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Stability of oscillatory solutions of impulsive differential equations with piecewise alternately advanced and retarded argument of generalized type

KUO-SHOU CHIU

ABSTRACT. In this study, we investigate scalar impulsive advanced and delayed differential equations with piecewise constant argument of generalized type, abbreviated as IDEPCAG, where the arguments are represented as general step functions. We propose criteria for the existence of oscillatory and non-oscillatory solutions, and derive sufficient conditions for the stability of the zero solution. Our results are novel, and extend and improve upon previous publications. Additionally, we provide several numerical examples and simulations to demonstrate the feasibility of our findings.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of natural, integer, and real numbers, respectively. Consider two real-valued sequences, t_k and γ_k, where $k \in \mathbb{Z}$, with the conditions that $t_k < t_{k+1}$, $t_k \leq \gamma_k \leq t_{k+1}$, and there exists a positive constant $\vartheta > 0$ such that $t_{k+1} - t_k \leq \vartheta$ for all $k \in \mathbb{Z}$, and $\lim_{k\to\pm\infty} t_k = \pm\infty$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a given general step function such that $\gamma(t) = \gamma_k$ for $t \in I_k = [t_k, t_{k+1})$, and $\mathbb{R} = \cup_{k\in\mathbb{Z}} I_k$.

Our investigation centers on examining the global asymptotic behavior and oscillation properties of solutions to a class of differential equations characterized by piecewise alternately advanced and retarded arguments of generalized type, which we refer to as the IDEPCAG:

$$y'(t) = a(t)y(t) + b(t)y(\gamma(t)), \quad y(\tau) = c_0, \quad t \neq t_k, \tag{1.1a}$$

$$\Delta y|_{t=t_k} = d_k x(t_k^-), \quad k \in \mathbb{Z}. \tag{1.1b}$$

Here, τ and c_0 belong to the set of real numbers, $a(t)$ is a continuous real-valued function on $\mathbb{R}$ with $a(t) \neq 0$, and $b(t)$ is also a real-valued continuous function on $\mathbb{R}$, $d_k \in \mathbb{R} \setminus \{-1\}$, $x(t_k^+) = \lim_{t\to t_k^+} x(t)$ and $x(t_k^-) = \lim_{t\to t_k^-} x(t)$.
The deviation argument $\ell(t) = t - \gamma(t)$ takes on negative values for $t_k < t < \gamma_k$ and positive values for $\gamma_k < t < t_{k+1}$, where $k \in \mathbb{Z}$. Consequently, the IDEPCAG (1.1a)-(1.1b) is of significant interest: it displays alternating characteristics on each interval $[t_k, t_{k+1})$, exhibiting an alternation between advanced and retarded behavior. More precisely, the IDEPCAG (1.1a)-(1.1b) assumes the advanced type on the interval $I_k^+ = [t_k, \gamma_k]$, and the retarded type on the interval $I_k^- = (\gamma_k, t_{k+1})$.

The initial subjects of investigation were differential equations with piecewise constant arguments (DEPCA), characterized by deviation arguments of a consistent sign, as explored in references such as [6, 24, 30, 32, 33, 34, 35]. These equations bear relevance to

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Corresponding author: Kuo-Shou Chiu; kschiu@umce.cl

impulse and loaded equations and exhibit properties common to specific models associated with vertically transmitted diseases, as discussed in [2].

The study of DEPCA of alternately of retarded and advanced type was initiated by A. R. Aftabizadeh and J. Wiener [1] in 1986. Subsequent advancements in this field were made by K. L. Cooke and J. Wiener [5] in 1987. Their observations unveiled that changes in the sign of the deviation argument not only introduced intriguing periodic properties but also complexities in the asymptotic and oscillatory behavior of solutions. Consequently, there arose a natural impetus to investigate oscillatory and stability properties within DEPCA, featuring a generalized deviation argument.

In dynamical models, deviation and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [19]. Many authors have established criteria for the existence of oscillatory solutions in differential equations with deviation argument, as referenced by [4, 18, 17, 20, 21, 22, 27, 28, 29, 30, 31, 33, 34]. Therefore, there is a significant interest in ascertaining the additional conditions required to ensure the stability of these oscillatory solutions. It is noteworthy that such inquiries have been addressed in the context of differential equations with piecewise constant argument, as seen in [1, 5, 7, 8, 9, 10, 30, 33, 34]. As an illustrative instance, in the work of A. R. Aftabizadeh et al. presented in reference [1], the following result was established: Let $a, b \in \mathbb{R}$ and $b \neq 0$ such that

$$a > 0 \quad \text{and} \quad \frac{-a(e^a+1)}{(e^{a/2}-1)^2} < b < \frac{-a}{e^{a/2}-1}e^{a/2},$$

$$a < 0 \quad \text{and} \quad b < \frac{-a}{e^{a/2}-1}e^{a/2} \quad \text{or} \quad b > \frac{-a(e^a+1)}{(e^{a/2}-1)^2}.$$

Then every oscillatory solution x of the following differential equation with piecewise constant argument

$$x'(t) = ax(t) + bx([t+\tfrac{1}{2}]), \quad x(0) = x_0, \tag{1.2}$$

converges to zero as $t \to \infty$.

In 2011, K.-S. Chiu [7], the author considered the first-order linear DEPCA of generalized type, referred to as the DEPCAG:

$$y'(t) = a(t)y(t) + b(t)y(\gamma(t)), \quad y(\tau) = y_0, \tag{1.3}$$

and the author investigated the sufficient conditions for the existence and global asymptotic stability of oscillatory and non-oscillatory solutions for certain specific cases.

In 2013, K.-S. Chiu and M. Pinto [8] considered the first-order linear DEPCAG (1.3) and demonstrated that the deviation argument generates oscillatory and non-oscillatory properties, as well as the existence of a unique periodic solution under certain conditions.

In 2013, F. Karakoc et al. [21] investigated the first order non-linear impulsive differential equation with piecewise constant arguments (IDEPCA):

$$\begin{cases} x'(t) + a(t)x(t) + x([t-1])f(x([t])) = 0, \quad x(-1) = x_{-1}, \quad x(0) = x_0 \quad t \neq n, \\ \Delta x|_{t=n} = d_k x(n^-), \quad n \in \mathbb{N}. \end{cases}$$

Moreover, in addition to establishing the existence and uniqueness of a solution, the authors have derived sufficient conditions for the oscillation of the solution. It is essential to emphasize that further research on the oscillation and asymptotic behavior of the IDEPCA with classical piecewise constant arguments can be located in the subsequent papers: [3, 16, 20, 21, 22, 23, 25, 26]. These papers provide further insights into the mentioned topic.

To the best of our knowledge, numerous studies have been undertaken to investigate the existence and global asymptotic stability of periodic solutions associated with the IDEPCAG [11, 12, 13, 14, 15, 16]. However, to date, none of these studies have presented straightforward criteria for ascertaining the existence and stability of oscillatory and non-oscillatory solutions, particularly for the IDEPCAG of alternately retarded and advanced types.

The main objective of this paper is to extend the classical results of the DEPCA from [1] and [32], the results of DEPCAG from [9, 10], and the results of the IDEPCA from [3, 16] to the IDEPCAG of alternately retarded and advanced types, represented by equations (1.1a)-(1.1b). The primary goal of this paper is to furnish concise criteria for determining the existence and stability of both oscillatory and non-oscillatory solutions within this extended framework.

For the convenience of the reader, we have included certain definitions that will be of importance in subsequent sections.

Definition 1.1. *A function y is a solution of the IDEPCAG* (1.1a)-(1.1b) *on $\mathbb{R}$ if the following conditions hold:*

i) *$y:\mathbb{R}\to\mathbb{R}$ is continuous for $t\in\mathbb{R}$, except possibly at the points t_k, $k\in\mathbb{Z}$.*
ii) *$y(t)$ is right-continuous and has left-hand limits at the points t_k, $k\in\mathbb{Z}$.*
iii) *The derivative $y'(t)$ exists at each point $t\in\mathbb{R}$, except possibly at the points t_k, $k\in\mathbb{Z}$, where the one-sided derivatives exist.*
iv) *$y(t)$ satisfies* (1.1a), *except possibly at the points t_k, $k\in\mathbb{Z}$.*
v) *$y(t_k)$ satisfies* (1.1b) *for $k\in\mathbb{Z}$.*

Definition 1.2. *A function $y(t)$ defined on $[\tau,\infty)$ is said to be oscillatory if there exist two real valued sequences $(\nu_k)_{k\geq 0}$, $(\nu_k')_{k\geq 0}\subset[\tau,\infty)$ such that $\nu_k\to\infty$, $\nu_k'\to\infty$ as $k\to\infty$ and $y(\nu_k)\leq 0\leq y(\nu_k')$ for $k\geq N$, where N is sufficiently large.*

Definition 1.3. *A function $y(t)$ is said to be non-oscillatory if it is not oscillatory.*

Definition 1.4. *A solution $\{x_k\}_{k\geq i(\tau)}$ of the difference equation is called oscillatory if $x_k\cdot x_{k+1}\leq 0$. Otherwise, $\{x_k\}_{k\in\mathbb{Z}}$ is called non-oscillatory.*

Our paper is organized as follows: In the next section, we establish criteria for the existence of oscillatory and non-oscillatory solutions in scalar impulsive advanced and delayed differential equations with piecewise constant argument of generalized type. In Section 3, the stability of the solutions of linear differential equations is treated. Furthermore, the last section presents appropriate examples, and numerical simulations are provided to demonstrate the validity of our results.

2. Existence of the oscillatory and non-oscillatory solutions

In this section, we establish sufficient conditions for oscillatory and non-oscillatory solutions in scalar impulsive advanced and delayed differential equations with piecewise constant argument of generalized type.

The following assumption will be used throughout the paper:

($\mathcal{N}$) For every $t\in\mathbb{R}$, let $i=i(t)\in\mathbb{Z}$ be the unique integer such that $t\in I_i=[t_i,t_{i+1})$, $\lambda(\tau,\gamma_{i(\tau)})\neq 0$, $\lambda(t_i,\gamma_i)\neq 0$ for all $i\in\{i(\tau)+j\}_{j\in\mathbb{N}}$, where

$$\lambda(t,s):=e^{\int_s^t a(\kappa)d\kappa}+\int_s^t e^{\int_u^t a(\kappa)d\kappa}b(u)du.$$

Moreover, impulsive effects $d_i\neq -1$ for all $i\in\mathbb{Z}$.

In the following theorem, we establish the conditions for the existence and uniqueness of solutions for the IDEPCAG (1.1a)-(1.1b) on the interval $[\tau, \infty)$. The proof of this statement is analogous to that of Theorem 2.2 in [16].

Theorem 2.1. *Assuming that* ($\mathcal{N}$) *holds, the IDEPCAG* (1.1a)-(1.1b) *has a unique solution on* $[\tau, \infty)$ *with the initial condition* $y(\tau) = y_0$. *Furthermore, for* $t \in [t_k, t_{k+1})$, $k > i(\tau)$, y *takes the form:*

$$y(t) = \frac{\lambda(t, \gamma_k)}{\lambda(t_k, \gamma_k)} x_k \tag{2.1}$$

where $x_k = y(t_k)$, *and the sequence* $\{x_k\}_{k \geq i(\tau)}$ *is the unique solution of the difference equation:*

$$x_{k+1} = (1 + d_{k+1}) \frac{\lambda(t_{k+1}, \gamma_k)}{\lambda(t_k, \gamma_k)} x_k, \tag{2.2}$$

for $k > i(\tau)$ *with the initial condition* $x_{i(\tau)} = y_0$.

Proof. Let $y_k(t)$ denote a solution of the IDEPCAG (1.1a)-(1.1b) on the interval $t_k \leq t < t_{k+1}$. On this interval, we have

$$y_k'(t) = a(t)y(t) + b(t)y_k(\gamma_k).$$

The general solution of this IDEPCAG within the specified interval is

$$\begin{aligned} y_k(t) &= \left[e^{\int_{\gamma_k}^t a(u)du} + \int_{\gamma_k}^t e^{\int_s^t a(u)du} b(s)ds \right] y_n(\gamma_k) \\ &= \lambda(t, \gamma_k)\, y_k(\gamma_k). \end{aligned} \tag{2.3}$$

For $t = t_k$ in (2.3), and when $\lambda(t_k, \gamma(t_k)) \neq 0$, we have

$$y_k(\gamma_k) = \frac{y_k(t_k)}{\lambda(t_k, \gamma_k)}. \tag{2.4}$$

Hence, by substituting (2.4) into (2.3), we obtain

$$y_k(t) = \frac{\lambda(t, \gamma_k)}{\lambda(t_k, \gamma_k)} y_k(t_k). \tag{2.5}$$

Due to the impulse conditions (1.1b), as t approaches t_{k+1} in (2.5), we have

$$y_{k+1}(t_{k+1}) = (1 + d_{k+1})y(t_{k+1}^-) = (1 + d_{k+1}) \frac{\lambda(t_{k+1}, \gamma_k)}{\lambda(t_k, \gamma_k)} y_k(t_k), \quad \text{for } k \geq i(\tau).$$

From (2.5) and the impulse conditions (1.1b), we derive the difference equation (2.2). Given the initial condition $x_{i(\tau)} = y(\tau) = y_0$, the solution of (2.2) can be uniquely obtained. Consequently, the unique solution of the IDEPCAG (1.1a)-(1.1b) with the initial condition $y(\tau) = y_0$ is given by (2.1). □

Note that, in general, it can be observed through a recurrence relation that the unique solution of the IDEPCAG (1.1a)-(1.1b) on $t \in [\tau, \infty)$ is expressed as

(2.6)

$$y(t) = y(\tau) \left(\frac{\lambda(t, \gamma_{i(t)})}{\lambda(t_{i(t)}, \gamma_{i(t)})} \right) \left(\prod_{k=i(\tau)+1}^{i(t)-1} (1 + d_{k+1}) \frac{\lambda(t_{k+1}, \gamma_k)}{\lambda(t_k, \gamma_k)} \right) \left(\frac{\lambda(t_{i(\tau)+1}, \gamma_{i(\tau)})}{\lambda(\tau, \gamma_{i(\tau)})} \right).$$

The following results are specific cases derived from Theorem 2.1.

Corollary 2.1. *Let* $\hat{\lambda}(t) = e^{at} + \frac{b}{a}(e^{at} - 1)$, $\vartheta_k^+ = \gamma(t_k) - t_k$, $\vartheta_k^- = t_{k+1} - \gamma(t_k)$ *for all* $k \in \{i(\tau) + j\}_{j\in\mathbb{N}}$ *and assume that* $\hat{\lambda}\left(\tau - \gamma(t_{i(\tau)})\right) \neq 0$ *and* $\hat{\lambda}\left(-\vartheta_k^+\right) \neq 0$ *for all* $k \in \{i(\tau) + j\}_{j\in\mathbb{N}}$. *When* $a(t) = a \neq 0$ *and* $b(t) = b$ *are constants, the IDEPCAG* (1.1a)-(1.1b) *possesses a unique solution, denoted as* y*, which is expressed as follows:*

$$y(t) = \frac{\hat{\lambda}\left(t - \gamma_k\right)}{\hat{\lambda}(-\vartheta_k^+)} x_k, \quad t_k \leq t < t_{k+1}$$

here, $x_k = y(t_k)$*, and the sequence* $\{x_k\}_{k\geq i(\tau)}$ *satisfies the following difference equation:*

$$x_{k+1} = (1 + d_{k+1}) \frac{\hat{\lambda}(\vartheta_n^-)}{\hat{\lambda}(-\vartheta_n^+)} x_k,$$

for $k > i(\tau)$ *with the initial condition* $x_{i(\tau)} = y_0$.

Corollary 2.2. *Let* $\beta\left(t\right) := \int_{\gamma(t)}^{t} b(s)ds$, $\beta_k^- := \int_{\gamma(t_k)}^{t_{k+1}} b(s)ds$, $\beta\left(\tau\right) \neq -1$ *and* $\beta\left(t_k\right) \neq -1$ *for all* $k \in \{i(\tau) + j\}_{j\in\mathbb{N}}$. *Then,* $u'(t) = b(t)u(\gamma(t))$ *with the initial condition* $u(\tau) = y_0$ *has a unique solution* u*, given by:*

$$u(t) = \frac{1 + \beta\left(t\right)}{1 + \beta\left(t_k\right)} u_k, \quad t_k \leq t < t_{k+1},$$

where $u_k = u(t_k)$ *and the sequence* $\{u_k\}_{k\geq i(\tau)}$ *satisfies the difference equation*

$$u_{k+1} = (1 + d_{k+1}) \frac{1 + \beta_k^-}{1 + \beta\left(t_k\right)} u_k,$$

for $k > i(\tau)$ *with the initial condition* $u_{i(\tau)} = y_0$.

The following theorem outlines the requisite criteria for the existence of both oscillatory and non-oscillatory solutions within the context of IDEPCAG, as defined by equations (1.1a)-(1.1b).

Theorem 2.2. *Suppose that* ($\mathcal{N}$) *holds and let* $y : [\tau, \infty) \to \mathbb{R}$ *be a solution of the IDEPCAG* (1.1a)-(1.1b).

i) *If the solution* $\{x_k\}_{k\geq i(\tau)}$ *of the difference equation (2.2) is oscillatory, then the solution* $y(t)$ *of the IDEPCAG* (1.1a)-(1.1b) *is also oscillatory.*
ii) *If the solution* $\{x_k\}_{k\geq i(\tau)}$ *of the difference equation (2.2) is non-oscillatory, then* $y(t)$ *is non-oscillatory if and only if*

(2.7)
$$\int_t^{\gamma_k} b(s) e^{\int_s^{\gamma_k} a(\kappa)d\kappa} ds < 1$$

holds true for $t_k \leq t < t_{k+1}$, $k \geq N$*, where* N *is sufficiently large.*

Proof. $i)$ The proof of $i)$ is evident; therefore, we will focus solely on proving $ii)$.
$ii)$ Let us now designate x_k as a non-oscillatory solution to the difference equation (2.2). In accordance with this, we may posit that $x_k > 0$ holds true for $k \geq N$, where N is suitably large. If $y(t)$ is identified as a non-oscillatory solution, it follows that we may assert $y(t) > 0$ for $t \geq T$, where T is adequately large. Consequently, in accordance with equation (2.1), we obtain:

(2.8)
$$y(t) = \frac{\lambda\left(t, \gamma_k\right)}{\lambda\left(t_k, \gamma_k\right)} x_k,$$

for $k \geq n$ where $n = \max\{N, T\}$. As $y(t) > 0$, it follows that:

$$\frac{\lambda\left(t, \gamma_k\right)}{\lambda\left(t_k, \gamma_k\right)} > 0,$$

which subsequently implies (2.7).

Let us now make the assumption that (2.7) holds true. Our objective is to demonstrate that $y(t)$ is non-oscillatory. To begin, let us assume, for the sake of contradiction, that $y(t)$ is an oscillatory solution. Consequently, there must exist two sequences (ν_n) and (ν'_n) such that $\nu_n \to \infty$, $\nu'_n \to \infty$ as $n \to \infty$ and $y(\nu_n) \leq 0 \leq y(\nu'_n)$. Consider $t_n < \nu_n < t_{n+1}$. It is evident that $\nu_n \to \infty$ as $n \to \infty$. Thus, by virtue of equation (2.1), we derive:

$$y(\nu_n) = \frac{\lambda\left(\nu_n, \gamma_{i(\nu_n)}\right)}{\lambda\left(t_{i(\nu_n)}, \gamma_{i(\nu_n)}\right)} x_{i(\nu_n)}.$$

Since $y(\nu_n) \leq 0$ and $x_{i(\nu_n)} = y(t_{i(\nu_n)}) > 0$, it follows that $\frac{\lambda\left(\nu_n, \gamma_{i(\nu_n)}\right)}{\lambda\left(t_{i(\nu_n)}, \gamma_{i(\nu_n)}\right)} < 0$, which contradicts (2.7). Therefore, the proof remains consistent if $x_k < 0$ for $k \geq N$. Thus, the proof is completed. □

By employing a methodology akin to the one elucidated above, we are able to deduce the subsequent outcomes regarding the oscillatory behavior of the IDEPCAG (1.1a)-(1.1b).

Theorem 2.3. *Suppose that ($\mathcal{N}$) holds. If*

$$\limsup_{k\to\infty} \int_{t_k}^{\gamma_k} b(s) e^{\int_s^{\gamma_k} a(u)du} ds > 1, \tag{2.9}$$

holds true, then every solution of the IDEPCAG (1.1a)-(1.1b) *is oscillatory.*

Proof. Let y be a solution of the IDEPCAG (1.1a)-(1.1b) such that $y(t) > 0$ (or $y(t) < 0$) for $t > t_n$, where $n \in \mathbb{N}$ is sufficiently large. If $t \in I_k$, where $k > n$, then, in accordance with (2.3), we obtain:

$$y(t_k) = \left(e^{\int_{\gamma_k}^{t_k} a(u)du} + \int_{\gamma_k}^{t_k} e^{\int_s^{t_k} a(u)du} b(s)ds\right) y(\gamma_k) = \lambda(t_k, \gamma_k) y(\gamma_k).$$

Given that $y(\gamma_k)$ and $y(t_k) > 0$, it follows that:

$$0 < \lambda(t_k, \gamma_k) \quad \text{if and only if} \quad \int_{t_k}^{\gamma_k} e^{\int_s^{\gamma_k} a(u)du} b(s)ds < 1,$$

or

$$\limsup_{k\to\infty} \int_{t_k}^{\gamma_k} e^{\int_s^{\gamma_k} a(u)du} b(s)ds \leq 1,$$

which contradicts condition (2.9). Therefore, the IDEPCAG (1.1a)-(1.1b) possesses exclusively oscillatory solutions. □

Theorem 2.4. *Suppose that ($\mathcal{N}$) holds. Then we have:*

i) *if* $1 + d_{k+1} > \kappa_+ > 0$, $k \in \mathbb{Z}$, *and*

$$\liminf_{k\to\infty} \int_{\gamma_k}^{t_{k+1}} b(s) e^{\int_s^{\gamma_k} a(u)du} ds < -1 \tag{2.10}$$

holds true, then every solution of the IDEPCAG (1.1a)-(1.1b) *is oscillatory.*

ii) *If* $1+d_{k+1}<\kappa_-<0$, $k\in\mathbb{Z}$, *and*

$$\liminf_{k\to\infty}\int_{\gamma_k}^{t_{k+1}} b(s)e^{\int_s^{\gamma_k} a(u)du}ds > -1 \tag{2.11}$$

holds true, then every solution of the IDEPCAG (1.1a)-(1.1b) *is oscillatory.*

Proof. Let y be a solution of the IDEPCAG (1.1a)-(1.1b) such that $y(t)>0$ (or $y(t)<0$) for $t>t_n$, where $n\in\mathbb{N}$ is sufficiently large. If $t\in I_k$, where $k>n$, then, in accordance with (2.3) and (1.1b), we obtain:

$$\begin{aligned} y(t_{k+1}) &= (1+d_{k+1})\left(e^{\int_{\gamma_k}^{t_{k+1}} a(u)du}+\int_{\gamma_k}^{t_{k+1}} e^{\int_s^{t_{k+1}} a(u)du}b(s)ds\right)y(\gamma_k)\\ &= (1+d_{k+1})\lambda(t_{k+1},\gamma_k)y(\gamma_k). \end{aligned}$$

Since $1+d_{k+1}>\kappa_+>0$, $y(\gamma_k)$ and $y(t_{k+1})>0$, thus

$$0<\lambda(t_{k+1},\gamma_k) \quad \text{if and only if} \quad \int_{\gamma_k}^{t_{k+1}} e^{\int_s^{\gamma_k} a(u)du}b(s)ds > -1,$$

or

$$\liminf_{k\to\infty}\int_{\gamma_k}^{t_{k+1}} e^{\int_s^{\gamma_k} a(u)du}b(s)ds \geq -1,$$

which contradicts condition (2.10).
Similarly, with $t\to t_{k+1}$ in (2.3) and by (1.1b) we obtain, after some simplifications and considering that $1+d_{k+1}<\kappa_+<0$, for $k\in\mathbb{Z}$, $y(\gamma_k)>0$, and that $y(t_{k+1})>0$,

$$\int_{\gamma_k}^{t_{k+1}} e^{\int_s^{\gamma_k} a(u)du}b(s)ds < -1,$$

or

$$\liminf_{k\to\infty}\int_{\gamma_k}^{t_{k+1}} e^{\int_s^{\gamma_k} a(u)du}b(s)ds \leq -1,$$

which contradicts (2.11). Therefore, the IDEPCAG (1.1a)-(1.1b) possesses exclusively oscillatory solutions. □

It is noteworthy that either condition (2.9) or (2.10) represents the conventional prerequisites for validating the presence of oscillatory solutions in the context of the DEPCA and the IDEPA. For additional references, we can refer to [1], [5], [7], [8], and [16].

Similarly, as demonstrated in Theorem 2.3 and Theorem 2.4, we derive the following result.

Theorem 2.5. *Suppose that* ($\mathcal{N}$) *holds and*

$$\limsup_{k\to\infty}\int_{t_k}^{\gamma_k} b(s)e^{\int_s^{\gamma_k} a(u)du}ds < 1.$$

If either of the conditions

$$1+d_{k+1}>\kappa_+>0, \quad k\in\mathbb{Z}, \quad \liminf_{k\to\infty}\int_{\gamma_k}^{t_{k+1}} b(s)e^{\int_s^{\gamma_k} a(u)du}ds > -1,$$

or

$$1+d_{k+1}<\kappa_-<0, \quad k\in\mathbb{Z}, \quad \liminf_{n\to\infty}\int_{\gamma_k}^{t_{k+1}} b(s)e^{\int_s^{\gamma_k} a(u)du}ds < -1$$

holds true, then every solution of the IDEPCAG (1.1a)-(1.1b) *is non-oscillatory.*

Now, we shall establish certain oscillation and non-oscillation outcomes for the IDEPCAG with constant coefficients. These conclusions will be derived from the preceding findings. Let us consider the IDEPCAG (1.1a)-(1.1b) with constant coefficients:

$$y'(t) = ay(t) + by(\gamma(t)), \quad y(\tau) = c_0, \quad t \neq t_k, \tag{2.12a}$$

$$\Delta y|_{t=t_k} = d_k x(t_k^-), \quad k \in \mathbb{Z}, \tag{2.12b}$$

where a, b are real constants.

Similarly to Theorem 2.3, we provide the following result for the IDEPCAG (2.12a)-(2.12b) with constant coefficients.

Corollary 2.3. *If $a \neq 0$, each one of the conditions*

$$\begin{aligned} &i) \quad b > \lim_{k\to\infty} \sup \frac{a}{e^{a(\gamma_k - t_k)} - 1}, \\ &ii) \quad 1 + d_{k+1} > \kappa_+ > 0, \quad k \in \mathbb{Z}, \quad b < -\lim_{k\to\infty} \inf \frac{ae^{a(t_{k+1}-\gamma_k)}}{e^{a(t_{k+1}-\gamma_k)} - 1}, \\ &iii) \quad 1 + d_{k+1} < \kappa_- < 0, \quad k \in \mathbb{Z}, \quad b > -\lim_{k\to\infty} \inf \frac{ae^{a(t_{k+1}-\gamma_k)}}{e^{a(t_{k+1}-\gamma_k)} - 1}, \end{aligned} \tag{2.13}$$

implies that every solution of the IDEPCAG (2.12a)-(2.12b) *with constant coefficients is oscillatory.*

Corollary 2.3 extends the findings of Theorem 2.3 in Aftabizadeh and Wiener [1] when considering $\gamma(t) = [t + \frac{1}{2}]$ and Corollary 2.3 in Chiu [7] when considering piecewise alternately advanced and retarded argument of generalized type.

The following corollary illustrates that (2.13) represents the "best possible" condition.

Corollary 2.4. *If $a \neq 0$, each one of the conditions*

$$\begin{aligned} &i) \quad 1 + d_{k+1} > \kappa_+ > 0,\ k \in \mathbb{Z}, \\ &\qquad -\lim_{k\to\infty} \inf \frac{ae^{a(t_{k+1}-\gamma_k)}}{e^{a(t_{k+1}-\gamma_k)} - 1} < b < \lim_{k\to\infty} \sup \frac{a}{e^{a(\gamma_k - t_k)} - 1}, \\ &ii) \quad 1 + d_{k+1} > \kappa_- < 0,\ k \in \mathbb{Z}, \\ &\qquad -\lim_{k\to\infty} \inf \frac{ae^{a(t_{k+1}-\gamma_k)}}{e^{a(t_{k+1}-\gamma_k)} - 1} > b,\ b < \lim_{k\to\infty} \sup \frac{a}{e^{a(\gamma_k - t_k)} - 1}, \end{aligned} \tag{2.14}$$

implies that the IDEPCAG (2.12a)-(2.12b) *with constant coefficients has no oscillatory solution.*

Proof. Condition (2.14) implies that $(1+d_{k+1})\frac{\hat{\lambda}(t_{k+1}-\gamma_k)}{\hat{\lambda}(t_k-\gamma_k)} > 0$ for all $k \geq i(\tau)$. Consequently, based on (2.6), we infer that the solution $y(t)$ of the IDEPCAG (2.12a)-(2.12b) with constant coefficients maintains a consistent sign on the interval $[\tau, \infty)$. □

Corollary 2.4 expands upon the conclusions drawn in Theorem 2.4 by Aftabizadeh and Wiener [1] when $\gamma(t) = [t + \frac{1}{2}]$. It also extends the outcomes of Theorem 3.2 in [5] for cases where $\gamma(t) = m[\frac{t+k}{m}]$, subject to the condition that $0 < k < m$. Furthermore, it expands the scope of the results delineated in Corollary 2.4 in [7] and Remark 3.3 in [8], particularly in scenarios where $\gamma(t)$ assumes a piecewise alternately advanced and retarded argument of generalized type. Moreover, it enhances the insights presented in Remark 5.4 of [16], especially when addressing impulsive effects and considering $\gamma(t) = m[\frac{t+k}{m}]$, where $0 < k < m$.

3. GLOBAL ASYMPTOTIC STABILITY

In this section, we establish the sufficient conditions for globally asymptotically stable solutions in scalar impulsive advanced and delayed differential equations with piecewise constant arguments of a generalized type.

Theorem 3.1. *Let $b(t)$ be locally integrable on $[\tau,\infty)$. Then,*

i) *the zero solution of the IDEPCAG* (1.1a)-(1.1b) *is stable, if*

$$\left|(1+d_{k+1})\frac{\lambda(t_{k+1},\gamma_k)}{\lambda(t_k,\gamma_k)}\right|\leq 1 \tag{3.1}$$

for all $k>i(\tau)$.

ii) *the zero solution of the IDEPCAG* (1.1a)-(1.1b) *is globally asymptotically stable, if*

$$\left|(1+d_{k+1})\frac{\lambda(t_{k+1},\gamma_k)}{\lambda(t_k,\gamma_k)}\right|\leq \ell<1 \tag{3.2}$$

for all $k>i(\tau)$.

Proof. Given that $t\in[t_{i(t)},t_{i(t)+1})$ and the continuity of $\frac{\lambda(t,\gamma_{i(t)})}{\lambda(t_{i(t)},\gamma_{i(t)})}$, it is evident that the function $\frac{\lambda(t,\gamma_{i(t)})}{\lambda(t_{i(t)},\gamma_{i(t)})}$ is bounded for all t. The subsequent proof readily ensues from (2.6). □

If we exclude the consideration of impulsive effects, Theorem 3.1 can yield results akin to those presented in Theorem 3.1 in [7], and also extend the findings in Theorem 5.5 of [16] while taking into account $\gamma(t)=m[\frac{t+k}{m}]$, where $0<k<m$.

The next theorem provides adequate conditions for the global asymptotic stability of zero solution of the IDEPCAG (2.12a)-(2.12b) with constant coefficients. To prove the last theorem, we require the following notation:

$$\vartheta_k^+=\gamma_k-t_k,\quad \vartheta_k^-=t_{k+1}-\gamma_k,$$

$$\varphi_+(a):=e^{-a\vartheta_k^+}+(1+d_{k+1})e^{a\vartheta_k^-}\quad\text{and}\quad \varphi_-(a):=e^{-a\vartheta_k^+}-(1+d_{k+1})e^{a\vartheta_k^-},\quad k\in\mathbb{Z}.$$

Theorem 3.2. *Suppose that $a\neq 0$, b are real constants. Then,*

i) *the zero solution of the IDEPCAG* (1.1a)-(1.1b) *with constant coefficients is stable, if*

$$\left\{\left(\frac{a+b}{a}\right)\varphi_+(a)-\left(\frac{b}{a}\right)(2+d_{k+1})\right\}\left\{\left(\frac{a+b}{a}\right)\varphi_-(a)+\frac{b}{a}d_{k+1}\right\}\geq 0. \tag{3.3}$$

ii) *the zero solution of the IDEPCAG* (2.12a)-(2.12b) *with constant coefficients is globally asymptotically stable, if*

$$\left\{\left(\frac{a+b}{a}\right)\varphi_+(a)-\left(\frac{b}{a}\right)(2+d_{k+1})\right\}\left\{\left(\frac{a+b}{a}\right)\varphi_-(a)+\frac{b}{a}d_{k+1}\right\}> 0. \tag{3.4}$$

Proof. *i*) The condition (3.1) can be expressed as follows:

$$-1\leq(1+d_{k+1})\,\frac{\hat\lambda(t_{j+1}-\gamma_j)}{\hat\lambda(t_j-\gamma_j)}\leq 1.$$

If $\hat\lambda(t_j-\gamma_j)>0$, then

(3.5)

$$-\left(e^{a(t_j-\gamma_j)}+\frac{b}{a}\left(e^{a(t_j-\gamma_j)}-1\right)\right)-(1+d_{k+1})\left(e^{a(t_{j+1}-\gamma_j)}+\frac{b}{a}\left(e^{a(t_{j+1}-\gamma_j)}-1\right)\right)\leq 0$$

and

(3.6)

$$0 \le \left(e^{a(t_j-\gamma_j)} + \frac{b}{a}\left(e^{a(t_j-\gamma_j)} - 1\right)\right) - (1+d_{k+1})\left(e^{a(t_{j+1}-\gamma_j)} + \frac{b}{a}\left(e^{a(t_{j+1}-\gamma_j)} - 1\right)\right).$$

From (3.5), we have

$$0 \le \left(\frac{a+b}{a}\right)\left(e^{-a\vartheta_k^+} + (1+d_{k+1})\,e^{a\vartheta_k^-}\right) - \left(\frac{b}{a}\right)(2+d_{k+1}),$$

that is,

$$0 \le \left(\frac{a+b}{a}\right)\varphi_+(a) - \left(\frac{b}{a}\right)(2+d_{k+1}), \tag{3.7}$$

and by (3.6), we have

$$0 \le \left(\frac{a+b}{a}\right)\left(e^{-a\vartheta_k^+} - (1+d_{k+1})\,e^{a\vartheta_k^-}\right) + \frac{b}{a}d_{k+1},$$

which is equal to

$$0 \le \left(\frac{a+b}{a}\right)\varphi_-(a) + \frac{b}{a}d_{k+1}. \tag{3.8}$$

The inequalities (3.7) and (3.8) imply (3.3).
If $\hat{\lambda}(t_j - \gamma_j) < 0$, then

(3.9)

$$-\left(e^{a(t_j-\gamma_j)} + \frac{b}{a}\left(e^{a(t_j-\gamma_j)} - 1\right)\right) - (1+d_{k+1})\left(e^{a(t_{j+1}-\gamma_j)} + \frac{b}{a}\left(e^{a(t_{j+1}-\gamma_j)} - 1\right)\right) \ge 0$$

and

(3.10)

$$0 \ge \left(e^{a(t_j-\gamma_j)} + \frac{b}{a}\left(e^{a(t_j-\gamma_j)} - 1\right)\right) - (1+d_{k+1})\left(e^{a(t_{j+1}-\gamma_j)} + \frac{b}{a}\left(e^{a(t_{j+1}-\gamma_j)} - 1\right)\right).$$

From (3.9), it follows that

$$0 \ge \left(\frac{a+b}{a}\right)\left(e^{-a\vartheta_k^+} + (1+d_{k+1})\,e^{a\vartheta_k^-}\right) - \left(\frac{b}{a}\right)(2+d_{k+1}),$$

that is,

$$0 \ge \left(\frac{a+b}{a}\right)\varphi_+(a) - \left(\frac{b}{a}\right)(2+d_{k+1}), \tag{3.11}$$

and by (3.10), we have

$$0 \ge \left(\frac{a+b}{a}\right)\left(e^{-a\vartheta_k^+} - (1+d_{k+1})\,e^{a\vartheta_k^-}\right) + \frac{b}{a}d_{k+1},$$

which is equal to

$$0 \ge \left(\frac{a+b}{a}\right)\varphi_-(a) + \frac{b}{a}d_{k+1}. \tag{3.12}$$

The inequalities (3.11) and (3.12) imply (3.3). Then, by Theorem 3.1, we can conclude that the zero solution of the IDEPCAG (1.1a)-(1.1b) with constant coefficients is stable.
$ii)$ The proof follows a technique analogous to that in part $i)$. □

In consideration of Theorem 3.2 and Corollary 2.3, we can deduce that:

Corollary 3.1. *If the condition* (3.4) *is satisfied, and each one of the conditions*

$$\begin{aligned} &i) \quad b > \lim_{k\to\infty} \sup \frac{a}{e^{a\vartheta_k^+} - 1}, \\ &ii) \quad 1 + d_{k+1} > \kappa_+ > 0, \quad k \in \mathbb{Z}, \quad b < -\lim_{k\to\infty} \inf \frac{ae^{a\vartheta_k^-}}{e^{a\vartheta_k^-} - 1}, \\ &iii) \quad 1 + d_{k+1} < \kappa_- < 0, \quad k \in \mathbb{Z}, \quad b > -\lim_{k\to\infty} \inf \frac{ae^{a\vartheta_k^-}}{e^{a\vartheta_k^-} - 1}, \end{aligned} \tag{3.13}$$

holds, then every oscillatory solution of the IDEPCAG (2.12a)-(2.12b) *with constant coefficients is globally asymptotically stable.*

Remark 3.1. *Corollary 3.1 yields results akin to those presented in* [1, Theorem 2.9] *and in* [7, Corollary 3.1]*, with the exception of impulsive effects. These derivations underscore the generality of our findings and their supplementary pertinence to the previously established results.*

4. Illustrative examples

We will introduce appropriate examples in this section. These examples will show the usefulness of our theory.

Consider the following scalar impulsive differential equations with piecewise alternately advanced and retarded argument of generalized type.

Example 4.1. Let us consider the IDEPCAG

$$y'(t) = -(\ln 3)y(t) - 2.5y(\gamma(t)), \quad y(0) = 2, \quad t \neq t_k, \tag{4.1a}$$

$$\Delta y|_{t=t_k} = d_k x(t_k^-), \quad k \in \mathbb{N}, \tag{4.1b}$$

where $d_k = 0.5$, $t_k = \frac{3\pi}{8}(k-1)$, $\gamma_k = \frac{3\pi}{8}k - \frac{\pi}{4}$ for all $k \in \mathbb{N}$. The IDEPCAG (4.1a)-(4.1b) is a special case of the IDEPCAG (2.12a)-(2.12b) with the parameters $a = -\ln 3$ and $b = -2.5$. It is not difficult to see that $\vartheta_k^- = t_{k+1} - \gamma_k = \frac{\pi}{4}$, $\vartheta_k^+ = \gamma_k - t_k = \frac{\pi}{8}$, $\hat{\lambda}(-\vartheta_k^+) \approx 2.76701 \neq 0$, $\varphi_+(a) \approx 2.17238$ and $\varphi_-(a) \approx 0.90651$.

We calculate

$$1 + d_{k+1} = 1.5 > 0, \quad -2.5 = b < -\lim_{k\to\infty} \inf \frac{ae^{a\vartheta_k^-}}{e^{a\vartheta_k^-} - 1} \approx 0.80196,$$

and

$$\left\{ \left(\frac{a+b}{a}\right)\varphi_+(a) - \left(\frac{b}{a}\right)(2 + d_{k+1}) \right\} \left\{ \left(\frac{a+b}{a}\right)\varphi_-(a) + \frac{b}{a}d_{k+1} \right\} \approx 5.86038 > 0$$

for $k \in \mathbb{N}$. In this case, the second hypothesis (3.13) of Corollary 3.1 holds. Therefore, every solution of the IDEPCAG (4.1a)-(4.1b) is oscillatory. Furthermore, each solution of the IDEPCAG (4.1a)-(4.1b) goes to zero as $t \to \infty$ by oscillating.

Figures 1 and 2 illustrate the simulation results, showcasing the global asymptotic stability of the oscillatory solution for the IDEPCAG (4.1a)-(4.1b) with and without impulses.

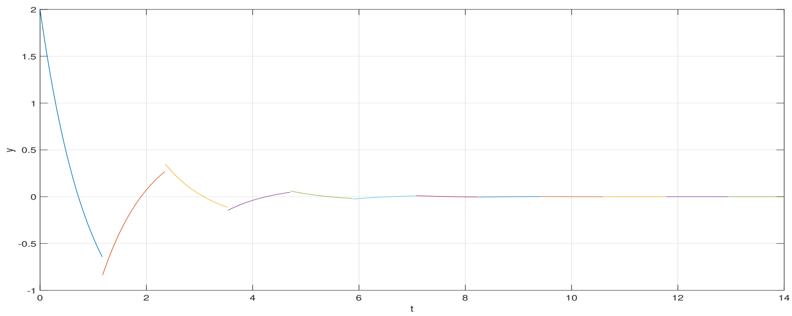

Fig. 1. The global asymptotic stability of the oscillatory solution for the IDEPCAG (4.1a)-(4.1b).

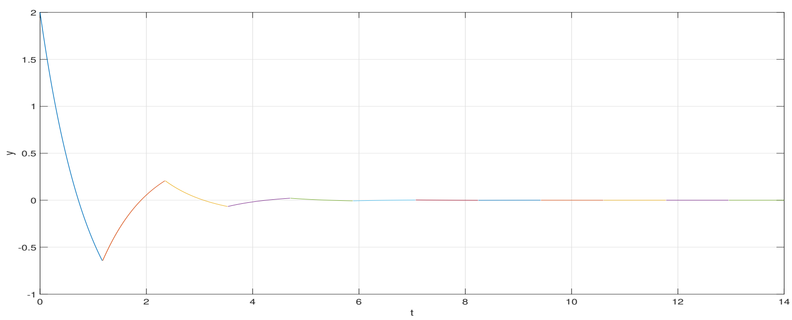

Fig. 2. The global asymptotic stability of the oscillatory solution for the DEPCAG (4.1a).

Example 4.2. Let us consider the IDEPCAG

$$y'(t) = 0.125y(t) - \sqrt{6}y(\gamma(t)), \quad y(0) = -1, \quad t \neq t_k, \tag{4.2a}$$

$$\Delta y|_{t=t_k} = d_k x(t_k^-), \quad k \in \mathbb{N}, \tag{4.2b}$$

where $d_k = -0.3$, $t_k = 3.5 \cdot (k-1)$, $\gamma_k = 3.5 \cdot k - 2.25$ for all $k \in \mathbb{N}$. The IDEPCAG (4.2a)-(4.2b) is a special case of the IDEPCAG (2.12a)-(2.12b) with $a = 0.125$ and $b = -\sqrt{6}$. It is not difficult to see that $\vartheta_k^- = t_{k+1} - \gamma_k = 2.25$, $\vartheta_k^+ = \gamma_k - t_k = 1.25$, $\hat{\lambda}(-\vartheta_k^+) \approx 3.68998 \neq 0$, $\varphi_+(a) \approx 1.78269$ and $\varphi_-(a) \approx -0.072004$.
We calculate

$$1 + d_{k+1} = 0.7 > 0, \quad -\sqrt{6} = b < -\liminf_{k\to\infty} \frac{ae^{a\vartheta_k^-}}{e^{a\vartheta_k^-} - 1} \approx 0.50987,$$

and

$$\left\{\left(\frac{a+b}{a}\right)\varphi_+(a) - \left(\frac{b}{a}\right)(2 + d_{k+1})\right\}\left\{\left(\frac{a+b}{a}\right)\varphi_-(a) + \frac{b}{a}d_{k+1}\right\} \approx 1.170842 > 0$$

for $k \in \mathbb{N}$. In this situation, the second hypothesis (3.13) of Corollary 3.1 is satisfied. Therefore, every solution of the IDEPCAG (4.2a)-(4.2b) is oscillatory. Furthermore, each solution of the IDEPCAG (4.2a)-(4.2b) goes to zero as $t \to \infty$ by oscillating.

Figure 3 illustrates the simulation result, demonstrating the global asymptotic stability of the oscillatory solution for the IDEPCAG (4.2a)-(4.2b) with impulses.

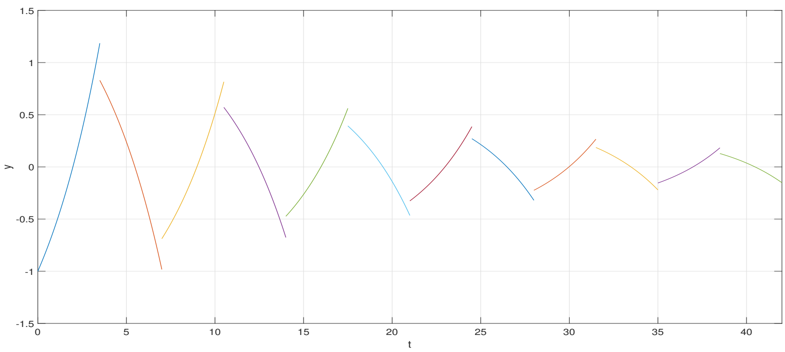

Fig. 3. The global asymptotic stability of the oscillatory solution for the IDEPCAG (4.2a)-(4.2b).

In the case without impulses, we have

$$\left|\frac{\hat{\lambda}(\vartheta_k^-)}{\hat{\lambda}(-\vartheta_k^+)}\right| \approx 1.36577 > 1.$$

According to Corollary 2.3 in [7], every solution of the DEPCAG (4.2a) without impulses is oscillatory. However, as per Theorem 3.1 in [7], the DEPCAG (4.2a) without impulses is not global asymptotic stability.

Figure 4 illustrates the simulation result, demonstrating the instability of the oscillatory solution for the IDEPCA (4.2a) without impulses.

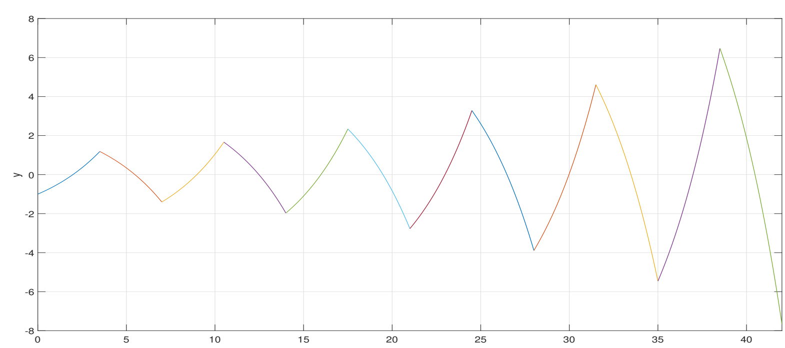

Fig. 4. The oscillatory solution for the DEPCAG (4.2a).

Example 4.2 illustrates that the presence or absence of impulsive effects significantly influences the system's stability. As indicated by the simulation results, the inclusion of impulsive effects in the IDEPCAG results in global asymptotic stability, while their absence in the DEPCAG leads to the lack of global asymptotic stability. This underscores the importance and impact of impulsive effects on the system's stability behavior.

Example 4.3. Let us consider the IDEPCAG

(4.3a) $$y'(t) = -0.4\sin(t)y(t) + 0.25\cos(t)y(\gamma(t)),\ y(0) = -1.8, \quad t \neq t_k,$$

(4.3b) $$\Delta y|_{t=t_k} = d_k x(t_k^-), \quad k \in \mathbb{N},$$

where $d_k = 0.3$, $t_k = 4\pi(k-1)$, $\gamma_k = 4\pi k - 2\pi$ for all $k \in \mathbb{N}$. The IDEPCAG (4.3a)-(4.3b) is a special case of the IDEPCA (1.1a)-(1.1b) with $a(t) = -0.4\sin(t)$ and $b(t) = 0.25\cos(t)$. It is not difficult to see that $\vartheta_k^- = t_{k+1} - \gamma_k = 2\pi$, $\vartheta_k^+ = \gamma_k - t_k = 2\pi$, $\lambda(t_k, \gamma_k) \approx 0.521893 \neq 0$. We calculate

$$1 + d_{k+1} = 1.3 > 0,$$

$$\int_{t_k}^{\gamma_k} 0.25\cos(s)e^{-0.4\int_u^{\gamma_k}\sin(u)du}ds \approx -0.4781061 < 1,$$

and

$$\int_{\gamma_k}^{t_{k+1}} 0.25\cos(s)e^{-0.4\int_u^{\gamma_k}\sin(u)du}ds \approx 0.4781061 > -1,$$

for $k \in \mathbb{N}$. In this case, the first hypothesis of Theorem 2.5 holds. So, every solution of the IDEPCAG (4.3a)-(4.3b) is non-oscillatory. Moreover,

$$\left|(1 + d_{k+1})\frac{\lambda(t_{k+1}, \gamma_k)}{\lambda(t_k, \gamma_k)}\right| = \left|\frac{e^{-0.4\int_{\gamma_k}^{t_{k+1}}\sin(s)ds} + 0.25\int_{\gamma_k}^{t_{k+1}} e^{-0.4\int_s^{t_{k+1}}\sin(u)du}\cos(s)ds}{e^{-0.4\int_{\gamma_k}^{t_k}\sin(s)ds} + 0.25\int_{\gamma_k}^{t_k} e^{-0.4\int_s^{t_k}\sin(u)du}\cos(s)ds}\right| \approx 0.4590071 < 1.$$

Then, the condition (3.2) is fulfilled. Therefore, according to Theorem 3.1, the zero solution of the IDEPCAG (4.3a)-(4.3b) is global asymptotic stability.

Figure 5 illustrates the simulation result, demonstrating the global asymptotic stability of the non-oscillatory solution for the IDEPCAG (4.3a)-(4.3b) with impulses.

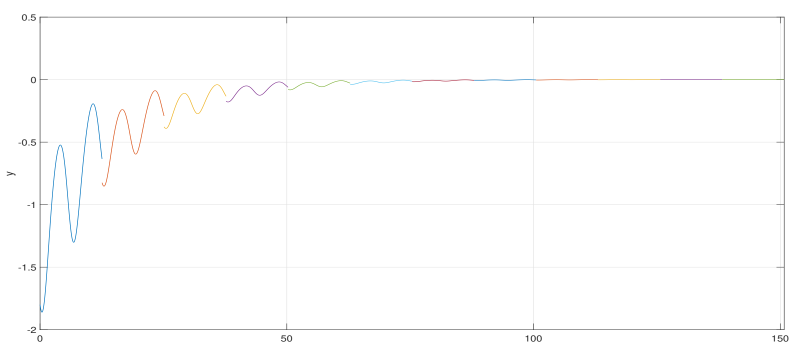

Fig. 5. The global asymptotic stability of the non-oscillatory solution for the IDEPCAG (4.3a)-(4.3b).

In the case without impulses, we have

$$\left|\frac{\lambda(t_{k+1}, \gamma_k)}{\lambda(t_k, \gamma_k)}\right| \approx 0.3530821 < 1.$$

According to Theorem 2.5 in [7], every solution of the DEPCA (4.3a) without impulses is non-oscillatory. However, as per Theorem 3.1 in [7], the zero solution of the DEPCAG (4.3a) without impulses is global asymptotic stability.

Figure 6 illustrates the simulation result, showcasing the stability of the non-oscillatory solution for the DEPCAG (4.3a) without impulses.

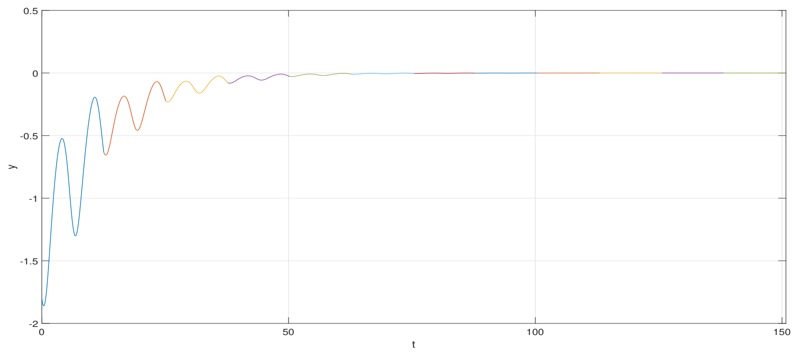

Fig. 6. The non-oscillatory solution for the DEPCAG (4.3a).

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Departamento de Matemática, Facultad de Ciencias Básicas, Universidad Metropolitana de Ciencias de la Educación, José Pedro Alessandri 774, Santiago, Chile.

Email address: kschiu@umce.cl