$$
$$\times\Big((ac^2+ac+a-2b)\mathfrak{f}_{1,\Omega}(\ell)-(ac^2+ac-2bc-a)\mathfrak{f}_{2,\Omega}(\ell)+(ac^2-2bc^2-ac-a)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}$$
$$\times\Big(-a(c^2+c-1)\mathfrak{f}_{2,\Omega}(\ell)+a(c^2-c-1)\mathfrak{f}_{3,\Omega}(\ell)+a(c^2+c+1)\mathfrak{f}_{1,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})$$
$$\times(-\mathfrak{f}_{1,\Omega}(\ell)+c\mathfrak{f}_{2,\Omega}(\ell)-c^2\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big),t\in[0,1],$$

where $\mathfrak{f}_{i,\Omega}(s)=\Psi_i(s,\Omega_1(s),\Omega_2(s),\Omega_3(s))$, and $\Omega=(\Omega_1,\Omega_2,\Omega_3)$, $i=1,2,3$.

Now, in view of Lemma 3.1, it important to say that the function $\Omega=(\Omega_1,\Omega_2,\Omega_3)$ is a solution of the problem (1.4), if and only if Ω is a fixed point of operator ψ.

Now, we introduce the following constant:

$$\xi:=\frac{6}{\Gamma(\nu+1)\Gamma(\zeta+1)}+\frac{17\Gamma(2-\nu)}{2\Gamma(\nu+1)\Gamma(\zeta-\nu+1)}+\frac{4}{\Gamma(\zeta+\nu+1)}. \tag{3.20}$$

In the following result, we prove the uniqueness of solution to the problem (1.4) by using Banach's contraction mapping theorem.

Theorem 3.2. *Let $\Psi_1,\Psi_2,\Psi_3\in\mathcal{C}([0,1]\times\mathbb{R}^3,\mathbb{R})$ and $\mu_i, i=1,2,3$, be non-negative constants such that*

(3.21)
$$|\Psi_i(t,\hat{z}_1,\hat{z}_2,\hat{z}_3)-\Psi_i(t,\hat{w}_1,\hat{w}_2,\hat{w}_3)|$$
$$\leq\mu_i\Big(|\hat{z}_1-\hat{w}_1|+|\hat{z}_2-\hat{w}_2|+|\hat{z}_3-\hat{w}_3|\Big),\ \forall\ t\in[0,1],\ \forall\ \hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{w}_1,\hat{w}_2,\hat{w}_3\in\mathbb{R}.$$

Then ψ is a contraction mapping in $\mathcal{C}[0,1]\times\mathcal{C}[0,1]\times\mathcal{C}[0,1]$ with Lipschitz constant $\xi(\mu_1+\mu_2+\mu_3)$. Moreover, if

$$\xi(\mu_1+\mu_2+\mu_3)<1, \tag{3.22}$$

then the problem (1.4) has a unique solution in $\mathcal{C}[0,1]\times\mathcal{C}[0,1]\times\mathcal{C}[0,1]$.

Proof. Let $\mathcal{X} := \mathcal{C}[0,1] \times \mathcal{C}[0,1] \times \mathcal{C}[0,1]$. Then, $(\mathcal{X}, \|.\|_{\mathcal{C}^3[0,1]})$ is a Banach space with norm

$$\|(\hat{z}, \hat{w}, \hat{v})\|_{\mathcal{C}^3[0,1]} = \|\hat{z}\|_\infty + \|\hat{w}\|_\infty + \|\hat{v}\|_\infty,$$

where $\|\Omega\|_\infty = \max_{t\in[0,1]} |\Omega(t)|$ for each $\Omega \in \mathcal{C}[0,1]$.

Now, for each $\Omega = (\Omega_1, \Omega_2, \Omega_3), \widetilde{\Omega} = (\widetilde{\Omega}_1, \widetilde{\Omega}_2, \widetilde{\Omega}_3) \in \mathcal{X}$ and $t \in [0,1]$, we have

$$\begin{aligned}
&|\psi_1(\Omega_1, \Omega_2, \Omega_3)(t) - \psi_1(\widetilde{\Omega}_1, \widetilde{\Omega}_2, \widetilde{\Omega}_3)(t)| \leq \\
&\quad t^\nu E_{\nu,\nu+1}(-\omega t^\nu)\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}(|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)| + |\mathfrak{f}_{2,\Omega}(\ell) - \mathfrak{f}_{2,\widetilde{\Omega}}(\ell)| \\
&+ |\mathfrak{f}_{3,\Omega}(\ell) - \mathfrak{f}_{3,\widetilde{\Omega}}(\ell)|)d\ell + \\
&\frac{1}{2\Gamma(\zeta)} \int_0^1 (1-\ell)^{\zeta-1}(|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)| + |\mathfrak{f}_{2,\Omega}(\ell) - \mathfrak{f}_{2,\widetilde{\Omega}}(\ell)| + |\mathfrak{f}_{3,\Omega}(\ell) - \mathfrak{f}_{3,\widetilde{\Omega}}(\ell)|)d\ell\Big) \\
&+ t^{\nu+1} E_{\nu,\nu+2}(-\omega t^\nu)\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1}|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)|d\ell\Big) \\
&+ \int_0^t (t-\ell)^{\nu+\zeta-1} E_{\nu,\nu+\zeta}(-\omega(t-\ell)^\nu)|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)|d\ell \\
&+ E_\nu(-\omega t^\nu)\Big(\frac{1}{2(c^3+1)} \frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)} \int_0^1 (1-\ell)^{\zeta-\nu-1} \\
&\times \Big((ac^2 + 2bc^2 + ac + a)|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)| + (ac^2 + ac + a + 2b)|\mathfrak{f}_{2,\Omega}(\ell) - \mathfrak{f}_{2,\widetilde{\Omega}}(\ell)| \\
&+ (ac^2 + ac + 2bc + a)|\mathfrak{f}_{3,\Omega}(\ell) - \mathfrak{f}_{3,\widetilde{\Omega}}(\ell)|\Big)d\ell\Big) \\
&+ E_\nu(-\omega t^\nu)\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)} \int_0^1 (1-\ell)^{\zeta-1}\Big(a(c^2+c+1)|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)| \\
&+ a(c^2+c+1)|\mathfrak{f}_{2,\Omega}(\ell) - \mathfrak{f}_{2,\widetilde{\Omega}}(\ell)| + a(c^2+c+1)|\mathfrak{f}_{3,\Omega}(\ell) - \mathfrak{f}_{3,\widetilde{\Omega}}(\ell)|\Big)d\ell\Big) \\
&+ E_\nu(-\omega t^\nu)\Big(\frac{1}{c^3+1} \int_0^1 (1-\ell)^{\zeta+\nu-1} E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)(c^2|\mathfrak{f}_{1,\Omega}(\ell) - \mathfrak{f}_{1,\widetilde{\Omega}}(\ell)| + |\mathfrak{f}_{2,\Omega}(\ell) - \mathfrak{f}_{2,\widetilde{\Omega}}(\ell)| \\
&+ c|\mathfrak{f}_{3,\Omega}(\ell) - \mathfrak{f}_{3,\widetilde{\Omega}}(\ell)|)d\ell\Big).
\end{aligned}$$

Here, by (3.21), Lemma 2.9, Lemma 2.10, Lemma 2.11 and from the fact that $(-1)^m \frac{d^m}{dx^m}(E_{v,w}(-x)) \geq 0$ for all $x > 0$ and for all $m = 0, 1, 2, \ldots,$

when $0 < v \leq 1$ and $w \geq v$, we can derive that

$$|\psi_1(\Omega_1,\Omega_2,\Omega_3)(t) - \psi_1(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)(t)| \leq$$
$$\left(\frac{2\mu_1+\mu_2+\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(7\mu_1+5\mu_2+5\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_1-\widetilde{\Omega}_1\|_\infty$$
$$+\left(\frac{2\mu_1+\mu_2+\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(7\mu_1+5\mu_2+5\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_2-\widetilde{\Omega}_2\|_\infty$$
$$\left(\frac{2\mu_1+\mu_2+\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(7\mu_1+5\mu_2+5\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_3-\widetilde{\Omega}_3\|_\infty.$$

Similarity, we can show that

$$|\psi_2(\Omega_1,\Omega_2,\Omega_3)(t) - \psi_2(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)(t)| \leq$$
$$\left(\frac{\mu_1+2\mu_2+\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(5\mu_1+7\mu_2+5\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_1-\widetilde{\Omega}_1\|_\infty$$
$$+\left(\frac{\mu_1+2\mu_2+\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(5\mu_1+7\mu_2+5\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_2-\widetilde{\Omega}_2\|_\infty$$
$$\left(\frac{\mu_1+2\mu_2+\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(5\mu_1+7\mu_2+5\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_3-\widetilde{\Omega}_3\|_\infty.$$

and

$$|\psi_3(\Omega_1,\Omega_2,\Omega_3)(t) - \psi_3(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)(t)|$$
$$\leq \left(\frac{\mu_1+\mu_2+2\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(5\mu_1+5\mu_2+7\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_1-\widetilde{\Omega}_1\|_\infty$$
$$+\left(\frac{\mu_1+\mu_2+2\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(5\mu_1+5\mu_2+7\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_2-\widetilde{\Omega}_2\|_\infty$$
$$+\left(\frac{\mu_1+\mu_2+2\mu_3}{\Gamma(\zeta+\nu+1)} + \frac{2(\mu_1+\mu_2+\mu_3)}{\Gamma(\nu+1)\Gamma(\zeta+1)}\frac{\Gamma(2-\nu)(5\mu_1+5\mu_2+7\mu_3)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}\right)\|\Omega_3-\widetilde{\Omega}_3\|_\infty.$$

From the above inequalities, we obtain

$$\begin{aligned}
&\|\psi(\Omega_1,\Omega_2,\Omega_3)(t)-\psi(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)\|_{\mathcal{C}^3[0,1]}\\
&\quad=\|\psi_1(\Omega_1,\Omega_2,\Omega_3)-\psi_1(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)\|_\infty+\|\psi_2(\Omega_1,\Omega_2,\Omega_3)-\psi_2(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)\|_\infty\\
&\qquad+\|\psi_3(\Omega_1,\Omega_2,\Omega_3)-\psi_3(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)\|_\infty\\
&\quad\le\xi(\mu_1+\mu_2+\mu_3)\|(\Omega_1,\Omega_2,\Omega_3)-(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)\|_{\mathcal{C}^3[0,1]}.
\end{aligned}$$

In view of the condition (3.22), we deduce that ψ is a contraction. Then we conclude by the Banach fixed point theory that the operator ψ has a unique fixed point Ω, which is the unique solution of problem (1.4). The theorem is proved. □

In the following, by applying Krasnoselskii's fixed point theorem, we present the existence result for the BVP (1.4).

Theorem 3.3. *Assume that*

H_1. $\Psi_i:[0,1]\times\mathbb{R}^3\to\mathbb{R}$, $i=1,2,3$, *are continuous.*

H_2. *There exist nonnegative functions* $p_i,q_i,r_i,k_i\in\mathcal{C}[0,1]$, *such that, for all* $(t,u,v,w)\in[0,1]\times\mathbb{R}^3$, *(i=1,2,3),*

$$|\Psi_i(t,u,v,w)|\le k_i(t)+p_i(t)|u|+q_i(t)|v|+r_i(t)|w|,$$

hold. Then the system (1.4) has at least one solution on $[0,1]$, *provided that*

$$\Big(\frac{6}{\Gamma(\nu+1)\Gamma(\zeta+1)}+\frac{17\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}+\frac{4}{\Gamma(\zeta+\nu+1)}\Big)\sum_{i=1}^{3}l_i<1,$$

where

$$p_i=\max_{t\in[0,1]}|p_i(t)|,q_i=\max_{t\in[0,1]}|q_i(t)|,r_i=\max_{t\in[0,1]}|r_i(t)|,$$
$$k_i=\max_{t\in[0,1]}|k_i(t)|,l_i=p_i+q_i+r_i,i=1,2,3.$$

Proof. Let

$$\epsilon\ge\frac{\Big(\frac{6}{\Gamma(\nu+1)\Gamma(\zeta+1)}+\frac{17\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}+\frac{4}{\Gamma(\zeta+\nu+1)}\Big)\sum_{i=1}^{3}k_i}{1-\Big(\frac{6}{\Gamma(\nu+1)\Gamma(\zeta+1)}+\frac{17\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}+\frac{4}{\Gamma(\zeta+\nu+1)}\Big)\sum_{i=1}^{3}l_i}$$

be a positive number and consider the closed ball

$$B_\epsilon=\{\Omega=(\Omega_1,\Omega_2,\Omega_3)\in\mathcal{X}:\|\Omega\|_{\mathcal{C}^3[0,1]}\le\epsilon\},$$

where $\mathcal{X}=\mathcal{C}[0,1]\times\mathcal{C}[0,1]\times\mathcal{C}[0,1]$ equipped $\|.\|_{\mathcal{C}^3[0,1]}$ is a Banach space with norm

$$\|(\hat{z},\hat{w},\hat{v})\|_{\mathcal{C}^3[0,1]}=\|\hat{z}\|_\infty+\|\hat{w}\|_\infty+\|\hat{v}\|_\infty,$$

where $\|z\|_\infty=\max_{t\in[0,1]}|z(t)|$ for each $z\in\mathcal{C}[0,1]$.

Next, let us define the operators $\mathcal{U},\mathcal{V}$ on B_ϵ as follows:

$$\mathcal{U}\Omega(t)=(\mathcal{U}_1(\Omega_1,\Omega_2,\Omega_3),\mathcal{U}_2(\Omega_1,\Omega_2,\Omega_3),\mathcal{U}_3(\Omega_1,\Omega_2,\Omega_3))(t),$$
$$\mathcal{V}\Omega(t)=(\mathcal{V}_1(\Omega_1,\Omega_2,\Omega_3),\mathcal{V}_2(\Omega_1,\Omega_2,\Omega_3),\mathcal{V}_3(\Omega_1,\Omega_2,\Omega_3))(t),$$

where

$$\mathcal{U}_1(\Omega_1,\Omega_2,\Omega_3)(t)=\mathcal{U}_2(\Omega_1,\Omega_2,\Omega_3)(t)=\mathcal{U}_3(\Omega_1,\Omega_2,\Omega_3)(t)=0, t\in[0,1],$$

$$\begin{aligned}
&\mathcal{V}_1(\Omega_1,\Omega_2,\Omega_3)(t)=\\
&t^{\nu}E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(\mathfrak{f}_{1,\Omega}(\ell)+\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell))d\ell+\\
&\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(\mathfrak{f}_{3,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{1,\Omega}(\ell))d\ell\Big)\\
&-t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\mathfrak{f}_{1,\Omega}(\ell)d\ell\Big)\\
&+\int_0^t(t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{f}_{1,\Omega}(\ell)d\ell\\
&+E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\\
&\times\Big((ac^2-2bc^2-ac-a)\mathfrak{f}_{1,\Omega}(\ell)+(ac^2+ac+a-2b)\mathfrak{f}_{2,\Omega}(\ell)-(ac^2+ac-2bc-a)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)\\
&+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}\\
&\times\Big(a(c^2-c-1)\mathfrak{f}_{1,\Omega}(\ell)+a(c^2+c+1)\mathfrak{f}_{2,\Omega}(\ell)-a(c^2+c-1)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)\\
&+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})\\
&\times(-c^2\mathfrak{f}_{1,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell)+c\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big), t\in[0,1],
\end{aligned}$$

$$\mathcal{V}_2(\Omega_1,\Omega_2,\Omega_3)(t) =$$
$$t^{\nu}E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(\mathfrak{f}_{2,\Omega}(\ell)+\mathfrak{f}_{3,\Omega}(\ell)-\mathfrak{f}_{1,\Omega}(\ell))d\ell+$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(\mathfrak{f}_{1,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big)$$
$$-t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\mathfrak{f}_{2,\Omega}(\ell)d\ell\Big)$$
$$+\int_0^t(t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{f}_{2,\Omega}(\ell)d\ell$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}$$
$$\times\Big(-(ac^2+ac-2bc-a)\mathfrak{f}_{1,\Omega}(\ell)+(ac^2-2bc^2-ac-a)\mathfrak{f}_{2,\Omega}(\ell)+(ac^2+ac+a-2b)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}$$
$$\times\Big(-a(c^2+c-1)\mathfrak{f}_{1,\Omega}(\ell)+a(c^2-c-1)\mathfrak{f}_{2,\Omega}(\ell)+a(c^2+c+1)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})$$
$$\times(c\mathfrak{f}_{1,\Omega}(\ell)-c^2\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big), t\in[0,1],$$

$$\mathcal{V}_3(\Omega_1,\Omega_2,\Omega_3)(t)$$
$$=t^{\nu}E_{\nu,\nu+1}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(\mathfrak{f}_{3,\Omega}(\ell)+\mathfrak{f}_{1,\Omega}(\ell)-\mathfrak{f}_{2,\Omega}(\ell))d\ell+$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(\mathfrak{f}_{2,\Omega}(\ell)-\mathfrak{f}_{3,\Omega}(\ell)-\mathfrak{f}_{1,\Omega}(\ell))d\ell\Big)$$
$$-t^{\nu+1}E_{\nu,\nu+2}(-\omega t^{\nu})\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}\mathfrak{f}_{3,\Omega}(\ell)d\ell\Big)$$
$$+\int_0^t(t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^{\nu})\mathfrak{f}_{3,\Omega}(\ell)d\ell$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{-1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}$$
$$\times\Big((ac^2+ac+a-2b)\mathfrak{f}_{1,\Omega}(\ell)-(ac^2+ac-2bc-a)\mathfrak{f}_{2,\Omega}(\ell)+(ac^2-2bc^2-ac-a)\mathfrak{f}_{3,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}$$
$$\times\Big(-a(c^2+c-1)\mathfrak{f}_{2,\Omega}(\ell)+a(c^2-c-1)\mathfrak{f}_{3,\Omega}(\ell)+a(c^2+c+1)\mathfrak{f}_{1,\Omega}(\ell)\Big)d\ell\Big)$$
$$+E_{\nu}(-\omega t^{\nu})\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^{\nu})$$
$$\times(-\mathfrak{f}_{1,\Omega}(\ell)+c\mathfrak{f}_{2,\Omega}(\ell)-c^2\mathfrak{f}_{3,\Omega}(\ell))d\ell\Big), t\in[0,1].$$

Here, we divide the proof into three steps.

(I). We prove that $\mathcal{V}\Omega_1 + \mathcal{U}\Omega_2 \in B_\epsilon$, for each $\Omega_1 = (\Omega_1^1, \Omega_1^2, \Omega_1^3), \Omega_2 = (\Omega_2^1, \Omega_2^2, \Omega_2^3) \in B_\epsilon$. For any $\Omega_1, \Omega_2 \in B_\epsilon$, we have $\|\Omega_1\|_{\mathcal{C}^3[0,1]} \leq \epsilon$ and $\|\Omega_2\|_{\mathcal{C}^3[0,1]} \leq \epsilon$. Now

$$
\begin{aligned}
&|\mathcal{V}_1\Omega_1(t)| \leq \\
&t^\nu E_{\nu,\nu+1}(-\omega t^\nu)\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}(|\mathfrak{f}_{1,\Omega_1}(\ell)| + |\mathfrak{f}_{2,\Omega_1}(\ell)| + |\mathfrak{f}_{3,\Omega_1}(\ell)|)d\ell + \\
&\frac{1}{2\Gamma(\zeta)}\int_0^1 (1-\ell)^{\zeta-1}(|\mathfrak{f}_{3,\Omega_1}(\ell)| + |\mathfrak{f}_{2,\Omega_1}(\ell)| + |\mathfrak{f}_{1,\Omega_1}(\ell)|)d\ell\Big) \\
&+ t^{\nu+1}E_{\nu,\nu+2}(-\omega t^\nu)\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1}|\mathfrak{f}_{1,\Omega_1}(\ell)|d\ell\Big) \\
&+ \int_0^t (t-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t-\ell)^\nu)|\mathfrak{f}_{1,\Omega_1}(\ell)|d\ell \\
&+ E_\nu(-\omega t^\nu)\Big(\frac{1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1 (1-\ell)^{\zeta-\nu-1} \\
&\times\Big((ac^2+2bc^2+ac+a)|\mathfrak{f}_{1,\Omega_2}(\ell)| + (ac^2+ac+a+2b)|\mathfrak{f}_{2,\Omega_2}(\ell)| + (ac^2+ac+2bc+a)|\mathfrak{f}_{3,\Omega_2}(\ell)|\Big)d\ell\Big) \\
&+ E_\nu(-\omega t^\nu)\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1 (1-\ell)^{\zeta-1} \\
&\times\Big(a(c^2+c+1)|\mathfrak{f}_{1,\Omega_2}(\ell)| + a(c^2+c+1)|\mathfrak{f}_{2,\Omega_2}(\ell)| + a(c^2+c+1)|\mathfrak{f}_{3,\Omega_2}(\ell)|\Big)d\ell\Big) \\
&+ E_\nu(-\omega t^\nu)\Big(\frac{1}{c^3+1}\int_0^1 (1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)(c^2|\mathfrak{f}_{1,\Omega_2}(\ell)| + |\mathfrak{f}_{2,\Omega_2}(\ell)| + c|\mathfrak{f}_{3,\Omega_2}(\ell)|)d\ell\Big).
\end{aligned}
$$

Then by applying (H_2), Lemma 2.9, Lemma 2.10, Lemma 2.11 and from the fact that $(-1)^m \frac{d^m}{dx^m}(E_{v,w}(-x)) \geq 0$ for all $x > 0$ and for all $m = 0, 1, 2, \ldots$, when $0 < v \leq 1$ and $w \geq v$, we obtain

$$
\begin{aligned}
&|\mathcal{V}_1\Omega_1(t)| \leq \\
&\frac{\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}}{2\Gamma(\zeta+1)\Gamma(\nu+1)} + \frac{\Gamma(2-\nu)\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)} + \frac{\Gamma(2-\nu)(k_1 + l_1\|\Omega_1\|_{\mathcal{C}^3[0,1]})}{\Gamma(\zeta-\nu+1)\Gamma(\nu+1)} \\
&+ \frac{k_1 + l_1\|\Omega_1\|_{\mathcal{C}^3[0,1]}}{\Gamma(\zeta+\nu+1)} + \frac{2\Gamma(2-\nu)}{\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}(\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}) \\
&+ \frac{3}{2\Gamma(\zeta+1)\Gamma(\nu+1)}(\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}) + \frac{1}{\Gamma(\zeta+\nu+1)}(\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}).
\end{aligned}
$$

By similar way, for $p = 2, 3$, we find

$$
\begin{aligned}
&|\mathcal{V}_p\Omega_1(t)| \leq \frac{\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}}{2\Gamma(\zeta+1)\Gamma(\nu+1)} + \frac{\Gamma(2-\nu)\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)} + \frac{\Gamma(2-\nu)(k_p + l_p\|\Omega_1\|_{\mathcal{C}^3[0,1]})}{\Gamma(\zeta-\nu+1)\Gamma(\nu+1)} \\
&+ \frac{k_p + l_p\|\Omega_1\|_{\mathcal{C}^3[0,1]}}{\Gamma(\zeta+\nu+1)} + + \frac{2\Gamma(2-\nu)}{\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}(\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}) \\
&+ \frac{3}{2\Gamma(\zeta+1)\Gamma(\nu+1)}(\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}) + \frac{1}{\Gamma(\zeta+\nu+1)}(\sum_{i=1}^3 k_i + l_i\|\Omega_1\|_{\mathcal{C}^3[0,1]}), t \in [0,1].
\end{aligned}
$$

On the other hand, for all $t \in [0, 1]$, we obtain the following inequality:

$$\|\mathcal{V}\Omega_1+\mathcal{U}\Omega_2\|_{\mathcal{C}^3[0,1]}=\|\mathcal{V}_1\Omega_1+\mathcal{U}_1\Omega_2\|_\infty+\|\mathcal{V}_2\Omega_1+\mathcal{U}_2\Omega_2\|_\infty+\|\mathcal{V}_3\Omega_1+\mathcal{U}_3\Omega_2\|_\infty$$
$$\leq\Big(\frac{6}{\Gamma(\nu+1)\Gamma(\zeta+1)}+\frac{17\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)\Gamma(\nu+1)}+\frac{4}{\Gamma(\zeta+\nu+1)}\Big)\sum_{i=1}^{3}(k_i+\epsilon l_i)\leq\epsilon.$$

Thus, $\mathcal{V}\Omega_1+\mathcal{U}\Omega_2\in B_\epsilon$, for each $\Omega_1=(\Omega_1^1,\Omega_1^2,\Omega_1^3),\Omega_2=(\Omega_2^1,\Omega_2^2,\Omega_2^3)\in B_\epsilon$.

(II). By the definition of $\mathcal{U}$, it is clear that $\mathcal{U}$ is a contraction mapping on B_ϵ.

(III). It is easy to see that $\mathcal{V}$ is continuous. Now we only prove that $\mathcal{V}$ is a completely continuous operator on $\mathcal{X}$.

Firstly, for any $\Omega=(\Omega^1,\Omega^2,\Omega^3)\in B_\epsilon, t\in[0,1]$, by using (H_1), we obtain $\mathcal{V}$ is uniformly bounded on B_ϵ. Secondly, for $\Omega=(\Omega^1,\Omega^2,\Omega^3)\in B_\epsilon$ and $t_1,t_2\in[0,1]$ with $0\leq t_1<t_2\leq 1$, we obtain

$$|\mathcal{V}_1\Omega(t_2)-\mathcal{V}_1\Omega(t_1)|\leq$$
$$|t_2^\nu E_{\nu,\nu+1}(-\omega t_2^\nu)-t_1^\nu E_{\nu,\nu+1}(-\omega t_1^\nu)|\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}(|\mathfrak{f}_{1,\Omega}(\ell)|+|\mathfrak{f}_{2,\Omega}(\ell)|+|\mathfrak{f}_{3,\Omega}(\ell)|)d\ell+$$
$$\frac{1}{2\Gamma(\zeta)}\int_0^1(1-\ell)^{\zeta-1}(|\mathfrak{f}_{3,\Omega}(\ell)|+|\mathfrak{f}_{2,\Omega}(\ell)|+|\mathfrak{f}_{1,\Omega}(\ell)|)d\ell\Big)$$
$$+|t_2^{\nu+1}E_{\nu,\nu+2}(-\omega t_2^\nu)-t_1^{\nu+1}E_{\nu,\nu+2}(-\omega t_1^\nu)|\Big(\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}|\mathfrak{f}_{1,\Omega}(\ell)|d\ell\Big)$$
$$+|\int_0^{t_2}(t_2-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t_2-\ell)^\nu)\mathfrak{f}_{1,\Omega}(\ell)d\ell-\int_0^{t_1}(t_1-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu)\mathfrak{f}_{1,\Omega}(\ell)d\ell|$$
$$+|E_\nu(-\omega t_2^\nu)-E_\nu(-\omega t_1^\nu)|\Big(\frac{1}{2(c^3+1)}\frac{\Gamma(2-\nu)}{\Gamma(\zeta-\nu)}\int_0^1(1-\ell)^{\zeta-\nu-1}$$
$$\times\Big((ac^2+2bc^2+ac+a)|\mathfrak{f}_{1,\Omega}(\ell)|+(ac^2+ac+a+2b)|\mathfrak{f}_{2,\Omega}(\ell)|+(ac^2+ac+2bc+a)|\mathfrak{f}_{3,\Omega}(\ell)|\Big)d\ell\Big)$$
$$+|E_\nu(-\omega t_2^\nu)-E_\nu(-\omega t_1^\nu)|$$
$$\times\Big(\frac{1}{2\Gamma(\zeta)(c^3+1)}\int_0^1(1-\ell)^{\zeta-1}\Big(a(c^2+c+1)|\mathfrak{f}_{1,\Omega}(\ell)|+a(c^2+c+1)|\mathfrak{f}_{2,\Omega}(\ell)|+a(c^2+c+1)|\mathfrak{f}_{3,\Omega}(\ell)|\Big)d\ell\Big)$$
$$+|E_\nu(-\omega t_2^\nu)-E_\nu(-\omega t_1^\nu)|$$
$$\times\Big(\frac{1}{c^3+1}\int_0^1(1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(1-\ell)^\nu)(c^2|\mathfrak{f}_{1,\Omega}(\ell)|+|\mathfrak{f}_{2,\Omega}(\ell)|+c|\mathfrak{f}_{3,\Omega}(\ell))|d\ell\Big).$$

Here, by appling by applying (H_2), Lemma 2.11, Lemma 2.12, Lemma 2.13, Lemma 2.14, and from the fact that $(-1)^m\frac{d^m}{dx^m}(E_{v,w}(-x))\geq 0$ for all $x>0$ and for all $m=0,1,2,\ldots$, when $0<v\leq 1$ and $w\geq v$, we conclude

$$
\begin{aligned}
&|\mathcal{V}_1\Omega(t_2) - \mathcal{V}_1\Omega(t_1)| \le \frac{1}{\omega}|E_\nu(-\omega t_2^\nu) - E_\nu(-\omega t_1^\nu)|\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)} + \frac{1}{2\Gamma(\zeta+1)}\Big)(\sum_{i=1}^{3} k_i + l_i\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ \frac{1}{\omega}(|t_2 E_{\nu,2}(-\omega t_2^\nu) - t_1 E_{\nu,2}(-\omega t_1^\nu)| + |t_2 - t_1|)\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)} + \frac{1}{2\Gamma(\zeta+1)}\Big)(k_1 + l_1\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&\Big(\int_0^{t_1} \Big((t_2-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t_2-\ell)^\nu) - (t_1-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu)\Big)d\ell\\
&+ \int_{t_1}^{t_2} (t_2-\ell)^{\nu+\zeta-1}E_{\nu,\nu+\zeta}(-\omega(t_2-\ell)^\nu)d\ell\Big)(k_1 + l_1\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ |E_\nu(-\omega t_2^\nu) - E_\nu(-\omega t_1^\nu)|(\frac{1}{\Gamma(\zeta+\nu+1)})(\sum_{i=1}^{3} k_i + l_i\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ |E_\nu(-\omega t_2^\nu) - E_\nu(-\omega t_1^\nu)|(\frac{3}{2\Gamma(\nu+1)\Gamma(\zeta+1)})(\sum_{i=1}^{3} k_i + l_i\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ |E_\nu(-\omega t_2^\nu) - E_\nu(-\omega t_1^\nu)|(\frac{5\Gamma(2-\nu)}{2\Gamma(\nu+1)\Gamma(\zeta-\nu+1)})(\sum_{i=1}^{3} k_i + l_i\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&\le \frac{t_2^\nu - t_1^\nu}{\Gamma(\nu+1)}\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)} + \frac{1}{2\Gamma(\zeta+1)})\Big)(\sum_{i=1}^{3} k_i + l_i\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ \frac{2}{\omega}(t_2 - t_1)\Big(\frac{\Gamma(2-\nu)}{2\Gamma(\zeta-\nu+1)} + \frac{1}{2\Gamma(\zeta+1)})\Big)(k_1 + l_1\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ \Big(\int_0^{t_1} ((t_2-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu) - (t_1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu))d\ell\\
&+ \int_{t_1}^{t_2} (t_2-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu)d\ell\Big)(k_1 + l_1\|\Omega\|_{\mathcal{C}^3[0,1]})\\
&+ \frac{\omega}{\Gamma(\nu+1)}(t_2^\nu - t_1^\nu)\Big(\frac{1}{\Gamma(\zeta+\nu+1)} + \frac{3}{\Gamma(\zeta+1)\Gamma(\zeta+1)} + \frac{5\Gamma(2-\nu)}{\Gamma(\nu+1)\Gamma(\zeta-\nu+1)}\Big)(\sum_{i=1}^{3} k_i + l_i\|\Omega\|_{\mathcal{C}^3[0,1]}).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\Big(\int_0^{t_1} \big((t_2-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu) - (t_1-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu)\big)d\ell\\
&+ \int_{t_1}^{t_2} (t_2-\ell)^{\zeta+\nu-1}E_{\nu,\nu+\zeta}(-\omega(t_1-\ell)^\nu)d\ell\Big) \le \frac{t_2^{\zeta+\nu} - t_1^{\zeta+\nu}}{\Gamma(\zeta+\nu+1)}.
\end{aligned}
$$

Since $t^{\nu+\zeta}$, t^ν and t are uniformly continuous on $[0,1]$, we conclude

$$|\mathcal{V}_1\Omega(t_2) - \mathcal{V}_1\Omega(t_1)| \to 0, \text{ as } t_2 \to t_1 \text{ independent of } \Omega.$$

In a similar manner, for each $t_1, t_2 \in [0,1]$ with $0 \le t_1 < t_2 \le 1$ we have we conclude

$$|\mathcal{V}_i\Omega(t_2) - \mathcal{V}_i\Omega(t_1)| \to 0, \text{ as } t_2 \to t_1 \text{ independent of } \Omega, i = 2, 3.$$

Hence, the operator $\mathcal{V}$ is equicontinuous on B_ϵ, and so, by using the Arzela-Ascoli theorem, we conclude that $\mathcal{V}$ is a compact operator, which implies that $\mathcal{V}$ is a completely continuous operator on $\mathcal{X}$.

Therefore, all the assumptions of Theorem 2.15 are satisfied, and then by Theorem 2.15 we deduce that there exists a fixed point of operator $\mathcal{V}+\mathcal{U}$, which is a solution of the boundary value problem (1.4) on $[0,1]$.

$\square$

4. Examples

To illustrate our results, let us consider the following simple examples.

Example 4.1. *Consider the system*

$$\begin{cases} \mathfrak{D}^{1.8}\left(\mathfrak{D}^{0.9}+30\right)\Omega_1(t)=\Psi_1(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0\le t\le 1, \\ \mathfrak{D}^{1.8}\left(\mathfrak{D}^{0.9}+30\right)\Omega_2(t)=\Psi_2(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0\le t\le 1, \\ \mathfrak{D}^{1.8}\left(\mathfrak{D}^{0.9}+30\right)\Omega_3(t)=\Psi_3(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0\le t\le 1, \\ \Omega_1(0)+\Omega_2(1)=0, \mathcal{D}^{0.9}\Omega_1(0)+\mathcal{D}^{0.9}\Omega_2(1)=0, \mathcal{D}^{2(0.9)}\Omega_1(0)+\mathcal{D}^{2(0.9)}\Omega_2(1)=0, \\ \Omega_2(0)+\Omega_3(1)=0, \mathcal{D}^{0.9}\Omega_2(0)+\mathcal{D}^{0.9}\Omega_3(1)=0, \mathcal{D}^{2(0.9)}\Omega_2(0)+\mathcal{D}^{2(0.9)}\Omega_3(1)=0, \\ \Omega_3(0)+\Omega_1(1)=0, \mathcal{D}^{0.9}\Omega_3(0)+\mathcal{D}^{0.9}\Omega_1(1)=0, \mathcal{D}^{2(0.9)}\Omega_3(0)+\mathcal{D}^{2(0.9)}\Omega_1(1)=0, \end{cases}$$

where

$$\Psi_1(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)) = \frac{|\Omega_1(t)|}{10\pi(1+|\Omega_1(t)|)}+\frac{1}{10\pi}\sin(|\Omega_2(t)|)+\frac{1+t}{(20\pi+t^2)}\frac{\Omega_3(t)}{1+|\Omega_3(t)|},$$

$$\Psi_2(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)) = \frac{\arctan(|\Omega_1(t)|)}{80}+\frac{|\Omega_2(t)|}{80(1+|\Omega_2(t)|)}+\frac{2|\Omega_3(t)|}{5(2+t)^5(1+|\Omega_3(t)|)},$$

$$\Psi_3(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)) = \frac{\arctan(|\Omega_1(t)|)}{30+t^2}+\frac{(t+1)\Omega_2(t)}{15(2+t)^2}+\frac{\Omega_3(t)}{30}.$$

For every $\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{w}_1,\hat{w}_2,\hat{w}_3\in\mathbb{R}$, it is clear that

$$|\Psi_1(t,\hat{z}_1,\hat{z}_2,\hat{z}_3)-\Psi_1(t,\hat{w}_1,\hat{w}_2,\hat{w}_3)|\le\frac{1}{10\pi}|\hat{z}_1-\hat{w}_1|+\frac{1}{10\pi}|\hat{z}_2-\hat{w}_2|+\frac{1}{10\pi}|\hat{z}_3-\hat{w}_3|,$$

$$|\Psi_2(t,\hat{z}_1,\hat{z}_2,\hat{z}_3)-\Psi_2(t,\hat{w}_1,\hat{w}_2,\hat{w}_3)|\le\frac{1}{80}|\hat{z}_1-\hat{w}_1|+\frac{1}{80}|\hat{z}_2-\hat{w}_2|+\frac{1}{80}|\hat{z}_3-\hat{w}_3|,$$

and

$$|\Psi_3(t,\hat{z}_1,\hat{z}_2,\hat{z}_3)-\Psi_3(t,\hat{w}_1,\hat{w}_2,\hat{w}_3)|\le\frac{1}{30}|\hat{z}_1-\hat{w}_1|+\frac{1}{30}|\hat{z}_2-\hat{w}_2|+\frac{1}{30}|\hat{z}_3-\hat{w}_3|.$$

We take

$$\mu_1=\frac{1}{10\pi},\mu_2=\frac{1}{80},\mu_3=\frac{1}{30}.$$

By simple computation on the given data, we find that

$$\xi \sum_{i=1}^{3} \mu_i = 0.4766490186 < 1,$$

where ξ *is defined in* (3.20). *Hence, Theorem 3.2 implies that the problem has a unique solution defined in* $[0,1]$.

Example 4.2. *Consider the system*

$$\begin{cases} \mathfrak{D}^{1.9}\left(\mathfrak{D}^{0.9}+60\right)\Omega_1(t) = \Psi_1(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0 \le t \le 1, \\ \mathfrak{D}^{1.9}\left(\mathfrak{D}^{0.9}+60\right)\Omega_2(t) = \Psi_2(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0 \le t \le 1, \\ \mathfrak{D}^{1.9}\left(\mathfrak{D}^{0.9}+60\right)\Omega_3(t) = \Psi_3(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)), \quad 0 \le t \le 1, \\ \Omega_1(0)+\Omega_2(1)=0, \mathcal{D}^{0.9}\Omega_1(0)+\mathcal{D}^{0.9}\Omega_2(1)=0, \mathcal{D}^{2(0.9)}\Omega_1(0)+\mathcal{D}^{2(0.9)}\Omega_2(1)=0, \\ \Omega_2(0)+\Omega_3(1)=0, \mathcal{D}^{0.9}\Omega_2(0)+\mathcal{D}^{0.9}\Omega_3(1)=0, \mathcal{D}^{2(0.9)}\Omega_2(0)+\mathcal{D}^{2(0.9)}\Omega_3(1)=0, \\ \Omega_3(0)+\Omega_1(1)=0, \mathcal{D}^{0.9}\Omega_3(0)+\mathcal{D}^{0.9}\Omega_1(1)=0, \mathcal{D}^{2(0.9)}\Omega_3(0)+\mathcal{D}^{2(0.9)}\Omega_1(1)=0, \end{cases}$$

where

$$\Psi_1(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)) = e^{-t} + \frac{|\Omega_1(t)|}{2\sqrt{t^2+25\pi^2}(1+|\Omega_1(t)|)} + \frac{1}{5(\pi+\pi e^t)}\sin(|\Omega_2(t)|) + \frac{1+t}{24\pi+t^2}\frac{\Omega_3(t)}{1+|\Omega_3(t)|},$$

$$\Psi_2(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)) = 1 + \frac{2\arctan(|\Omega_1(t)|)}{4\pi\sqrt{5e^t+20}} + \frac{|\Omega_2(t)|}{30(1+|\Omega_2(t)|)} + \frac{2|\Omega_3(t)|}{5\pi(2+t)^2(1+|\Omega_3(t)|)},$$

$$\Psi_3(t,\Omega_1(t),\Omega_2(t),\Omega_3(t)) = 3\ln(2+t) + \frac{\arctan(|\Omega_1(t)|)}{10\pi+t^2} + \frac{\Omega_2(t)}{10\pi(1+t)^3} + \frac{\Omega_3(t)}{e^t+62}.$$

If

$$\begin{aligned} &k_1(t) = e^{-t}, k_2(t) = 1, k_3(t) = 3\ln(2+t), \\ &p_1(t) = \frac{1}{2\sqrt{t^2+25\pi^2}}, p_2(t) = \frac{2}{4\pi\sqrt{5e^t+20}}, p_3(t) = \frac{1}{10\pi+t^2}, \\ &q_1(t) = \frac{1}{5(\pi+\pi e^t)}, q_2(t) = \frac{1}{30}, q_3(t) = \frac{1}{10\pi(1+t)^3}, \\ &r_1(t) = \frac{1+t}{24\pi+t^2}, r_2(t) = \frac{2}{5\pi(2+t)^2}, r_3(t) = \frac{1}{e^t+62}, \end{aligned}$$

then we conclude that

$$\begin{aligned} &p_1 = \frac{1}{10\pi}, p_2 = \frac{1}{10\pi}, p_3 = \frac{1}{10\pi}, \\ &q_1 = \frac{1}{10\pi}, q_2 = \frac{1}{30}, q_3 = \frac{1}{10\pi}, \\ &r_1 = \frac{1}{12\pi}, r_2 = \frac{1}{10\pi}, r_3 = \frac{1}{62}, \\ &l_1 = 0.02833, l_2 = 0.0533, l_3 = 0.0358. \end{aligned}$$

Hence, we obtain

$$\xi \sum_{i=1}^{3} l_i = 0.6655239192 < 1,$$

where ξ is defined in (3.20)*. Therefore, by Theorem 3.3, we conclude that the BVP has at least one solution on [0,1].*

5. Conclusion

The current study introduces a new tripled system of fractional Langevin differential equations that incorporate cyclic antiperiodic boundary conditions and Mittag-Leffler functions. In order to guarantee the existence and uniqueness of solutions, the Krasnoselskii fixed point theorem, the Banach contraction mapping theorem, and specific properties of the Mittag-Leffler functions are implemented.

The primary problem differs from previous research in the literature in that it entails distinct boundary conditions, which facilitates the application of generalizations to the problem. Additionally, the main findings are established without the need for of the friction coefficient ω, which necessitates an alternative methodology. This has been proven by generating two examples with large values of ω, indicating that the primary conclusions of this work are distinct.

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References

[1] Ahmad, M.; Zada, A.; Alzabut, J. Stability analysis of a nonlinear coupled implicit switched singular fractional differential equations with p-Laplacian, *Adv. Differ. Equ.* 2019, 436 (2019).

[2] Ahmad, M.; Zada, A.; Alzabut, J. Hyers-Ulam stability of a coupled system of fractional differential equations of Hilfer-Hadamard type, *Demonstr. Math.* **52** (2019), 283-295.

[3] Abbas, S.; Benchohra, M.; Eddine Lazreg, J.; Nieto, J.J.; Zhou, Y. Fractional Differential Equations and Inclusions, *Classical and Advanced Topics. World Scientific*, 2023.

[4] Agarwal, R. P.; Benchohra, M.; Hamani, S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109** (2010), 973–1033.

[5] Ahmadova, A.; Mahmudov, N. Langevin differntial equations with general fractional orders and their applications to electric circuit theory, *J. Comput. Appl. Math.* **388** (2021) 2021:113299.

[6] Baitiche. Z.; Derbazi, C.; Benchohra, M.; Nieto, J.J. Monotone Iterative Technique for a New Class of Nonlinear Sequential Fractional Differential Equations with Nonlinear Boundary Conditions under the Ψ-Caputo Operator, *Mathematics,* **10** 2022, 1173.

[7] Baghani, H. An analytical improvement of a study of nonlinear Langevin equation involving two fractional orders in different intervals, *J. Fixed Point Theory Appl.* 21:95 (2019).

[8] Baghani, H. Existence and uniqueness of solutions to fractional Langevin equations involving two fractional orders, *J. Fixed Point Theory Appl.* 20:63 (2018).

[9] Baghani, H.; Alzabut, J.; Nieto, J.J. A coupled system of Langevin differential equations of fractional order and associated to antiperiodic boundary conditions, *Math. Meth. Appl. Sci.* **43** (2020), 675–681.

[10] Baghani, H.; Nieto, J.J. Applications of the Mittag-Leffler Function in Solvability and Stability of a Class of Fractional Langevin Equations with Two Fractional Orders, *J. Anal.* **32** (2024), 915-929.

[11] Baghani, H.; Nieto, J.J. Some new properties of the Mittag-Leffler functions and their applications to solvability and stability of a class of fractional Langevin differential equations, *Qual. Theor. Dyn. Syst.* 23:18 (2024).

[12] Baghani, H.; Salem, A. Solvability and stability of a class of fractional Langevin differential equations using the Laplace transform method, Submitted.

[13] Diethelm, K.; Ford, N.J. Analysis of fractional differential equations, *J. Math. Anal. Appl.*, *265* (2002), 229–248.

[14] Fazli, H.; Nieto, J.J. Fractional Langevin equation with anti-periodic boundary conditions, *Chaos, Solit. Fract. 114* (2018), 332-337.

[15] Granas. A.; Dugundji, J. *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[16] Hilfer, R. *Applications of Fractional Calculus in Physics*, Singapore, World Scientific, 2000.

[17] Kiyamehr, Z.; Baghani, H. Existence of solutions of BVPs for fractional Langevin equations involving Caputo fractional derivatives, *J. Appl. Anal. 27* (2021), 47-55.

[18] Kilbas, A.A.; Srivastava, H. M.; Trujillo, J. J. *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[19] Krim, S.; Salim, A.; Benchohra, M. On implicit Caputo tempered fractional boundary value problems with delay. *Lett. Nonlinear Anal. Appl.* **1** (2023), 12-29.

[20] Lakshmikantham, V.; Leela, S.; Vasundhara Devi, J. *Theory of Fractional Dynamic Systems,* Cambridge Academic, Cambridge, UK, 2009.

[21] Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley and Sons, New York, 1993.

[22] Podlubny, I. *Fractional Differential Equations*, New York, Academic Press, 1999.

[23] Salim, A.; Alzabut, J.; Sudsutad, W.; Thaiprayoon, C. On impulsive implicit ψ-Caputo hybrid fractional differential equations with retardation and anticipation. *Mathematics.* **10** (2022).

[24] Salim, A.; Benchohra, M. Existence and Uniqueness Results for Generalized Caputo Iterative Fractional Boundary Value Problems. *Fract. Differ. Calc.* **12** (2022), 197-208.

[25] Zada, A.; Alzabut, J.; Waheed, I.; Popa, I.L. Ulam-Hyers stability of impulsive integrodifferential equations with Riemann-Liouville boundary conditions, *Adv. Differ. Equ.* **64** (2020).

[26] Zhang, W.; Ni, J. Qualitative analysis of tripled system of fractional Langevin equations with cyclic anti-periodic boundary conditions, *Fract. Calc. Appl. Anal.*, **26** (2023), 2392–2420.

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