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Stability of generalized P-harmonic maps

AHMED MOHAMMED CHERIF[1]

ABSTRACT. In this paper, we prove that any stable $P(x)$-harmonic map ψ from $\mathbb{S}^2$ to N is a holomorphic or anti-holomorphic map, where N is a Kählerian manifold with non-positive holomorphic bisectional curvature and $P(x) \geq 2$ is a smooth function on the sphere $\mathbb{S}^2$ satisfying some condition. We study the existence of stable $P(x)$-harmonic map ψ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold N, and the stability of $P(x)$-harmonic identity. We also study the case of a product $\mathbb{S}^{n_1} \times ... \times \mathbb{S}^{n_k}$.

1. INTRODUCTION

Let $\psi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, $\tau(\psi)$ the tension field of ψ (see [1, 6]), and let $P : M \to [2, \infty)$, $x \longmapsto P(x)$ be a smooth function. The $P(x)$-tension field of ψ is defined by

$$\tau_{P(x)}(\psi) = |d\psi|^{P(x)-2}\tau(\psi) + d\psi(\operatorname{grad} |d\psi|^{P(x)-2}), \tag{1.1}$$

where $|d\psi|$ is the Hilbert-Schmidt norm of the differential $d\psi$, and grad denotes the gradient operator with respect to g. The map ψ is called $P(x)$-harmonic if the $P(x)$-tension field vanishes, that is $\tau_{P(x)}(\psi) = 0$. It is the Euler-Lagrange equation of the $P(x)$-energy functional (see [10])

$$E_P(\psi; D) = \int_D \frac{|d\psi|^{P(x)}}{P(x)} dx. \tag{1.2}$$

$P(x)$-harmonic maps is a natural generalization of harmonic map (see [1, 6]) and p-harmonic map (see [2, 3, 7]).
We define the index form for $P(x)$-harmonic maps by (see [10])

$$I(v, w) = \int_M h(J^{\psi}_{P(x)}(v), w) dx, \tag{1.3}$$

for all $v, w \in \Gamma(\psi^{-1}TN)$ where $J^{\psi}_{P(x)}$ is the generalized Jacobi operator of ψ defined by

$$\begin{aligned} J^{\psi}_{P(x)}(v) \quad = \quad & -|d\psi|^{P(x)-2} \operatorname{trace}_g R^N(v, d\psi)d\psi - \operatorname{trace}_g \nabla^{\psi} |d\psi|^{P(x)-2} \nabla^{\psi} v \\ & - \operatorname{trace}_g \nabla(P(x) - 2)|d\psi|^{P(x)-4} \langle \nabla^{\psi} v, d\psi \rangle d\psi, \end{aligned} \tag{1.4}$$

where $\langle \, , \rangle$ denote the inner product on $T^*M \otimes \psi^{-1}TN$, and R^N is the curvature tensor of (N, h) defined by

$$R^N(U, V)W = \nabla^N_U \nabla^N_V W - \nabla^N_V \nabla^N_U W - \nabla^N_{[U,V]} W, \tag{1.5}$$

for all $U, V, W \in \Gamma(TN)$, ∇^N is the Levi-Civita connection of (N, h), ∇^{ψ} denote the pull-back connection on $\psi^{-1}TN$, and dx is the volume form of (M, g) (see [1]). Let ψ be a $P(x)$-harmonic map such that for any vector field v along ψ the index form satisfies $I(v, v) \geq 0$,

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Corresponding author: Ahmed Mohammed Cherif; a.mohammedcherif@univ-mascara.dz

thus ψ is called a stable $P(x)$-harmonic map. Note that, if $P(x) \geq 2$, then every $P(x)$-harmonic map from a compact Riemannian manifold (M, g) without boundary to a Riemannian manifold (N, h) has $\mathrm{Sect}^N \leq 0$ is stable (see [10]).

Let M be a differentiable manifold. An almost Hermitian structure on M is by definition a pair (J, g) of an almost complex structure J and a Riemannian metric g satisfying

$$J^2 X = -X, \quad g(JX, JY) = g(X, Y) \tag{1.6}$$

for all $X, Y \in \Gamma(TM)$. A manifold with such a structure (J, g) is called an almost Hermitian manifold. An almost Hermitian manifold (M, J, g) is Kählerian if and only if its almost complex structure J is parallel with respect to the Levi-Civita connection (see [17]), that is

$$\nabla_X JY = J\nabla_X Y, \quad X, Y \in \Gamma(TM). \tag{1.7}$$

Let X, Y be two unit vectors at a point in M. The holomorphic bisectional curvature is defined by

$$\mathrm{BHR}(X, Y) = g(R^M(X, JX)Y, JY). \tag{1.8}$$

Let $\psi : (M, J, g) \to (N, J', h)$ be a smooth map between two almost Hermitian manifolds. The map ψ is called $\pm$holomorphic (holomorphic or anti-holomorphic) if $d\psi \circ J = \pm J' \circ d\psi$ (see [17]).

The paper extends the results from stable harmonic map to stable $P(x)$-harmonic map. It gives some properties of stable $P(x)$-harmonic map from $\mathbb{S}^n$ into a Riemannian manifold. The stability of the $P(x)$-harmonic identity map of a compact Riemannian manifold without boundary, and of $P(x)$-harmonic maps from a compact Riemannian manifold without boundary into $\mathbb{S}^{n_1} \times ... \times \mathbb{S}^{n_k}$ is studied. The paper extends some results proved by Y. T. Siu and S. T. Yau in [15], and by L. F. Cheung, P. F. Leung in [4]. The search results provide additional information on harmonic maps, and the stability of Riemannian manifolds (see [9]).

2. Main results

2.1. **Holomorphicity of $P(x)$-harmonic maps.** In the theorem below, we examine the conditions that determine whether a $P(x)$-harmonic map from the standard sphere $\mathbb{S}^2$ into a special Kähler manifold is a holomorphic or anti-holomorphic map (see [11, 15]).

Theorem 2.1. *Let (N, J', h) be a Kähler manifold with non-positive holomorphic bisectional curvature, and let $P(x) \geq 2$ be a smooth function on the sphere $\mathbb{S}^2$. Then, any stable $P(x)$-harmonic map $\psi : \mathbb{S}^2 \to N$ satisfying the inequality*

$$\frac{1}{2}\int_{\mathbb{S}^2} |d\psi|^{P(x)-2} \Delta |d\psi|^2 dx + 2\int_{\mathbb{S}^2} (P(x) - 2)|d\psi|^{P(x)-2} |\nabla d\psi|^2 dx \leq 0, \tag{2.9}$$

is a holomorphic or anti-holomorphic map. Moreover, ψ is harmonic map and $\operatorname{grad} |d\psi|^{P(x)-2} \in \ker d\psi$.

Proof. Note that, the unit sphere $\mathbb{S}^2$ admits a complex structure, and every Riemannian metric on an oriented 2-dimensional manifold is a Kähler metric with respect to the naturally induced complex structure.

Take $f(x) = |d_x\psi|^{P(x)-2}$ for all $x \in \mathbb{S}^2$. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^2$. Let $v \in \Gamma(T\mathbb{S}^2)$. We compute

$$\sum_{i=1}^{2} \nabla^\psi_{e_i} f \nabla^\psi_{e_i} J' d\psi(v) = J' \nabla^\psi_{\operatorname{grad} f} d\psi(v) + f \sum_{i=1}^{2} J' \nabla^\psi_{e_i} \nabla^\psi_{e_i} d\psi(v). \tag{2.10}$$

By using the property $\nabla^{\psi}_X d\psi(Y) = \nabla^{\psi}_Y d\psi(X) + d\psi([X,Y])$ and the definition of the curvature tensor, the equation (2.10) becomes

$$\begin{aligned}
\sum_{i=1}^{2}\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i} J'd\psi(v) &= J'\nabla^{\psi}_v d\psi(\operatorname{grad} f) + J'd\psi([\operatorname{grad} f, v]) \\
&\quad + \sum_{i=1}^{2}\Big[fJ'\nabla^{\psi}_{e_i}\nabla^{\psi}_v d\psi(e_i) + fJ'\nabla^{\psi}_{e_i} d\psi([e_i,v])\Big] \\
&= J'\nabla^{\psi}_v d\psi(\operatorname{grad} f) + J'd\psi(\nabla_{\operatorname{grad} f} v) - J'd\psi(\nabla_v \operatorname{grad} f) \\
&\quad + \sum_{i=1}^{2}\Big[fJ'R^N(d\psi(e_i), d\psi(v))d\psi(e_i) + fJ'\nabla^{\psi}_v\nabla^{\psi}_{e_i} d\psi(e_i) \\
&\quad + 2fJ'\nabla^{\psi}_{\nabla_{e_i} v} d\psi(e_i) + fJ'd\psi([e_i,[e_i,v]])\Big]. \qquad (2.11)
\end{aligned}$$

By the antisymmetry of the curvature tensor and the definition of tension field of ψ, we obtain the following

$$\begin{aligned}
\sum_{i=1}^{2}\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i} J'd\psi(v) &= J'\nabla^{\psi}_v d\psi(\operatorname{grad} f) + J'd\psi(\nabla_{\operatorname{grad} f} v) - J'd\psi(\nabla_v \operatorname{grad} f) \\
&\quad - \sum_{i=1}^{2} fJ'R^N(d\psi(v), d\psi(e_i))d\psi(e_i) + fJ'\nabla^{\psi}_v \tau(\psi) \\
&\quad + \sum_{i=1}^{2}\Big[fJ'\nabla^{\psi}_v d\psi(\nabla_{e_i} e_i) + fJ'\nabla^{\psi}_{\nabla_{e_i} v} d\psi(e_i) \\
&\quad + fJ'd\psi(\nabla_{e_i}\nabla_{e_i} v) - fJ'd\psi(\nabla_{e_i}\nabla_v e_i)\Big] \\
&= J'\nabla^{\psi}_v d\psi(\operatorname{grad} f) + J'd\psi(\nabla_{\operatorname{grad} f} v) - J'd\psi(\nabla_v \operatorname{grad} f) \\
&\quad - \sum_{i=1}^{2} fJ'R^N(d\psi(v), d\psi(e_i))d\psi(e_i) + J'\nabla^{\psi}_v f\tau(\psi) \\
&\quad - v(f)J'\tau(\psi) + \sum_{i=1}^{2}\Big[fJ'd\psi(\nabla_v\nabla_{e_i} e_i) + fJ'\nabla^{\psi}_{\nabla_{e_i} v} d\psi(e_i) \\
&\quad + fJ'd\psi(\nabla_{e_i}\nabla_{e_i} v) - fJ'd\psi(\nabla_{e_i}\nabla_v e_i)\Big]. \qquad (2.12)
\end{aligned}$$

By using the $P(x)$-harmonicity of ψ and the definition of the Ricci tensor $\operatorname{Ricci} v = \sum_{i=1}^{2} R(v,e_i)e_i$, we have from equation (2.12) the following

$$\begin{aligned}
\sum_{i=1}^{2}\nabla^{\psi}_{e_i} f\nabla^{\psi}_{e_i} J'd\psi(v) &= J'd\psi(\nabla_{\operatorname{grad} f} v) - J'd\psi(\nabla_v \operatorname{grad} f) \\
&\quad - \sum_{i=1}^{2} fJ'R^N(d\psi(v), d\psi(e_i))d\psi(e_i) - v(f)J'\tau(\psi) \\
&\quad + fJ'd\psi(\operatorname{Ricci} v) + fJ'd\psi(\operatorname{trace}\nabla^2 v). \qquad (2.13)
\end{aligned}$$

From equation (2.13), we conclude that

$$
\begin{aligned}
-h(\operatorname{trace}\nabla^{\psi} f\nabla^{\psi} J'd\psi(v), J'd\psi(v)) &= -h(d\psi(\nabla_{\operatorname{grad} f} v), d\psi(v)) \\
&\quad +h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\
&\quad +\sum_{i=1}^{2} f h(R^N(d\psi(v), d\psi(e_i))d\psi(e_i), d\psi(v)) \\
&\quad +v(f)h(\tau(\psi), d\psi(v)) \\
&\quad -f h(d\psi(\operatorname{Ricci} v), d\psi(v)) \\
&\quad -f h(d\psi(\operatorname{trace}\nabla^2 v), d\psi(v)).
\end{aligned}
\tag{2.14}
$$

By the definition of generalized Jacobi operator (1.4) and equation (2.14), we have the following

$$
\begin{aligned}
h(J^{\psi}_P(J'd\psi(v)), J'd\psi(v)) &= -\sum_{i=1}^{2} f h(R^N(J'd\psi(v), d\psi(e_i))d\psi(e_i), J'd\psi(v)) \\
&\quad -h(d\psi(\nabla_{\operatorname{grad} f} v), d\psi(v)) \\
&\quad +h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\
&\quad +\sum_{i=1}^{2} f h(R^N(d\psi(v), d\psi(e_i))d\psi(e_i), d\psi(v)) \\
&\quad +v(f)h(\tau(\psi), d\psi(v)) - f h(d\psi(\operatorname{Ricci} v), d\psi(v)) \\
&\quad -f h(d\psi(\operatorname{trace}\nabla^2 v), d\psi(v)) - \operatorname{div}\theta \\
&\quad +(P(x)-2)|d\psi|^{P(x)-4}\langle\nabla^{\psi} J'd\psi(v), d\psi\rangle^2,
\end{aligned}
\tag{2.15}
$$

where $\theta(X) = (P(x)-2)|d\psi|^{P(x)-4}\langle\nabla^{\psi} J'd\psi(v), d\psi\rangle h(d\psi(X), J'd\psi(v))$.
Let $\lambda(x) = \langle\alpha, x\rangle$ for all $x \in \mathbb{S}^2$ where $\alpha \in \mathbb{R}^3$, and let $v = \operatorname{grad}\lambda$. It is easy to show that $v = \sum_{i=1}^{2}\langle\alpha, e_i\rangle e_i$, $\nabla_X v = -\lambda X$ for all $X \in \Gamma(T\mathbb{S}^2)$, and $\operatorname{trace}\nabla^2 v = -v$ (see [17]).
From equation (2.15) with $\operatorname{Ricci} v = v$, we have at point x_0

$$
\begin{aligned}
h(J^{\psi}_P(J'd\psi(v)), J'd\psi(v)) &= h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\
&\quad +\lambda h(d\psi(\operatorname{grad} f), d\psi(v)) \\
&\quad +\sum_{i=1}^{2} f h(R^N(d\psi(v), d\psi(e_i))d\psi(e_i), d\psi(v)) \\
&\quad -\sum_{i=1}^{2} f h(R^N(J'd\psi(v), d\psi(e_i))d\psi(e_i), J'd\psi(v)) \\
&\quad +v(f)h(\tau(\psi), d\psi(v)) - \operatorname{div}\theta \\
&\quad +(P(x)-2)|d\psi|^{P(x)-4}\langle\nabla^{\psi} J'd\psi(v), d\psi\rangle^2.
\end{aligned}
\tag{2.16}
$$

Let $e_1 = e$ and $e_2 = Je$. From the $P(x)$-harmonicity of ψ and equation (2.16), we obtain

$$\begin{aligned}\operatorname{trace}_\alpha h(J_P^\psi(J'd\psi(v)), J'd\psi(v)) &= \operatorname{trace}_\alpha h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ &+ fh(R^N(d\psi(Je), d\psi(e))d\psi(e), d\psi(Je)) \\ &+ fh(R^N(d\psi(e), d\psi(Je))d\psi(Je), d\psi(e)) \\ &- fh(R^N(J'd\psi(e), d\psi(e))d\psi(e), J'd\psi(e)) \\ &- fh(R^N(J'd\psi(Je), d\psi(e))d\psi(e), J'd\psi(Je)) \\ &- fh(R^N(J'd\psi(e), d\psi(Je))d\psi(Je), J'd\psi(e)) \\ &- fh(R^N(J'd\psi(Je), d\psi(Je))d\psi(Je), J'd\psi(Je)) \\ &- fh(\tau(\psi), \tau(\psi)) - \operatorname{trace}_\alpha \operatorname{div} \theta \\ &+ (P(x)-2)|d\psi|^{P(x)-4} \operatorname{trace}_\alpha \langle \nabla^\psi J'd\psi(v), d\psi\rangle^2.\end{aligned}$$

We set $K(U, V) = h(R^N(U, V)U, V)$ for $U, V \in \Gamma(TN)$. The last equation is equivalent to the following

$$\begin{aligned}\operatorname{trace}_\alpha h(J_P^\psi(J'd\psi(v)), J'd\psi(v)) &= \operatorname{trace}_\alpha h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ &- 2fK(d\psi(e), d\psi(Je)) + 2fK(d\psi(e), J'd\psi(Je)) \\ &+ fK(d\psi(e), J'd\psi(e)) + fK(d\psi(Je), J'd\psi(Je)) \\ &- fh(\tau(\psi), \tau(\psi)) - \operatorname{trace}_\alpha \operatorname{div} \theta \\ &+ (P(x)-2)|d\psi|^{P(x)-4} \operatorname{trace}_\alpha \langle \nabla^\psi J'd\psi(v), d\psi\rangle^2. \qquad (2.17)\end{aligned}$$

Take $\omega = d\psi(Je) - J'd\psi(e)$ and $\eta = d\psi(Je) + J'd\psi(e)$. We have

$$\begin{aligned}h(R^N(\omega, J'\omega)\eta, J'\eta) &= -2K(d\psi(e), d\psi(Je)) + 2K(d\psi(e), J'd\psi(Je)) \\ &+ K(d\psi(e), J'd\psi(e)) + K(d\psi(Je), J'd\psi(Je)). \qquad (2.18)\end{aligned}$$

Substituting the formula (2.18) in (2.17), we get

$$\begin{aligned}\operatorname{trace}_\alpha h(J_P^\psi(J'd\psi(v)), J'd\psi(v)) &= \operatorname{trace}_\alpha h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) - \operatorname{trace}_\alpha \operatorname{div} \theta \\ &+ fh(R^N(\omega, J'\omega)\eta, J'\eta) - fh(\tau(\psi), \tau(\psi)) \\ &+ (P(x)-2)|d\psi|^{P(x)-4} \operatorname{trace}_\alpha \langle \nabla^\psi J'd\psi(v), d\psi\rangle^2. \qquad (2.19)\end{aligned}$$

The first term of (2.19) is given by (see [5, 14])

$$\begin{aligned}\operatorname{trace}_\alpha h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) &= \sum_{i=1}^2 h(d\psi(\nabla_{e_i} \operatorname{grad} f), d\psi(e_i)) \\ &= \sum_{i=1}^2 h(\nabla_{e_i}^\psi d\psi(\operatorname{grad} f), d\psi(e_i)) \\ &- \frac{1}{2}\sum_{i=1}^2 e_i(f) e_i(|d\psi|^2). \qquad (2.20)\end{aligned}$$

Let $\alpha(X) = h(d\psi(\operatorname{grad} f), d\psi(X))$ and $\beta(X) = fX(|d\psi|^2)$. From equation (2.20) and the $P(x)$-harmonicity condition, we obtain

$$\begin{aligned}\operatorname{trace}_\alpha h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) &= \operatorname{div} \alpha + fh(\tau(\psi), \tau(\psi)) \\ &- \frac{1}{2}\operatorname{div} \beta + \frac{1}{2} f\Delta |d\psi|^2. \qquad (2.21)\end{aligned}$$

By using the definition of $\nabla d\psi$ and the property $\nabla_X v = -\lambda X$, we find that

$$\langle \nabla^{\psi} J' d\psi(v), d\psi\rangle = -\sum_{i=1}^{2} h((\nabla d\psi)(v, e_i), J' d\psi(e_i)). \tag{2.22}$$

According to Cauchy-Schwarz inequality and equation (2.22) we get

$$\begin{aligned}
\operatorname{trace}_\alpha \langle \nabla^{\psi} J' d\psi(v), d\psi\rangle^2 &\leq 2\sum_{i,j=1}^{2} h((\nabla d\psi)(e_j, e_i), J' d\psi(e_i))^2 \\
&\leq 2\sum_{i,j=1}^{2} |(\nabla d\psi)(e_j, e_i)|^2 |d\psi(e_i)|^2 \\
&\leq 2|d\psi|^2 \sum_{i,j=1}^{2} |(\nabla d\psi)(e_i, e_j)|^2 \\
&= 2|d\psi|^2 |\nabla d\psi|^2.
\end{aligned} \tag{2.23}$$

From (2.19), (2.21), (2.23), the stability condition of ψ, the Green Theorem, and the assumption (2.9), we conclude that

$$\begin{aligned}
\int_{\mathbb{S}^2} f h(R^N(\omega, J'\omega)\eta, J'\eta)dx &\geq \operatorname{trace}_\alpha I(J' d\psi(v), J' d\psi(v)) \\
&\quad -\frac{1}{2}\int_{\mathbb{S}^2} f \Delta |d\psi|^2 dx \\
&\quad -2\int_{\mathbb{S}^2} (P(x) - 2) f |\nabla d\psi|^2 dx \geq 0.
\end{aligned} \tag{2.24}$$

But by the condition of non-positive holomorphic bisectional curvature

$$\int_{\mathbb{S}^2} f h(R^N(\omega, J'\omega)\eta, J'\eta)dx \leq 0.$$

Therefore, $\omega = 0$ or $\eta = 0$. So that the map ψ is $\pm$holomorphic. Consequently ψ is harmonic (see [17]). From the $P(x)$-harmonicity condition of ψ, we obtain $\operatorname{grad} f \in \operatorname{Ker} d\psi$. □

If the function $P(x) = 2$ on M, we deduce the following Corollary.

Corollary 2.1. [17] *Let N be a Kähler manifold with non-positive holomorphic bisectional curvature. Then any stable harmonic map $\psi : \mathbb{S}^2 \to N$ is a holomorphic or anti-holomorphic map.*

2.2. **Stable $P(x)$-harmonic maps from $\mathbb{S}^n$.** Y. L. Xin proved in [16] that any stable harmonic map from $\mathbb{S}^n$ $(n > 2)$ into any Riemannian manifold must be a constant map. This result is a specific case of a more general Theorem.

Theorem 2.2. *Any stable $P(x)$-harmonic map ψ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant, where $2 \leq P(x) < n$ is a smooth function on $\mathbb{S}^n$ satisfying*

$$\langle \operatorname{grad} |d\psi|^2, \operatorname{grad} P\rangle = 0,$$

on $\mathbb{S}^n$.

Proof. Let $\{e_i\}$ be a normal orthonormal frame at x_0 in $\mathbb{S}^n$, $\lambda(x) =< \alpha, x >_{\mathbb{R}^{n+1}}$ for all $x \in \mathbb{S}^n$ where $\alpha \in \mathbb{R}^{n+1}$, and let $v = \operatorname{grad} \lambda$. Note that

$$v = \sum_{i=1}^{n} < \alpha, e_i > e_i, \quad \nabla_X v = -\lambda X,$$

for all $X \in \Gamma(T\mathbb{S}^n)$, and $\operatorname{trace} \nabla^2 v = -v$, where ∇ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric of the sphere. Take $f = |d\psi|^{P-2}$. We compute

$$\sum_{i=1}^{n} \nabla^{\psi}_{e_i} f \nabla^{\psi}_{e_i} d\psi(v) = \nabla^{\psi}_{\operatorname{grad} f} d\psi(v) + \sum_{i=1}^{n} f \nabla^{\psi}_{e_i} \nabla^{\psi}_{e_i} d\psi(v). \tag{2.25}$$

By using the properties of ∇^{ψ}, the first term of (2.25) is given by

$$\begin{aligned} \nabla^{\psi}_{\operatorname{grad} f} d\psi(v) &= \nabla^{\psi}_{v} d\psi(\operatorname{grad} f) + d\psi([\operatorname{grad} f, v]) \\ &= \nabla^{\psi}_{v} d\psi(\operatorname{grad} f) - \lambda d\psi(\operatorname{grad} f) - d\psi(\nabla_v \operatorname{grad} f), \end{aligned} \tag{2.26}$$

and the second term of (2.25) is given by

$$\begin{aligned} \sum_{i=1}^{n} f \nabla^{\psi}_{e_i} \nabla^{\psi}_{e_i} d\psi(v) &= \sum_{i=1}^{n} \Big[f \nabla^{\psi}_{e_i} \nabla^{\psi}_{v} d\psi(e_i) + f \nabla^{\psi}_{e_i} d\psi([e_i, v]) \Big] \\ &= \sum_{i=1}^{n} \Big[f R^N(d\psi(e_i), d\psi(v)) d\psi(e_i) + f \nabla^{\psi}_{v} \nabla^{\psi}_{e_i} d\psi(e_i) \\ &\quad + 2 f \nabla^{\psi}_{[e_i, v]} d\psi(e_i) + f d\psi([e_i, [e_i, v]]) \Big] \\ &= -\sum_{i=1}^{n} f R^N(d\psi(v), d\psi(e_i)) d\psi(e_i) + f \nabla^{\psi}_{v} \tau(\psi) - 2\lambda f \tau(\psi) \\ &\quad + \sum_{i=1}^{n} \Big[f \nabla^{\psi}_{v} d\psi(\nabla_{e_i} e_i) + f d\psi(\nabla_{e_i} \nabla_{e_i} v) - f d\psi(\nabla_{e_i} \nabla_v e_i) \Big] \\ &= -\sum_{i=1}^{n} f R^N(d\psi(v), d\psi(e_i)) d\psi(e_i) + \nabla^{\psi}_{v} f \tau(\psi) - v(f) \tau(\psi) \\ &\quad -2\lambda f \tau(\psi) + \sum_{i=1}^{n} \Big[f d\psi(\nabla_v \nabla_{e_i} e_i) + f d\psi(\nabla_{e_i} \nabla_{e_i} v) \\ &\quad - f d\psi(\nabla_{e_i} \nabla_v e_i) \Big]. \end{aligned} \tag{2.27}$$

Substituting the equations (2.26) and (2.27) in (2.25), with the $P(x)$-harmonicity of ψ, we have the following

$$\begin{aligned} \sum_{i=1}^{n} \nabla^{\psi}_{e_i} f \nabla^{\psi}_{e_i} d\psi(v) &= -\lambda d\psi(\operatorname{grad} f) - d\psi(\nabla_v \operatorname{grad} f) \\ &\quad - \sum_{i=1}^{n} f R^N(d\psi(v), d\psi(e_i)) d\psi(e_i) - v(f) \tau(\psi) \\ &\quad -2\lambda f \tau(\psi) + \sum_{i=1}^{n} \Big[f d\psi(\nabla_v \nabla_{e_i} e_i) \\ &\quad + f d\psi(\nabla_{e_i} \nabla_{e_i} v) - f d\psi(\nabla_{e_i} \nabla_v e_i) \Big]. \end{aligned} \tag{2.28}$$

From equations (1.4) and (2.28), we have

$$\begin{aligned} J_P^{\psi}(d\psi(v)) &= \lambda d\psi(\operatorname{grad} f) + d\psi(\nabla_v \operatorname{grad} f) + v(f)\tau(\psi) + 2\lambda f \tau(\psi) \\ &\quad - \sum_{i=1}^{n} \Big[f d\psi(\nabla_v \nabla_{e_i} e_i) + f d\psi(\nabla_{e_i}\nabla_{e_i} v) - f d\psi(\nabla_{e_i}\nabla_v e_i) \\ &\quad + \nabla^{\psi}_{e_i}(P(x)-2)|d\psi|^{P(x)-4}\langle \nabla^{\psi} d\psi(v), d\psi\rangle d\psi(e_i)\Big], \end{aligned} \tag{2.29}$$

it is equivalent to the following equation

$$\begin{aligned} J_P^{\psi}(d\psi(v)) &= \lambda d\psi(\operatorname{grad} f) + d\psi(\nabla_v \operatorname{grad} f) + v(f)\tau(\psi) + 2\lambda f \tau(\psi) \\ &\quad - f d\psi(\operatorname{Ricci} v) - f d\psi(\operatorname{trace} \nabla^2 v) \\ &\quad - \sum_{i=1}^{n} \nabla^{\psi}_{e_i}(P(x)-2)|d\psi|^{P(x)-4}\langle \nabla^{\psi} d\psi(v), d\psi\rangle d\psi(e_i). \end{aligned} \tag{2.30}$$

A direct calculation shows that

$$\langle \nabla^{\psi} d\psi(v), d\psi\rangle = \frac{1}{2} v\left(|d\psi|^2\right) - \lambda |d\psi|^2. \tag{2.31}$$

Since $\operatorname{Ricci} v = (n-1)v$ and $\operatorname{trace} \nabla^2 v = -v$, from (2.30) we conclude that

$$\begin{aligned} h(J_P^{\psi}(d\psi(v)), d\psi(v)) &= \lambda h(d\psi(\operatorname{grad} f), d\psi(v)) + h(d\psi(\nabla_v \operatorname{grad} f), d\psi(v)) \\ &\quad + v(f) h(\tau(\psi), d\psi(v)) + 2\lambda f h(\tau(\psi), d\psi(v)) \\ &\quad - (n-2) f h(d\psi(v), d\psi(v)) - \operatorname{div}\theta \\ &\quad + (P(x)-2)|d\psi|^{P(x)-4}\langle \nabla^{\psi} d\psi(v), d\psi\rangle^2, \end{aligned} \tag{2.32}$$

where $\theta(X) = (P(x)-2)|d\psi|^{P(x)-4}\langle \nabla^{\psi} d\psi(v), d\psi\rangle h(d\psi(X), d\psi(v))$. By using the $P(x)$-harmonicity of ψ and equations (2.31), (2.32), we find that

$$\begin{aligned} \operatorname{trace}_{\alpha} h(J^{\psi}_{P(x)}(d\psi(v)), d\psi(v)) &= \langle d\psi(\nabla \operatorname{grad} f), d\psi\rangle \\ &\quad - f|\tau(\psi)|^2 - (n-2)|d\psi|^{P(x)} - \operatorname{trace}_{\alpha} \operatorname{div}\theta \\ &\quad + \frac{P(x)-2}{4}|d\psi|^{P(x)-4}|\operatorname{grad}|d\psi|^2|^2 \\ &\quad + (P(x)-2)|d\psi|^{P(x)}. \end{aligned} \tag{2.33}$$

By using the following formula (see [5, 14])

$$\big\langle \nabla^{\psi} d\psi(\operatorname{grad} f), d\psi\big\rangle = \frac{1}{2}\big(\operatorname{grad} f\big)\big(|d\psi|^2\big) + \big\langle d\psi(\nabla \operatorname{grad} f), d\psi\big\rangle,$$

the $P(x)$-harmonicity of ψ, and the assumption $\langle \operatorname{grad}|d\psi|^2, \operatorname{grad} P\rangle = 0$ we obtain

$$\begin{aligned} \int_{\mathbb{S}^n} \langle d\psi(\nabla \operatorname{grad} f), d\psi\rangle dx &= \int_{\mathbb{S}^n} f|\tau(\psi)|^2 dx \\ &\quad - \frac{1}{4}\int_{\mathbb{S}^n} (P(x)-2)|d\psi|^{P(x)-4}|\operatorname{grad}|d\psi|^2|^2 dx. \end{aligned} \tag{2.34}$$

From the stable $P(x)$-harmonic condition, the Green Theorem, and equations (2.33), (2.34), we get the following inequality

$$\int_{\mathbb{S}^n} (P(x)-n)|d\psi|^{P(x)} dx \geq 0. \tag{2.35}$$

Consequently ψ is constant because $2 \leq P(x) < n$ for all $x \in \mathbb{S}^n$. □

Corollary 2.2. [17] *Any stable harmonic map ψ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant.*

B. Merdji and A. Mohammed Cherif proved the following result in [10] for the case where the codomain of the stable $P(x)$-harmonic map is the standard sphere $\mathbb{S}^n$.

Theorem 2.3. *Let (M, g) be a compact Riemannian manifold without boundary. When $n > 2$, any stable $P(x)$-harmonic map $\varphi : (M, g) \to \mathbb{S}^n$ must be constant, where $P(x)$ is a smooth function on M such that $2 \leq P(x) < n$.*

2.3. **Stability of $P(x)$-harmonic identity map.** For the identity map of a compact Riemannian manifold without boundary, we have the following Theorem.

Theorem 2.4. *Let (M, g) be a compact Riemannian manifold without boundary of dimension n. If one of the following conditions holds*

(1) $n = 1$ and $P(x) \geq 2$ for all $x \in M$.
(2) $n = 2$ and $P(x) = C^{st} \geq 2$ for all $x \in M$.

Then, the identity map $Id : (M, g) \to (M, g)$ is stable $P(x)$-harmonic.

Proof. Let $\{e_i\}$ be an orthonormal frame on (M, g) such that $\nabla_{e_j} e_i = 0$ at $x_0 \in M$ for all $i, j = 1, ..n$. Let $v \in \Gamma(TM)$, we have at x_0

$$\begin{aligned} g(J^{Id}_{P(x)}(v), v) &= \sum_{i=1}^{n} \Big[- n^{\frac{P(x)-2}{2}} g(R(v, e_i)e_i, v) - g(\nabla_{e_i} n^{\frac{P(x)-2}{2}} \nabla_{e_i} v, v) \\ &\quad - g(\nabla_{e_i} (P(x) - 2) n^{\frac{P(x)-4}{2}} (\operatorname{div} v) e_i, v) \Big]. \end{aligned} \tag{2.36}$$

Let $\eta_1, \eta_2 \in \Gamma(T^*M)$ defined by

$$\begin{aligned} \eta_1(X) &= n^{\frac{P(x)-2}{2}} g(\nabla_X v, v), \\ \eta_2(X) &= (P(x) - 2) n^{\frac{P(x)-4}{2}} (\operatorname{div} v) g(X, v). \end{aligned}$$

So that, the equation (2.36) becomes

$$\begin{aligned} g(J^{Id}_{P(x)}(v), v) &= -n^{\frac{P(x)-2}{2}} \operatorname{Ric}(v, v) - \operatorname{div} \eta_1 + n^{\frac{P(x)-2}{2}} |\nabla v|^2 \\ &\quad - \operatorname{div} \eta_2 + (P(x) - 2) n^{\frac{P(x)-4}{2}} (\operatorname{div} v)^2. \end{aligned} \tag{2.37}$$

As the manifold M is compact without boundary, by the Green Theorem we get the following inequality

$$\begin{aligned} I(v, v) &= \int_M \Big[- n^{\frac{P(x)-2}{2}} \operatorname{Ric}(v, v) dx + n^{\frac{P(x)-2}{2}} |\nabla v|^2 \\ &\quad + (P(x) - 2) n^{\frac{P(x)-4}{2}} (\operatorname{div} v)^2 \Big] dx \\ &\geq \int_M n^{\frac{P(x)-2}{2}} \Big[|\nabla v|^2 - \operatorname{Ric}(v, v) \Big] dx. \end{aligned} \tag{2.38}$$

According to the following Yano's formula (see [18])

$$\int_M \left[|\nabla v|^2 - \operatorname{Ric}(v, v) \right] dx = \int_M \Big[\frac{1}{2} |L_v g|^2 - (\operatorname{div} v)^2 \Big] dx,$$

where $L_v g$ is the Lie derivative of the metric g, with the following inequality

$$\begin{aligned}|L_v g|^2 &= \sum_{i,j=1}^n [(L_v g)(e_i, e_j)]^2 \\ &= \sum_{i,j=1}^n \big[g(\nabla_{e_i} v, e_j) + g(\nabla_{e_j} v, e_i)\big]^2 \\ &\geq 4\sum_{i=1}^n g(\nabla_{e_i} v, e_i)^2 \\ &\geq \frac{4}{n}\Big[\sum_{i=1}^n g(\nabla_{e_i} v, e_i)\Big]^2 \\ &\geq \frac{4}{n}(\operatorname{div} v)^2,\end{aligned}$$

we conclude that

$$\int_M \big[|\nabla v|^2 - \operatorname{Ric}(v,v)\big]\, dx \quad \geq \quad \frac{2-n}{n}\int_M (\operatorname{div} v)^2 dx. \tag{2.39}$$

If $n = 1$, from equations (2.38) and (2.39), we find that

$$I(v,v) \quad \geq \quad \frac{2-n}{n}\int_M (\operatorname{div} v)^2 dx. \tag{2.40}$$

Thus, Id is stable $P(x)$-harmonic map. If $n = 2$, and the function $P(x)$ is constant on M, we have $I(v,v) \geq 0$. Hence Id is also stable $P(x)$-harmonic map. □

Corollary 2.3. *For any smooth function $P(x) \geq 2$ on $\mathbb{S}^1$, the identity map of $\mathbb{S}^1$ is stable $P(x)$-harmonic.*

2.4. **Stability of $P(x)$-harmonic maps into $\mathbb{S}^{n_1} \times \ldots \times \mathbb{S}^{n_k}$.** Let (N,h) be a complete n-dimensional submanifold in the Euclidean space $\mathbb{R}^{n+r}$. Define the function h by

$$h(x) = \max\left\{|B(u,u)|^2, u \in T_x N,\ |u| = 1\right\},$$

for all $x \in N$ where B denote the second fundamental form of (N,h) in $\mathbb{R}^{n+r}$. We define also the function ϕ by

$$\phi(u) = \sum_{a=1}^n |B(u, v_a)|^2, \quad \forall u \in T_x N,$$

where $\{v_a\}$ is an orthonormal frame on (N,h). Note that, ϕ is independent of the choice of this orthonormal frame (see [8, 4]). Under the above notation, we obtain the following results.

Theorem 2.5. *We assume that for any unit vector $u \in T_x N$, we have*

$$(P(x) - 2)h(x) + \phi(u) - \operatorname{Ric}^N(u,u) < 0. \tag{2.41}$$

Then, any stable $P(x)$-harmonic map ψ from a compact Riemannian manifold (M,g) without boundary to (N,h) is constant where $P(x) \geq 2$ is a smooth function on M.

Proof. Consider a parallel vector field v in $\mathbb{R}^{n+r}$. We assume that ψ is not a constant map. From (1.3) and (1.4) the second variation corresponding to $v^\top$ is given by

$$\begin{aligned} I(v^\top, v^\top) &= -\int_M |d\psi|^{P(x)-2} \sum_{i=1}^m h(R^N(v^\top, d\psi(e_i))d\psi(e_i), v^\top)dx \\ &\quad + \int_M |d\psi|^{P(x)-2} \sum_{i=1}^m h(\nabla^\psi_{e_i} v^\top, \nabla^\psi_{e_i} v^\top)dx \\ &\quad + \int_M (P(x)-2)|d\psi|^{P(x)-4} \Big[\sum_{i=1}^m h(d\psi(e_i), \nabla^\psi_{e_i} v^\top)\Big]^2 dx. \end{aligned} \tag{2.42}$$

Let $i = 1, ..., m$. We compute the following term

$$\begin{aligned} \nabla^\psi_{e_i} v^\top &= \nabla^N_{d\psi(e_i)} v^\top \\ &= (\nabla^{\mathbb{R}^{n+r}}_{d\psi(e_i)} v^\top)^\top \\ &= -(\nabla^{\mathbb{R}^{n+r}}_{d\psi(e_i)} v^\perp)^\top \\ &= A_{v^\perp}(d\psi(e_i)). \end{aligned} \tag{2.43}$$

Now, we consider the quadratic form Q on $\mathbb{R}^{n+r}$ defined by

$$\begin{aligned} Q(v) &= -\int_M |d\psi|^{P(x)-2} \sum_{i=1}^m h(R^N(v^\top, d\psi(e_i))d\psi(e_i), v^\top)dx \\ &\quad + \int_M |d\psi|^{P(x)-2} \sum_{i=1}^m h(A_{v^\perp}(d\psi(e_i)), A_{v^\perp}(d\psi(e_i)))dx \\ &\quad + \int_M (P(x)-2)|d\psi|^{P(x)-4} \Big[\sum_{i=1}^m h(B(d\psi(e_i), d\psi(e_i)), v^\perp)\Big]^2 dx. \end{aligned} \tag{2.44}$$

We choose an orthonormal frame $\{v_a, v_b\}$, $a = 1, ..., n$, $b = n+1, ..., n+r$, such that the v_a are tangent to (N, h) and the v_b are normal to (N, h). The trace of Q is given by

$$\begin{aligned} \operatorname{trace} Q &= -\int_M |d\psi|^{P(x)-2} \sum_{i=1}^m \sum_{a=1}^n h(R^N(v_a, d\psi(e_i))d\psi(e_i), v_a)dx \\ &\quad + \int_M |d\psi|^{P(x)-2} \sum_{i=1}^m \sum_{a=1}^n |B(d\psi(e_i), v_a)|^2 dx \\ &\quad + \int_M (P(x)-2)|d\psi|^{P(x)-4} \Big|\sum_{i=1}^m B(d\psi(e_i), d\psi(e_i))\Big|^2 dx. \end{aligned} \tag{2.45}$$

By using the Schwarz inequality, we have

$$\begin{aligned} \Big|\sum_{i=1}^m B(d\psi(e_i), d\psi(e_i))\Big|^2 &= \sum_{i,j=1}^m \langle B(d\psi(e_i), d\psi(e_i)), B(d\psi(e_j), d\psi(e_j))\rangle_{\mathbb{R}^{n+r}} \\ &\leq \sum_{i,j=1}^m |B(d\psi(e_i), d\psi(e_i))||B(d\psi(e_j), d\psi(e_j))| \\ &\leq h(x) \sum_{i,j=1}^m |d\psi(e_i)|^2 |d\psi(e_j)|^2. \end{aligned} \tag{2.46}$$

We put $d\psi(e_i) = |d\psi(e_i)|u_i$ for $d\psi(e_i) \neq 0$ at x. From equation (2.45) and inequality (2.46), we find that

$$\begin{aligned}\operatorname{trace} Q &\leq -\int_M |d\psi|^{P(x)-2} \sum_{i=1}^m |d\psi(e_i)|^2 \operatorname{Ric}^N(u_i, u_i)dx \\ &+ \int_M |d\psi|^{P(x)-2} \sum_{i=1}^m |d\psi(e_i)|^2 \phi(u_i)dx \\ &+ \int_M (P(x)-2)|d\psi|^{P(x)-2} h(x) \sum_{i=1}^m |d\psi(e_i)|^2 dx. \qquad (2.47)\end{aligned}$$

By the assumption (2.41) and (2.47), we conclude that $\operatorname{trace} Q < 0$. Hence ψ is not stable. □

Corollary 2.4. *Let $\mathbb{S}^{n_1} \times ... \times \mathbb{S}^{n_k}$ be a product of k unit spheres. We assume that $2 \leq P(x) < \min\{n_1, ..., n_k\}$ for all x in a compact Riemannian manifold (M, g) without boundary. Then, any stable $P(x)$-harmonic map $\psi : (M, g) \to \mathbb{S}^{n_1} \times ... \times \mathbb{S}^{n_k}$ is constant.*

Proof. Note that $N = \mathbb{S}^{n_1} \times ... \times \mathbb{S}^{n_k} \subset \mathbb{R}^{n_1+...+n_k+k}$. Since the second fundamental form of $\mathbb{S}^{n_i}$ in $\mathbb{R}^{n_i+1}$ $(i = 1, ..., k)$ is given by

$$B_i(X, Y) = -\langle X, Y\rangle_{\mathbb{R}^{n_i+1}}\xi_i,$$

where ξ_i is the position vector field of $\mathbb{R}^{n_i+1}$. The function h is given by

$$\begin{aligned}h(x) &= \max\left\{\left|B_i(u_i, u_i)\right|^2, u_i \in T_{x_i}\mathbb{S}^{n_i}, |u_i| = 1\right\} \\ &= \max\left\{\left|\langle u_i, u_i\rangle_{\mathbb{R}^{n_i+1}}\xi_i\right|^2, u_i \in T_{x_i}\mathbb{S}^{n_i}, |u_i| = 1\right\}. \qquad (2.48)\end{aligned}$$

As $|\xi_i| = 1$ on $\mathbb{S}^{n_i}$ for all $i = 1, ..., k$, from (2.48) we get $h(x) = 1$ for all $x \in N$. Using the same method, we find that $\phi(u) = 1$ for any unit vector $u \in T_xN$. Since the Ricci curvature of $\mathbb{S}^{n_i}$ satisfies $\operatorname{Ric}^{\mathbb{S}^{n_i}}(u_i, u_i) = n_i - 1$ for any unit vector $u_i \in T_{x_i}\mathbb{S}^{n_i}$, we obtain for all $i = 1, ..., k$ the inequality

$$\operatorname{Ric}^N(u, u) \geq \min\{n_1, ..., n_k\} - 1.$$

The Corollary 2.4 follows from Theorem 2.5. □

Example 2.1. *Let $M = \mathbb{T}^2 \subset \mathbb{R}^3$ the torus of revolution equipped with the Riemannian metric $g = a^2dx^2 + (b + a\cos x)^2dy^2$, for $b > a > 0$. A straightforward calculation shows that for all $\alpha \in \mathbb{S}^3$, the map ψ from (M, g) to $\mathbb{S}^3 \times \mathbb{S}^1$ defined by $\psi(x, y) = (\alpha, x)$ is non-constant $P(x, y)$-harmonic, where*

$$P(x, y) = \frac{\ln\left((b + a\cos x)(1 + x^2)\right) + c}{\ln\left(\frac{a}{2}(1 + x^2)\right)}, \quad \forall(x, y) \in \mathbb{T}^2,$$

for some $c \in \mathbb{R}$. Here, $n_1 = 3$ and $n_2 = 1$, thus it is impossible to make $2 \leq P(x) < \min\{n_1, n_2\}$.

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[1]DEPARTMENT MATHEMATICS, UNIVERSITY OF MASCARA, MASCARA 29000, ALGERIA
Email address: a.mohammedcherif@univ-mascara.dz