CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 351-360

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.06

A Voronovskaya type theorem associated to geometric series of Bernstein - Durrmeyer operators

ȘTEFAN - LUCIAN GAROIU

ABSTRACT. In this paper we give a Voronovskaya type theorem for the operators introduced by U. Abel, which are defined as the geometric series of Bernstein - Durrmeyer operators.

1. INTRODUCTION

A first study considering the geometric series of some positive and linear operators is due to Păltănea, see [14]. Namely, he studied the properties of geometric series associated to Bernstein operators ([5]) $B_n : C[0,1] \to C[0,1]$, given as follows:

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ x \in [0,1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1],\ k = 0,\ 1,\ \dots,\ n.$$

In paper [14], Păltănea introduced the operators

$$A_n = \frac{1}{n}\sum_{k=0}^{\infty} (B_n)^k . \tag{1.1}$$

However, there are cases when operators A_n aren't well defined, so the domain of definition cannot be $C[0,1]$ rather, it is chosen as the space of functions

$$\psi C[0,1] = \{f : C[0,1] \to C[0,1] : \exists\, g \in C[0,1],\ f = \psi g\},$$

which, endowed with the norm $||f||_\psi = \sup\limits_{x\in(0,1)} \frac{|f(x)|}{\psi(x)}$, is a Banach space. Throughout the paper the function $\psi : [0,1] \to \mathbb{R}$, is given as $\psi(x) = x(1-x)$.

In [14] it was proved that $\lim\limits_{n\to\infty} ||A_n(\psi f) - 2G(f)||_\psi = 0$, for any $f \in C[0,1]$, where $G(f)(x) := (1-x)\int\limits_0^x t f(t)\,dt + x\int\limits_0^x (1-t) f(t)dt$, and $(G(f)(x))'' = -f(x),\ x \in [0,1]$. Also, a new proof of this result was given in [2].

A generalization of the operators A_n was introduced by Abel et al. ([3]), namely, in formula (1.1) operators B_n were replaced by positive linear operators L_n belonging to a general class of operators. If we denote by G_{L_n} the geometric series attached to these operators L_n, then the following result was proved: $\lim\limits_{n\to\infty} ||G_{L_n}(f) - 2G(f/\psi)||_\psi = 0$, which holds for functions f belonging to the space $C_\psi[0,1] = \{f : C[0,1] \to C[0,1] : \exists\, g \in B[0,1] \cap C(0,1),\ f = \psi g\}$ which together with the norm $||\cdot||_\psi$ is a Banach space. The

Received: 05.03.2024. In revised form: 05.11.2024. Accepted: 10.11.2024
2010 *Mathematics Subject Classification.* 41A36, 47D06, 47A58, 41A35, 41A45, 41A81, 47A67, 40A30.
Key words and phrases. *Voronovskaya type theorem, Geometric series of Bernstein - Durrmeyer operators, Positive and linear operators.*
Corresponding author: Ștefan - Lucian GAROIU; stefan.garoiu@unitbv.ro

operators A_n were also studied on the space $C_0[0,1] = \{f \in C[0,1] : f(0) = f(1) = 0\}$ endowed with the usual sup-norm, in paper [15]. There a Voronovskaya theorem was obtained.

U. Abel, in paper [1], introduced the geometric series associated to Bernstein-Durrmeyer operators (which first appeared in paper [8] and independently in [12] and their properties were later studied in [6], [7], [13] etc.)

$$M_n(f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt, \qquad f \in L^\infty[0,1].$$

Namely, the operators he studied are defined as follows:

$$P_n = \frac{1}{n}\sum_{k=0}^{\infty} (M_n)^k .$$

These operators are well defined on the space V, which is

$$V = \{f \in L^\infty([0,1]) : ||f||_* < \infty\}, \tag{1.2}$$

where by $\|\cdot\|_*$ we mean the norm:

$$||f||_* = \sup_{y\in(0,1)} \left| (\psi(y))^{-1} \int_0^y f(x)\, dx \right| .$$

Also, V endowed with the norm $||\cdot||_*$ is a Banach space.

For $f \in V$, define the function F on $(0,1)$ by

$$F(y) = (\psi(y))^{-1} \int_0^y f(x)\, dx, \qquad y \in (0,1). \tag{1.3}$$

Then, $f = (\psi F)'$ a. e. on $[0,1]$ and $||f||_* = ||F||_\infty$.

Further, the operator $P : V \to V$, was defined as

$$P(f)(x) = \int_0^1 \int_x^t F(u)\, du dt, \qquad x \in [0,1],\ f \in V, \tag{1.4}$$

Integrating by parts in (1.4) it can be seen that

$$P(f)(x) = -\int_0^x tF(t)\, dt + \int_x^1 (1-t)F(t) dt, \qquad x \in [0,1],\ f \in V \tag{1.5}$$

and here, if we differentiate, we find

$$P'(f)(x) = -F(x). \tag{1.6}$$

In his paper, Abel proved that operators P_n satisfy the following convergence result

Theorem 1.1. *If $f \in V$, then, in $(V, ||\cdot||_*)$, the convergence*

$$\lim_{n\to\infty} ||P_n(f) - P(f)||_* = 0, \tag{1.7}$$

holds.

Also, Abel obtained the following two results concerning the norm of operators M_n and P_n on the space V.

Proposition 1.1. *For each $n \in \mathbb{N}$, the operators M_n map V to V, that is, $M_n(V) \subset V$, and*

$$||M_n||_{\mathcal{L}(V,V)} = \frac{n}{n+2}. \tag{1.8}$$

Proposition 1.2. *For each $n \in \mathbb{N}$ the operators P_n map V to V, that is, $P_n(V) \subset V$, and*

$$||P_n||_{\mathcal{L}(V,V)} = \frac{1}{2} + \frac{1}{n}. \tag{1.9}$$

More recent results concerning the power series of approximation operators can be seen in [4], [9], [10], [11] and [16].

The aim of our paper will be to provide an estimation of the convergence of operators P_n in the form of a Voronovskaya type theorem.

2. A Voronovskaya type result

In this section we will provide our main result, namely we will prove our Voronovskaya type theorem associated to the operators P_n. First, we will denote by $G_{M_n} = \sum\limits_{k=0}^{\infty} (M_n)^k$ the geometric series associated to Bernstein - Durrmeyer operators M_n, where the convergence holds on V. Next, we will prove that the following identities hold.

Lemma 2.1. *Operator $G_{M_n} \in V$ and it verifies the identities:*

$$(I - M_n) \circ G_{M_n} = I, \tag{2.10}$$

and

$$G_{M_n} \circ (I - M_n) = I, \tag{2.11}$$

where I denotes the identity operator.

Proof. Identities (2.10) and (2.11) follow from the general properties of operators algebra $\mathcal{L}(V, V)$ since from (1.8) we have that $\|M_n\|_{\mathcal{L}(V,V)} < 1$. □

In the following we will work on space $V_1 = V \cap C[0,1]$. Note that condition $f \in V_1$ is equivalent with conditions $f \in C[0,1]$ and the relation below holds

$$\int_0^1 f(t)dt = 0. \tag{2.12}$$

On this space, we define the operator $U : V_1 \to C[0,1]$ through

$$U(f)(y) = \begin{cases} (\psi(y))^{-1} \int\limits_0^y f(x)\,dx, & y \in (0,1) \\ f(0), & y = 0 \\ -f(1), & y = 1 \end{cases} \qquad f \in V_1, \tag{2.13}$$

and norm $\|\cdot\|_*$ as:

$$\|f\|_* = \sup_{y\in[0,1]} |U(f)(y)|. \tag{2.14}$$

Here, we have $U(f)(1) = \lim_{y\to 1} \frac{\int_0^y f(t)dt}{\psi(y)}$ and $U(f)(0) = \lim_{y\to 0} \frac{\int_0^y f(t)dt}{\psi(y)}$, so, since $\psi(0) = \psi(1) = 0$ using l' Hospital's rule we will have that $U(f)(1) = -f(1)$ and $U(f)(0) = f(0)$.

Next, we will define the operator

$$\Theta(h)(x) = -\int_0^x th(t)dt + \int_x^1 (1-t)h(t)dt, \tag{2.15}$$

where $h \in C[0,1]$ and $x \in [0,1]$. This operator has the following property.

Proposition 2.3. *The operator* Θ *maps* $C[0,1]$ *to* V_1, *i. e.*

$$\Theta(C[0,1]) \subset V_1.$$

Proof. We have that:

$$\begin{aligned}\int_0^1 \Theta(h)(x)dx &= \int_0^1 \left[-\int_0^x th(t)dt + \int_x^1 (1-t)h(t)dt\right] dx\\ &= -\int_0^1 \left[\int_t^1 th(t)dx\right] dt + \int_0^1 \left[\int_0^t (1-t)h(t)dx\right] dt\\ &= -\int_0^1 t(1-t)h(t)dt + \int_0^1 t(1-t)h(t)dt\\ &= 0.\end{aligned}$$

So, since $\Theta(h)$ satisfies condition (2.12) our assertion is true. □

From above and from (2.15), we have that

$$P(f) := \Theta(U(f)), \qquad f \in V_1. \tag{2.16}$$

Next, on the space V_1 the following result concerning the norm $\|\cdot\|_*$ holds.

Lemma 2.2. *For any function* $f \in V_1$ *we have that:*

$$\|f\|_* \leq 2\|f\|_\infty. \tag{2.17}$$

Proof. Let U be as in (2.13) and $\|f\|_* = \sup_{y\in[0,1]} |U(y)|$. There is a sequence $(y_n)_n$, $y_n \in [0,1]$ such that $\lim_{n\to\infty} |U(y_n)| = \|f\|_*$. From Bolzano - Weierstrass theorem there exists a covergent subsequence $(y_{n_k})_k$ of the sequence $(y_n)_n$. Let $y^* \in [0,1]$ be such that $\lim_{k\to\infty} y_{n_k} = y^*$. Then we have the following cases:

I. If $y^* = 0$ then

$$\|f\|_* = \lim_{k\to\infty} |U(y_{n_k})| = \lim_{y\to 0} |U(y)| = |f(0)| \leq \|f\|_\infty.$$

II. If $y^* \in (0, \frac{1}{2})$ then

$$\|f\|_* = \lim_{k\to\infty} |U(y_{n_k})| = |F(y^*)| = \left|\frac{1}{y^*(1-y*)} \int_0^{y^*} f(t)dt\right| \leq \|f\|_\infty \frac{1}{1-y^*} \leq 2\|f\|_\infty.$$

III. If $y^* \in (\frac{1}{2}, 1)$ then

$$\|f\|_* = \lim_{k\to\infty} |U(y_{n_k})| = |F(y^*)| = \left|-\frac{1}{y^*(1-y*)} \int_{y^*}^1 f(t)dt\right| \leq \|f\|_\infty \frac{1}{y^*} \leq 2\|f\|_\infty.$$

IV. If $y^* = 1$ then

$$\|f\|_* = \lim_{k\to\infty} |U(y_{n_k})| = \lim_{y\to 1} |U(y)| = |f(1)| \leq \|f\|_\infty.$$

So, our proof is complete. □

Now, we can prove our main result.

Theorem 2.2. *Let* $f \in V_1$ *be a ten times differentiable function on* $[0,1]$ *and which satisfies the following conditions* $\int_0^1 f(y)\log\psi(y)dy = 0$, $f(0) + f(1) = 0$, $f'(0) - f'(1) = 0$, $f''(0) + f''(1) = 0$ *and* $f'''(0) - f'''(1) = 0$. *Then:*

$$\lim_{n\to\infty} n(P_n(f) - P(f)) = 2P(f) - \Theta(T'\psi') - \frac{1}{2}\Theta(T''\psi), \tag{2.18}$$

with regard to the norm $\|\cdot\|_*$.

Proof. Let us fix $f \in V_1$ and then for simplicity we denote $P(t) = P(f)(t)$, $t \in (0,1)$ and $T(t) = U(f)(t)$.

One can prove that $T \in C^5[0,1]$. Indeed, one can obtain by induction that $T \in C^k[0,1]$, $1 \le k \le 5$ as follows. If we suppose by induction that $T \in C^{k-1}[0,1]$ and we have that T admits continuous k-derivative on interval $(0,1)$ it suffices that $T^{(k)}$ has finite limits at points 0 and 1. Note that the k-th derivative of T on interval $(0,1)$ can be expressed, after simplification, as a fraction having the denominator equal to ψ^{k+1} and the numerator given in terms of the derivatives of f up to order $k-1$. Then, the existence of the finite limits of $T^{(k)}$ at points 0 and 1 can be obtained by applying l'Hosptial's rule $k+1$ times, which is possible since f is $2k = k-1+k+1$ times differentiable.

Now, to prove our result we will use Taylor's polynomial with integral remainder, up to the fifth degree, associated to P, where we will keep in mind that $P'(t) = -T(t)$. Let $s,\ t \in [0,1]$, then

$$\begin{aligned} P(s) &= P(t) - T(t)(s-t) - \frac{1}{2}T'(t)(s-t)^2 - \frac{1}{6}T''(t)(s-t)^3 \\ &\quad - \frac{1}{24}T'''(t)(s-t)^4 - \frac{1}{120}T^{(4)}(t)(s-t)^5 - R_5(t,s), \end{aligned} \tag{2.19}$$

where $R_5(t,s) = \frac{1}{120}\int\limits_t^s (s-u)^5 T^{(5)}(u)du$. Since $P(s) \in V_1$ and any polynomial belongs to V_1 also $R_5(t,\cdot) \in V_1$, for each $t \in [0,1]$.

Further, we will need the j^{th} order moments of operators M_n which are denoted by $m_j(t) = M_n\left((e_1 - t)^j\right)(t)$, $j = 0,\ 1,\ 2, \ldots$ and $e_1(t) = t$. It is well known that the following recurrence formula (see [6]) holds

$$(j+n+2)m_{j+1}(t) = \psi(t)\left[2jm_{j-1}(t) + m_j'(t)\right] + (j+1)\psi'(t)m_j(t), \tag{2.20}$$

and

$$\begin{aligned} m_1(t) &= \frac{1}{n+2}\psi'(t), \\ m_2(t) &= \frac{2n-6}{(n+2)(n+3)}\psi(t) + \frac{2}{(n+2)(n+3)}, \\ m_3(t) &= \frac{12(n-1)}{(n+2)(n+3)(n+4)}\psi'(t)\psi(t) + \frac{6}{(n+2)(n+3)(n+4)}\psi'(t), \\ m_4(t) &= \frac{12\left(n^2 - 21n + 10\right)}{(n+2)(n+3)(n+4)(n+5)}\psi^2(t) \\ &\quad + \frac{24(3n-5)}{(n+2)(n+3)(n+4)(n+5)}\psi(t) + \frac{24}{(n+2)(n+3)(n+4)(n+5)}, \\ m_5(t) &= O\left(\frac{1}{n^3}\right)\psi^2(t)\psi'(t) + O\left(\frac{1}{n^4}\right)\psi(t)\psi'(t) + O\left(\frac{1}{n^5}\right)\psi'(t) \\ m_6(t) &= O\left(\frac{1}{n^3}\right)\psi^3(t) + O\left(\frac{1}{n^4}\right)\psi^2 + O\left(\frac{1}{n^5}\right)\psi(t) + O\left(\frac{1}{n^6}\right). \end{aligned} \tag{2.21}$$

Now, since $I - M_n : V_1 \to V_1$, from (2.19) we get:

$$\begin{aligned} (I - M_n)(P)(t) &= T(t)m_1(t) + \frac{1}{2}T'(t)m_2(t) + \frac{1}{6}T''(t)m_3(t) \\ &\quad + \frac{1}{24}T'''(t)m_4(t) + \frac{1}{120}T^{(4)}(t)m_5(t) + M_n(R_5(t,\cdot))(t). \end{aligned} \tag{2.22}$$

In order to obtain our Voronovskaya type result we will need to use Lemma 2.1 and to relation (2.22) we will apply operator P_n. But, Lemma 2.1 only works for operators from the space V_1 so first we will prove that $(I - M_n)(P)$ belongs to the space V_1. Namely, we will prove that

$$\int_0^1 (I - M_n)(P)(t)dt = 0.$$

We have that:

$$\int_0^1 (I - M_n)(P)(t)dt = \int_0^1 P(f)(t)dt - \int_0^1 M_n((P)(f))(t)dt.$$

From Proposition 2.3 and from (2.16) we have that:

$$\int_0^1 P(f)(t)dt = 0,$$

so we only have to prove that $\int_0^1 M_n((P)(f))(t)dt = 0$. However,

$$\int_0^1 M_n((P)(f))(t)dt = (n+1)\sum_{k=0}^{n} \int_0^1 p_{n,k}(t)\,dt \cdot \int_0^1 p_{n,k}(x)\,P(f)(x)\,dx$$

$$= \sum_{k=0}^{n} \int_0^1 p_{n,k}(x)\,P(f)(x)\,dx = \int_0^1 \sum_{k=0}^{n} p_{n,k}(x)\,P(f)(x)\,dx = \int_0^1 P(f)(x)\,dx = 0.$$

So, $(I - M_n)(P)$ belongs to V_1.

Also, operators P_n can only be applied to functions belonging to V_1 so we will need to prove that terms from the right hand side of (2.22) belong to V_1.

First, we have that

$$\begin{aligned}
\int_0^1 T(y)\psi'(y)dy &= \int_0^1 \frac{\psi'(y)}{\psi(y)} \left(\int_0^y f(t)dt\right) dy \\
&= \int_0^1 (\log \psi(y))' \left(\int_0^y f(t)dt\right) dy \\
&= -\int_0^1 \log \psi(y) f(y)dy \\
&= 0.
\end{aligned} \tag{2.23}$$

so we have that $T\psi' \in V_1$ which means that $Tm_1 \in V_1$. Next, integrating by parts we have that

$$\int_0^1 T'(y)\psi(y)dy = -\int_0^1 T(y)\psi'(y)dy.$$

Therefore, from (2.23) it follows that $\int_0^1 T'(y)\psi(y)dy = 0$, so $T'\psi \in V_1$. Now, we will prove that $\int_0^1 T'(y)dy = 0$. Here,

$$\int_0^1 T'(y)dy = T(1) - T(0) = -(f(1) + f(0)).$$

Then using the hypothesis it follows that $\int_0^1 T'(y)dy = 0$ so $T' \in V_1$ which together with $T'\psi \in V_1$ imply that $T'm_2 \in V_1$. Further, integrating by parts, we have that

$$\begin{aligned}
\int_0^1 T''(y)\psi'(y)dy &= T'(y)\psi'(y)|_0^1 - \int_0^1 T'(y)\psi''(y)dy \\
&= -T'(1) - T'(0) + 2(T(1) - T(0)).
\end{aligned} \tag{2.24}$$

Since,

$$T'(y) = \frac{\psi(y)f(y) - \psi'(y)\int_0^y f(t)dt}{\psi^2(y)},$$

using l'Hospital rule for $\frac{0}{0}$ we have that $T'(1) = -\frac{1}{2}f'(1) + f(1)$ and $T'(0) = \frac{1}{2}f'(0) + f(0)$. So, condition (2.24) becomes

$$\int_0^1 T''(y)\psi'(y)dy = \frac{1}{2}(f'(1) - f'(0)) - 3(f(0) + f(1)),$$

which, as we can see from the hypothesis, means that $\int_0^1 T''(y)\psi'(y)dy = 0$ so $T''\psi' \in V_1$. Next, integration by parts yields:

$$\begin{aligned}\int_0^1 T''(y)\psi(y)\psi'(y)dy &= T'(y)\psi(y)\psi'(y)|_0^1 - \int_0^1 T'(y)[(\psi'(y))^2 + \psi(y)\psi''(y)]dy \\ &= -\int_0^1 T'(y)(1 - 6\psi(y))dy = -\int_0^1 T'(y)dy + 6\int_0^1 T'(y)\psi(y)dy = 0,\end{aligned}$$

which implies that $T''\psi\psi'$ belong to V_1. Because $T''\psi' \in V_1$ and $T''\psi\psi' \in V_1$ then $T''m_3 \in V_1$. Proceeding in a similar fashion we can prove that

$$\int_0^1 T'''(y)\psi^2(y)dy = -2\int_0^1 T''(y)\psi(y)\psi'(y)dy = 0,$$

so $T'''\psi^2$ is in the space V_1. Again, we have that $\int_0^1 T'''(y)\psi(y)dy = -\int_0^1 T''(y)\psi'(y)dy = 0$, so $T'''\psi \in V_1$. Now, $\int_0^1 T'''(y)dy = T''(1) - T''(0)$. But,

$$T''(y) = \frac{2(1 - 3\psi(y))\int_0^y f(t)dt - \psi(y)(2\psi'(y)f(y) - \psi(y)f'(y))}{\psi^3(y)},$$

so after using l'Hospital's rule for $\frac{0}{0}$ and the assumptions made in the statement of our theorem we get

$$\int_0^1 T'''(y)dy = -\frac{1}{3}[f''(0) + f''(1)] + [f'(1) - f'(0)] - 2[f(1) + f(0)] = 0,$$

which means that $T''' \in V_1$. Therefore we have that $T'''m_4 \in V_1$.

Next, integrating by parts we obtain that

$$\begin{aligned}\int_0^1 T^{(4)}(y)\psi(y)\psi'(y)dy &= T'''(y)\psi(y)\psi'(y)|_0^1 - \int_0^1 T'''(y)(1 - 6\psi(y))dy \\ &= 6\int_0^1 T'''(y)\psi(y)dy - \int_0^1 T'''(y)dy \\ &= 6T''(y)\psi(y)|_0^1 - 6\int_0^1 T''(y)\psi'(y)dy \\ &= 0,\end{aligned}$$

and also,

$$\begin{aligned}\int_0^1 T^{(4)}(y)\psi^2(y)\psi'(y)dy &= T'''(y)\psi^2(y)\psi'(y)|_0^1 - 2\int_0^1 T'''(y)\psi(y)dy + 10\int_0^1 T'''(y)\psi^2(y)dy \\ &= 0,\end{aligned}$$

so it follows that $T^{(4)}\psi\psi'$ and $T^{(4)}\psi^2\psi'$ belong to V_1. Again, integration by parts yields

$$\int_0^1 T^{(4)}(y)\psi'(y)dy = T'''(y)\psi'(y)|_0^1 + 2\int_0^1 T'''(y)dy$$
$$= -T'''(1) - T'''(0).$$

We have that:

$$T'''(y) = \frac{1}{\psi^4(y)}\left\{6\psi'(y)(2\psi(y)-1)\int_0^y f(t)dt \right.$$
$$\left. + \psi(y)[6(1-3\psi(y))f(y) - \psi(y)(3\psi'(y)f'(y) - \psi(y)f''(y))]\right\}.$$

Using l'Hospital's rule for $\frac{0}{0}$ we get:

$$T'''(1) + T'''(0) = \frac{1}{4}[f'''(0) - f'''(1)] + [f''(1) + f''(0)] + 3[f'(0) - f'(1)] + 6[f(1) + f(0)],$$

so, from the hypothesis, we obtain $\int_0^1 T^{(4)}(y)\psi'(y) = 0$ so $T^{(4)}\psi'$ is in V_1. So $T^{(4)}m_5 \in V_1$. Also, from (2.19) it follows that $M_n(R_5(t,\cdot))(t) \in V_1$ because all the other terms are from V_1.

Now, we will apply operators nP_n to relation (2.22), and since we have that $(I - M_n)(P) \in V_1$ we can use Lemma 2.1 and for $x \in [0,1]$ we obtain

$$P(x) = \frac{n}{n+2}P_n(T\psi')(x) + \frac{n}{2}P_n\left(\frac{2n-6}{(n+2)(n+3)}T'\psi + \frac{2}{(n+2)(n+3)}T'\right)(x)$$
$$+ \frac{n}{6}P_n(T''m_3)(x) + \frac{n}{24}P_n(T'''m_4)(x) + \frac{n}{120}P_n\left(T^{(4)}m_5\right)(x) \tag{2.25}$$
$$+ nP_n(K_n)(x),$$

where $K_n(t) = M_n(R_5(t,\cdot))(t)$.

Then, because $f = \psi'T + T'\psi$, we have

$$n(P(f) - P_n(f))(x) = \frac{-2n}{n+2}P_n(T\psi')(x) - \frac{8n^2+6n}{(n+2)(n+3)}P_n(T'\psi)(x)$$
$$+ \frac{n^2}{(n+2)(n+3)}P_n(T')(x) + \frac{n^2}{6}P_n(T''m_3)(x) \tag{2.26}$$
$$+ \frac{n^2}{24}P_n(T'''m_4)(x) + \frac{n^2}{120}P_n\left(T^{(4)}m_5\right)(t)$$
$$+ n^2P_n(K_n)(x).$$

Now, we will prove that remainder $n^2P_n(K_n)$, from (2.26), converges to 0 in the norm $\|\cdot\|_*$. First, for $s,\ t \in [0,1]$ we have

$$|R_5(t,s)| = \left|\frac{1}{120}\int_t^s (s-u)^5 T^{(5)}(u)du\right|$$
$$\le \frac{\|T^{(5)}(u)\|_\infty}{120}\left|\int_t^s (s-u)^5 du\right|$$
$$= \frac{\|T^{(5)}(u)\|_\infty}{6!}(s-u)^6.$$

Then for $t \in [0,1]$ we obtain

$$|K_n(t)| = |M_n(R_5(t,\cdot))(t)| \le \frac{\|T^{(5)}(u)\|_\infty}{6!}m_6(t) = O\left(\frac{1}{n^3}\right).$$

Hence, $n^2\|K_n\|_\infty = O\left(\frac{1}{n}\right)$, which together with Lemma 2.2 imply that $P_n(K_n)$ converges to 0 in the norm $\|\cdot\|_*$.

Next, using Theorem 1.1 and (2.21), returning to (2.26) we see that

$$\begin{aligned}\lim_{n\to\infty} n(P_n(f) - P(f)) &= 2P(T\psi') + 8P(T'\psi) - P(T') \\ &\quad - 2P(T''\psi'\psi) - \frac{1}{2}P(T'''\psi^2),\end{aligned} \tag{2.27}$$

with regard to the norm $\|\cdot\|_*$, which is,

$$\begin{aligned}\lim_{n\to\infty} n(P_n(f) - P(f)) &= 2P(f) + 6P(T'\psi) - P(T') \\ &\quad - 2P(T''\psi'\psi) - \frac{1}{2}P(T'''\psi^2),\end{aligned} \tag{2.28}$$

with regard to the norm $\|\cdot\|_*$. Now, we have that

$$P(T''\psi'\psi)(x) =$$
$$-\int_0^x \frac{t}{\psi(t)}\left(\int_0^t T''(y)\psi(y)\psi'(y)dy\right)dt + \int_x^1 \frac{1-t}{\psi(t)}\left(\int_0^t T''(y)\psi(y)\psi'(y)dy\right)dt,$$

where, if we integrate by parts and use the fact that $(\psi'(y))^2 = 1 - 4\psi(y)$, we get that:

$$\begin{aligned}\int_0^t T''(y)\psi(y)\psi'(y)dy &= \int_0^t (T'(y))'\psi(y)\psi'(y)dy \\ &= \psi(t)\psi'(t)T'(t) - \int_0^t T'(y)dy + 6\int_0^t T'(y)\psi(y)dy,\end{aligned}$$

so,

$$\begin{aligned}P(T''\psi'\psi)(x) &= -\int_0^x t\psi'(t)T'(t)dt + \int_x^1 (1-t)\psi'(t)T'(t)dt \\ &\quad + \int_0^x \frac{t}{\psi(t)}\left(\int_0^t T'(y)dy\right)dt - \int_x^1 \frac{1-t}{\psi(t)}\left(\int_0^t T'(y)dy\right)dt \\ &\quad -6\int_0^x \frac{t}{\psi(t)}\left(\int_0^t T'(y)\psi(y)dy\right)dt + 6\int_x^1 \frac{1-t}{\psi(t)}\left(\int_0^t T'(y)\psi(y)dy\right)dt,\end{aligned} \tag{2.29}$$

which becomes

$$P(T''\psi'\psi)(x) = \Theta(T'\psi')(x) - P(T')(x) + 6P(T'\psi)(x). \tag{2.30}$$

Next,

$$P(T'''\psi^2)(x) = \tag{2.31}$$
$$-\int_0^x \frac{t}{\psi(t)}\left(\int_0^t T'''(y)\psi^2(y)dy\right)dt + \int_x^1 \frac{1-t}{\psi(t)}\left(\int_0^t T'''(y)\psi^2(y)dy\right)dt,$$

and since integration by parts yields

$$\int_0^t T'''(y)\psi^2(y)dy = \psi^2(t)T(t) - 2\int_0^t T''(y)\psi'(y)\psi(y)dy,$$

we can see that (2.31) becomes

$$P(T'''\psi^2)(x) = \\ -\int_0^x t\psi(t)T''(t)dt + \int_x^1 (1-t)\psi(t)T''(t)dt \\ -2\left[-\int_0^x \frac{t}{\psi(t)}\left(\int_0^t T''(y)\psi'(y)\psi(y)dy\right)dt + \int_x^1 \frac{1-t}{\psi(t)}\left(\int_0^t T''(y)\psi'(y)\psi(y)dy\right)dt\right] \\ = \Theta(T''\psi)(x) - 2P(T''\psi\psi')(x). \tag{2.32}$$

Therefore, from (2.30) we see that (2.32) becomes

$$P(T'''\psi^2)(x) = \Theta(T''\psi)(x) - 2\Theta(T'\psi')(x) + 2P(T')(x) - 12P(T'\psi)(x). \tag{2.33}$$

Now, replacing (2.30) and (2.33) in (2.28) we obtain

$$\lim_{n\to\infty} n(P_n(f) - P(f)) = 2P(f) - \Theta(T'\psi') - \frac{1}{2}\Theta(T''\psi) \tag{2.34}$$

with regard to the norm $\|\cdot\|_*$, which is our Voronovskaya type result. □

REFERENCES

[1] Abel, U. Geometric series of Bernstein-Durrmeyer operators. *East J. on Approx.* **15** (2009), no. 4, 439–450.
[2] Abel, U.; Ivan, M.; Păltănea, R. Geometric series of Bernstein operators revisited. *J. Math. Anal. Appl.* **400** (2013), no. 1, 22–24.
[3] Abel, U.; Ivan, M.; Păltănea, R. Geometric series of positive linear operators and the inverse Voronovskaya theorem on a compact interval. *J. Approx. Theory.* **184** (2014), 163–175.
[4] Acar, T.; Aral, A.; Raşa, I. Power series of positive linear operators. *Mediterr. J. Math.* **16** (2019), no. 2, Paper no. 43, 11pp.
[5] Bernstein S. Démonstration du Théorème de Weierstrass fondée sur le calcul des Probabilités. *Communications of The Kharkov Mathematical Society*, **13** (1912/13), 1–2.
[6] Derriennic, M. -M. Sur l'approximation de functions integrable sur [0, 1] par des polynomes de Bernstein modifies. *J. Approx. Theory* **31** (1981), 323–343.
[7] DeVore, R. A.; Lorentz, G. G. Constructive Approximation. *Grundlehren* **303** (1993). Springer, Heidelberg.
[8] Durrmeyer, J. L. Une formule d'inversion de la Transformee de Laplace, Applications a la Theorie des Moments. *These de 3e Cycle, Faculte des Sciences de l'Universite de Paris* (1967).
[9] Garoiu, Ş.; Păltănea, R. Generalized Voronovskaya theorem and the convergence of power series of positive linear operators. *J. Math. Anal. Appl.* **531** (2024), no. 2, part 2, Paper No. 127868, 14pp.
[10] Garoiu, Ş.; Păltănea, R. The representation of the limit of power series of positive linear operators by using the semigroup of operators generated by their iterates. *Dolomites Res. Notes Approx.*, **16** (2023), no. 3, Special Issue FAATNA 20>22, 39–47.
[11] Gonska, H.; Raşa, I.; Stanilă, E.D. Power series of operators U_n^ρ. *Positivity* **19** (2015), no. 2, 237–249.
[12] Lupaș, A. Die Folge der Betaoperatoren. Dissertation. *Univ. Stuttgart* (1972), Stuttgart.
[13] Păltănea, R. Approximation Theory Using Positive Linear Operators. *Birkhäuser* (2004).
[14] Păltănea, R. The power series of Bernstein operators. *Automat. Comput. Appl. Math.* **15** (2006), no.2, 247–253 (2007).
[15] Păltănea, R. On the geometric series of linear positive operators. *Constr. Math. Anal.* **2** (2019), no. 2, 49–56.
[16] Raşa, I. Power series of Bernstein operators and approximation of resolvents. *Mediterr. J. Math.* **9** (2012), no. 4, 635–644.

TRANSILVANIA UNIVERSITY OF BRAȘOV
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
STR IULIU MANIU NO 50, CODE 500091, TOWN BRAȘOV, COUNTRY ROMANIA
Email address: stefan.garoiu@unitbv.ro