CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 393-408

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.08

On statistical convergence of topological Henstock-Kurzweil integral

HEMANTA KALITA[1], SUMIT SOM[2], AND BIPAN HAZARIKA[3]

ABSTRACT. In this paper, we introduce Henstock-Kurzweil type integrable function (in brief, topological Henstock-Kurzweil integrable function) on a topological vector space associate with a Radon measure μ. Basic results of topological Henstock-Kurzweil integrable function are discussed. Also, the relationship between topological Henstock-Kurzweil integral and Lebesgue integral is discussed. Moreover, we investigate several convergence theorems for μ-measurable topological Henstock-Kurzweil integrable function on a topological vector space. Finally, we extent the notion of statistical convergence for topological Henstock-Kurzweil integrable function on a μ-subcell of a topological vector space.

1. INTRODUCTION

Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. In the year 1854, Riemann first introduced the concept of Riemann integral of f through the concept of tagged partitions in the interval $[a, b]$ and the limit of Riemann sums as the norm of the tagged partitions goes to zero. During 1950, Henstock and Kurzweil developed an integral, known as Henstock-Kurzweil integral which uses the concept of δ-fine tagged partition of the interval $[a, b]$ where $\delta : [a, b] \to \mathbb{R}_+$. Henstock-Kurzweil integral is more general than Lebesgue integral as well as similar to that of Riemann integral (see [8, 10, 18]). Moreover, the Henstock-Kurzweil integral is non-absolute, in the sense that, some functions are Henstock-Kurzweil integrable but not absolutely Henstock-Kurzweil integrable. G. Carrao [1] investigated Henstock-Kurzweil type integral on a complete measure metric space endowed with a Radon measure μ and with a family $\mathcal{F}$ of "intervals" satisfying the Vitali covering theorem called μ-cell (see [1, Definition 2.14]). Recently, H. Kalita et al. expanded the idea of μ-cell for Henstock-Kurzweil integrals in various settings in [12, 13].

In the context of linear topological vector space, Ch. Klein et al. [14] presented the Riemann integral with respect to any non-atomic measure of functions. R. Paluga et al. [15] presented several properties of Henstock-Kurzweil integrals on a topological space. Recently, H. Kalita et al. have introduced ap-Henstock-Kurzweil integrals on topological vector space in [11].

On the other hand, the concept of statistical convergence for sequences of real or complex numbers has been investigated by Fast in [3]. It uses the notion of natural density of subsets of natural numbers. If $K \subseteq \mathbb{N}$, then the natural density of K is denoted by $d(K)$ and is defined by $d(K) = \lim\limits_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in K\}|$. Since then, it has been the subject of investigation in other articles, including [4, 5, 6], and [7]. Maio et al. [9] examined statistical convergence in uniform and topological spaces, and illustrate the applications of this convergence to function spaces, hyperspaces, and selection principles. Also, the

Received: 22.01.2024. In revised form: 20.07.2024. Accepted: 14.09.2024
2020 *Mathematics Subject Classification.* 26A39, 26A46, 54A20, 54B20.
Key words and phrases. Topological Henstock-Kurzweil integral, Cauchy's criterion, statistical convergence, topological vector spaces.
Corresponding author: Hemanta Kalita; hemanta30kalita@gmail.com; hemantakalita@vitbhopal.ac.in

notion of μ-statistically convergent function sequences was presented in [2]. Recently, statistical Riemann and Lebesgue integrable sequences of functions have been studied by Srivastava et al. in [17] and investigated Korovkin type approximation theorems for such class of functions. On the other hand, in [16], J. Sokolowski et al. defined the concept of topological derivatives.

The idea of [9] motivated us to introduce topological Henstock-Kurzweil integrals on a μ-cell of a topological vector space. The fascinating ideas of [17] impels us to extent the convergence of topological Henstock-Kurzweil integrable functions to statistical convergence in our setting.

The paper is organized as follows: in Section 2, some basic notions and terminologies has been introduced which will be needed throughtout the paper. In Section 3, we introduce the notion of topological Henstock-Kurzweil integral (denoted as THK integral) of a μ-cell-valued functions along with some properties. In Section 4, we extend the theory of convergence of THK integral to statistically convergence of THK integrable functions. Moreover, we introduce statistically equi-integrability for THK integrable functions to prove sequence of equi-integrable THK integrable functions are statistically Cauchy.

2. Preliminaries

Let δ be a positive function on the closed interval $[a, b]$. We say $P = \Big\{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\Big\}$ is δ-fine tagged partition of $[a, b]$ if $\Big\{[x_{i-1}, x_i] : 1 \leq i \leq n\Big\}$ is a partition of $[a, b]$, $t_i \in [a, b]$ and $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for every i, $1 \leq i \leq n$. Riemann sum is defined as $S(f, P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ if exists.

Definition 2.1. [8, Definition 9.3] *A function $f : [a, b] \to \mathbb{R}$ is called Henstock-Kurzweil integrable on $[a, b]$ if there exists a real number L with the following property: for each $\epsilon > 0$, there exists a positive function δ such that $|S(f, P) - L| < \epsilon$ whenever P is tagged partition of $[a, b]$. We called $L = (HK) \int_a^b f$.*

Recall Henstock-Kurzweil equi-integrable functions as follows.

Definition 2.2. *A sequence $(f_n)_{n=1}^{\infty}$ of Henstock-Kurzweil integrable functions on $[a, b]$ is said to be Henstock-Kurzweil equi-integrable on $[a, b]$ if for each $\epsilon > 0$ there exists a gauge δ independent of n, on $[a, b]$ such that*

$$\sup_{n \in \mathbb{N}} \left| S(f_n, P) - (HK) \int_a^b f_n \right| < \epsilon$$

for each δ-fine partition P of $[a, b]$.

Let $\mathcal{X}$ be a compact Hausdorff topological vector space over real numbers. We denote support of μ by $supp(\mu) = \{x \in \mathcal{X} : \mu(\mathcal{U}) > 0$ for every θ nbd $\mathcal{U}$ of $x\}$. Throughout our work $\mu(\mathcal{U}) > 0$ is understood. If $A \subset \mathcal{X}$, then $\overline{A}$ denotes the closure of A in $\mathcal{X}$. ∂A denotes boundary of A where $\partial A = \overline{A} \cap \overline{\mathcal{X} \setminus A}$. Let $Bo(\mathcal{X})$ denotes the Borel sigma algebra of $\mathcal{X}$ containing all compact subsets of $\mathcal{X}$. Recalling an element $A \in Bo(\mathcal{X})$ is called μ-continuity set if $\mu(\partial A) = 0$.

Lemma 2.1. [11] *Let $\mathcal{X}$ be a topological vector space. Then there is a local base $\mathcal{B}$ of θ (the zero vector), satisfying the following:*

(1) If $\mathcal{U}, \mathcal{V} \in \mathcal{B}$, then there is a $\mathcal{W} \in \mathcal{B}$ with $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$.
(2) If $\mathcal{U} \in \mathcal{B}$ and $x \in \mathcal{U}$, there is a $\mathcal{V} \in \mathcal{B}$ such that $x + \mathcal{V} \subseteq \mathcal{U}$.

(3) *If $\mathcal{U} \in \mathcal{B}$, there is a $\mathcal{V} \in \mathcal{B}$ such that $\mathcal{V} + \mathcal{V} \subseteq \mathcal{U}$.*
(4) *If $\mathcal{U} \in \mathcal{B}$ and $x \in \mathcal{X}$, then there is $k \in \mathbb{R}$ such that $x \in k\mathcal{U}$*
(5) *If $\mathcal{U} \in \mathcal{B}$ and $0 < |k| \leq 1$, then $k\mathcal{U} \subseteq \mathcal{U}$ and $k\mathcal{U} \in \mathcal{B}$.*
(6) *$T\{\mathcal{U} : \mathcal{U} \in \mathcal{B}\} = \{\theta\}$.*

Conversely, given a collection $\mathcal{B}$ of subsets containing θ and satisfying the above conditions, there is a topology for $\mathcal{X}$ making $\mathcal{X}$ a topological vector space and having B as a local base at θ.

Let $\left(\mathcal{X}, \mathcal{T}\right)$ be a topological space. A family $\mathcal{F} = \{\mathcal{Q}_i : i \in \mathbb{N}\}$ of subsets of $\mathcal{X}$ is a filter in $\mathcal{X}$ if the following are satisfied:

(1) For every $i \in \mathbb{N}$, $\mathcal{Q}_i \neq \emptyset$.
(2) For $\mathcal{Q}_i, \mathcal{Q}_j \in \mathcal{F}$ then $\mathcal{Q}_i \cap \mathcal{Q}_j \in \mathcal{F}$
(3) If $\mathcal{Q} \in \mathcal{F}$, and $\mathcal{Q} \subseteq \mathcal{B}$ then $\mathcal{B} \in \mathcal{F}$.

The filter $\mathcal{F}$ converges to $x \in \mathcal{X}$ if for every θ-nbd $\mathcal{U}$ (θ is zero vector of $\mathcal{X}$) there exists $\mathcal{Q} \in \mathcal{F}$ such that $\mathcal{Q} - x \subseteq \mathcal{U}$. We say $\mathcal{F}$ is Cauchy if for every θ-nbd $\mathcal{U}$ there exists $\mathcal{Q} \in \mathcal{F}$ such that $\mathcal{Q} \subseteq \mathcal{U}$. Let $\mathcal{X}$ be a topological vector space. We say that $\mathcal{X}$ is complete if every Cauchy filter in $\mathcal{X}$ converges. We say that $\mathcal{X}$ is locally convex if there is a local base at θ whose members are convex.

Definition 2.3. *(1) Let $\mathcal{Q}, \mathcal{R} \in \mathcal{F}$. We say $\mathcal{Q}, \mathcal{R}$ are non overlapping if $\mathcal{Q} \cap \mathcal{R} = \emptyset$.*
(2) Let $\mathcal{G}$ be a subfamily of $\mathcal{F}$. We say that $\mathcal{G}$ is a fine cover of $E \subset \mathcal{X}$ if $\mu(\mathcal{Q}) \to 0$ whenever $x \in \mathcal{Q}$ for all $x \in E$.

Definition 2.4. *We call $\mathcal{F}$ is a family of μ-cells if it satisfies the following conditions:*

(1) *Given $\mathcal{Q} \in \mathcal{F}$ and a constant $a > 0$ there exists a division $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\}$ and there exists $\mathcal{U}$, such that $\inf\left[\mu\left(\mathcal{U}(\mathcal{Q}_i)\right)\right] < a$ for $i = 1, 2, ..., m$;*
(2) *Given $A,\ \mathcal{Q} \in \mathcal{F}$ and $A \subset \mathcal{Q}$, there exists a division $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\}$ of $\mathcal{Q}$ and there exists $\mathcal{U}$, such that $\mathcal{U}(A) = \mathcal{U}(\mathcal{Q}_1)$;*
(3) *$\mu(\partial\mathcal{Q}) = 0$ for each $\mathcal{Q} \in \mathcal{F}$ where $\partial\mathcal{Q}$ is the boundary of $\mathcal{Q}$.*

Let us construct an example supporting Definition 2.4 as below.

Example 2.1. *Consider $n > 1$. Let $\mathcal{X}$ be the unit cube $[0,1]^n$ of $\mathbb{R}^n$ endowed with the Euclidean distance in $\mathbb{R}^n$ and with the n-dimensional Lebesgue measure μ_n. Let $0 < a \leq 1$, the system of $\mathcal{F}$ of all non empty closed sub-intervals $\mathcal{Q}$ of $[0,1]^n$ such that $\mu_n(\mathcal{Q}) > a\mu_n(B)$ for some ball B containing $\mathcal{Q}$ is the family of μ_n-cell.*

We call $\mathcal{F}$ is μ-filter if for each subset E of $\mathcal{X}$ and for each subfamily $\mathcal{G}$ of $\mathcal{F}$ that is a fine cover of E, there exists a countable system $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_j, ..\right\}$ of pairwise non-overlapping cells of $\mathcal{G}$ such $\mu(E) \setminus \mu(\cup\mathcal{Q}_j) \geq 0$.

Definition 2.5. *Let $\mathcal{Q} \in \mathcal{F}$, $E \subset \mathcal{Q}$ and δ be a gauge on $\mathcal{Q}$. A collection $P = \left\{(x_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ of finite ordered pairs of points and cells is said to be*

(1) *partition of $\mathcal{Q}$ if $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\} \subseteq Bo(\mathcal{X})$ is a division of $\mathcal{Q}$ and $x_i \in \mathcal{Q}_i$ for $i = 1, 2, ..., m$;*

(2) a partial partition of $\mathcal{Q}$ if $\left\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\right\} \subseteq Bo(\mathcal{X})$ is a subsystem of a division of $\mathcal{Q}$ and $x_i \in \mathcal{Q}_i$ for $i = 1, 2, ..., m$;

(3) δ-fine if $\mu\left(\mathcal{Q}_i\right) < \delta(x_i)$ for $i = 1, 2, ..., m$;

(4) E-anchored if the points $x_1, x_2, ..., x_m$ belongs to E.

3. Henstock-Kurzweil integrals on topological vector spaces

In this section, we introduce topological Henstock-Kurzweil integral on a μ-cell $\mathcal{Q}$ of a topological vector space $\mathcal{X}$. The following Cousin's type lemma addresses the existence of δ-fine partitions of a given cell $\mathcal{Q}$.

Lemma 3.2. *If δ is a gauge on $\mathcal{Q}$, then there exists a δ-fine partition of $\mathcal{Q}$.*

Proof. The proof follows similar technique of [1, Lemma 2.2.1], so omitted. □

Let $P = \left\{(x_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ be a partition of $\mathcal{Q} \in \mathcal{F}$ and $f : \mathcal{Q} \to \mathcal{X}$ be a given function. We define Riemann sum as $S(f, P) = \sum_{i=1}^{m} f(x_i)\mu\left(\mathcal{Q}_i\right)$. We are ready to define topological Henstock-Kurzweil integral as follows:

Definition 3.6. *A function $f : \mathcal{Q} \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable on $\mathcal{Q}$ with respect to μ if there exists a real number A such that for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$, with $|S(f, P) - A| < \mu(\mathcal{U})$ whenever $P = \left\{(x_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ is δ-fine partition on $\mathcal{Q}$ and $\mu(\mathcal{U}) > 0$.*

The number A is said to be the Henstock-Kurzweil integral of f on $\mathcal{Q}$ with respect to μ or topological Henstock-Kurzweil integral (in short, THK integral) on $\mathcal{Q}$ with respect to μ and we write $A = \int_{\mathcal{Q}} f d\mu$. Now-onwards, we denote THK for topological Henstock-Kurzweil integral of f on $\mathcal{Q}$ with respect to μ. It is not hard to find the uniqueness of the integral value of a given THK integrable function $f : \mathcal{Q} \to \mathbb{R}$. The collection of all THK integrable functions on $\mathcal{Q}$ with respect to μ shall be denoted by THK($\mathcal{Q}$). The Alexiewicz semi-norm on THK($\mathcal{Q}$) can be defined by

$$||f|| = \sup_{\mathcal{Q} \in \mathcal{F}} \left| \int_{\mathcal{Q}} f \right|$$

where the integral is in the sense of THK. It is easy to see $(THK(\mathcal{Q}), ||.||)$ is a linear space.

Example 3.2. *The function $f(x) = c$ with $c \in \mathbb{R}$, for all $x \in \mathcal{Q}$ is THK integrable on $\mathcal{Q}$ with $\int_{\mathcal{Q}} f d\mu = c\mu(\mathcal{Q})$.*

Proof. Let $\mathcal{U}$ be given. Since $f(x) = c,\ \forall\ x \in \mathcal{Q}$. In this situation for any tagged partition P of $\mathcal{Q}$, we have $S(f, P) = \sum_{i=1}^{n} f(t_i)\mu(\mathcal{Q}_i) = c\mu(\mathcal{Q})$. Hence for any tagged partition P,

$$\begin{aligned} |S(f, P) - c\mu(\mathcal{Q})| &= |c\mu(\mathcal{Q}) - c\mu(\mathcal{Q})| \\ &= 0 < \mu(\mathcal{U}). \end{aligned}$$

Let P be a tagged partition, that is δ-fine, and $\mathcal{U}$ is any θ-nbd, it holds for all $\mathcal{U}$. Thus $f(x) = c$ is THK on $\mathcal{Q}$ with $\int_{\mathcal{Q}} f d\mu = c\mu(\mathcal{Q})$. □

Few simple properties of THK integrals are as follows.

Theorem 3.1. *Let $f, g \in THK(\mathcal{Q})$, then $f + g \in THK(\mathcal{Q})$, and $\int_{\mathcal{Q}}(f+g)d\mu = \int_{\mathcal{Q}} f d\mu + \int_{\mathcal{Q}} g d\mu$.*

Theorem 3.2. *If $f \in THK(\mathcal{Q})$ and $k \in \mathbb{R}$, then $kf \in THK(\mathcal{Q})$ and $\int_{\mathcal{Q}} kf d\mu = k \int_{\mathcal{Q}} f d\mu$.*

Theorem 3.3. *If $f \in THK(\mathcal{Q})$ and $f(x) \geq 0$ for each $x \in \mathcal{Q}$, then $\int_{\mathcal{Q}} f d\mu \geq 0$.*

Corollary 3.1. *Let $f, g \in THK(\mathcal{Q})$. If $f \geq g$ for each $x \in \mathcal{Q}$, then $\int_{\mathcal{Q}} f d\mu \geq \int_{\mathcal{Q}} g d\mu$.*

Proposition 3.1. *If $f : \mathcal{Q} \to \mathbb{R}$ be THK integrable on $\mathcal{Q}$ and $|f(x)| < M$ with $M \in \mathbb{R}$ for all $x \in \mathcal{Q}$, then $|\int_{\mathcal{Q}} f| \leq M\big(\mu(\mathcal{Q})\big)$.*

Next, we prove Cauchy criterion for THK integrable functions on $\mathcal{Q}$.

Theorem 3.4. (The Cauchy Criterion) A function $f : \mathcal{Q} \to \mathbb{R}$ is THK integrable on $\mathcal{Q}$ if and only if for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ such that $|S(f, P_1) - S(f, P_2)| < \mu(\mathcal{U})$ whenever P_1 and P_2 are δ-fine partitions of $\mathcal{Q}$.

Proof. Let $f : \mathcal{Q} \to \mathbb{R}$ be a THK integrable function on $\mathcal{Q}$. By definition, for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$, $|S(f, P) - \int_{\mathcal{Q}} f d\mu| < \mu(\mathcal{U})$ whenever P is δ-fine partition of $\mathcal{Q}$. Consider $P_1,\ P_2$ are δ-fine partitions of $\mathcal{Q}$, then

$$\begin{aligned}|S(f, P_1) - S(f, P_2)| &= |S(f, P_1) - \int_{\mathcal{Q}} f d\mu + \int_{\mathcal{Q}} f d\mu - S(f, P_2)| \\ &\leq |S(f, P_1) - \int_{\mathcal{Q}} f d\mu| + |S(f, P_2) - \int_{\mathcal{Q}} f d\mu| \\ &< \frac{\mu(\mathcal{U})}{2} + \frac{\mu(\mathcal{U})}{2} = \mu(\mathcal{U}).\end{aligned}$$

Conversely, let $n \in \mathbb{N}$ and δ_n be a gauge on $\mathcal{Q}$ such that $|S(f, P_n) - S(f, P'_n)| < \mu(\mathcal{U})$ whenever P_n, P'_n are δ_n-fine partitions of $\mathcal{Q}$. Let $\rho_n(x) = \min\big\{\delta_1(x), \delta_2(x), ..., \delta_n(x)\big\}$ be a gauge on $\mathcal{Q}$. By Lemma 3.2, there exists an ρ_n-fine partition P_n of $\mathcal{Q}$, for each $n \in \mathbb{N}$. Let $\mathcal{U}$ be given and choose a positive natural number N such that $\frac{1}{N} < \mu(\mathcal{U})$. If m, n are positive natural $(n < m)$ such that $n \geq N$, then P_n, P_m are ρ_N-fine partitions on $\mathcal{Q}$. Hence $|S(f, P_n) - S(f, P_m)| < \frac{1}{N} < \mu(\mathcal{U})$. Clearly, $\Big\{S(f, P_n)\Big\}_{n=1}^{\infty}$ is Cauchy sequence of real numbers, and convergent. Let $\lim\limits_{n\to\infty} S(f, P_n) = \Delta$. Then $|S(f, P_n) - \Delta| < \mu(\mathcal{U})$, for each $n \geq N$. Let P be a ρ_N-fine partitions on $\mathcal{Q}$, then

$$\begin{aligned}|S(f, P) - \Delta| &\leq |S(f, P) - S(f, P_N)| + |S(f, P_N) - \Delta| \\ &< \frac{\mu(\mathcal{U})}{2} + \frac{\mu(\mathcal{U})}{2} = \mu(\mathcal{U}).\end{aligned}$$

Thus $f \in THK(\mathcal{Q})$ and $\Delta = \int_{\mathcal{Q}} f d\mu$. □

In the following theorem, we shall prove that THK integrability of f on a set $\mathcal{Q}$ implies its THK integrability on each subcells of $\mathcal{Q}$.

Theorem 3.5. *If $f \in THK(\mathcal{Q})$, and if $\mathcal{A}$ is a subcell of $\mathcal{Q}$, then $f \in THK(\mathcal{Q})$ and $\int_{\mathcal{A}} f d\mu = \int_{\mathcal{Q}} f\chi_{\mathcal{A}} d\mu$.*

Proof. Let $\mathcal{U}$ be given. By Theorem 3.4, there exists a gauge δ on $\mathcal{Q}$ so that $|S(f, P_1) - S(f, P_2)| < \mu(\mathcal{U})$ for each pair of δ-fine partitions P_1 and P_2 of $\mathcal{Q}$. Given that there exists a division $P = \big\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\big\}$ of $\mathcal{Q}$ and $A \subset \mathcal{Q}$, such that $A = \mathcal{Q}_1$. For each $k \in$

$\{2,3,...,m\}$ we fix a δ-fine partition P_k of $\mathcal{Q}_k$. If R_1 and R_2 are δ-fine partitions of A, then $R_1 \cup \bigcup\limits_{k=2}^{m} P_k$ and $R_2 \cup \bigcup\limits_{k=2}^{m} P_k$ are δ-fine partitions of $\mathcal{Q}$. Thus,

$$|S(f,R_1) - S(f,R_2)| = |S(f,R_1) + \sum_{k=2}^{m} S(f,P_k) - S(f,R_2) - \sum_{k=2}^{m} S(f,P_k)|$$

$$\leq \left|S\Big(f, R_1 \cup \bigcup_{k=2}^{m} P_k\Big) - S\Big(f, R_2 \cup \bigcup_{k=2}^{m} P_k\Big)\right| < \mu(\mathcal{U}).$$

Thus by Theorem 3.4, $f \in THK(A)$. □

Proposition 3.2. *Let $f : \mathcal{Q} \to \mathbb{R}$ be a THK integrable function on $\mathcal{Q}$. If $\Big\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\Big\}$ is a division of $\mathcal{Q}$, then $f \in THK(\mathcal{Q}_1) \cap ... \cap THK(\mathcal{Q}_m)$ and $\int_{\mathcal{Q}} f d\mu = \sum\limits_{i=1}^{m} \int_{\mathcal{Q}_i} f d\mu$.*

Proof. Given $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ such that $|S(f,P) - \int_{\mathcal{Q}} f d\mu| < \mu(\mathcal{U})$, for each δ-fine partition P of $\mathcal{Q}$. By Theorem 3.5, $f \in THK(\mathcal{Q}_i)$ for $i = 1,2,...,m$ such that $\delta_i(x) < \delta(x)$ for each $x \in \mathcal{Q}_i$ and such that $|S(f,P_i) - \int_{\mathcal{Q}_i} f d\mu| < \frac{\mu(\mathcal{U})}{m}$, for each δ_i-fine partitions P_i of $\mathcal{Q}$. Therefore $P = P_1 \cup ... \cup P_m$ is a δ-fine partition of $\mathcal{Q}$. Consequently,

$$|S(f,P) - \sum_{i=1}^{m} \int_{\mathcal{Q}_i} f d\mu| \leq |S(f,P_1) - \int_{\mathcal{Q}_1} f d\mu| + ... + |S(f,P_m) - \int_{\mathcal{Q}_m} f dm| < \mu(\mathcal{U}).$$

Thus $\int_{\mathcal{Q}} f d\mu = \sum\limits_{i=1}^{m} \int_{\mathcal{Q}_i} f d\mu$. □

We define indefinite THK integral of a given THK integrable function f as follows:

Definition 3.7. *Let M be the collection of all subcells of $\mathcal{Q}$ and A be any subcell of $\mathcal{Q}$. A function $F : M \to \mathbb{R}$ defined by $F(A) = \int_A f d\mu$ is called an indefinite THK integral of f.*

It is easy to see by Proposition 3.2, each additive THK integrable function is an indefinite THK integrable on $\mathcal{Q}$.

Lemma 3.3. *(Saks-Henstock Lemma) A function $f : \mathcal{Q} \to \mathbb{R}$ is THK integrable on $\mathcal{Q}$ if and only if there exists an additive cell function π defined on the family of subcells of $\mathcal{Q}$ such that for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ with $\sum\limits_{(x_i,\mathcal{Q}_i)\in P} |\pi(\mathcal{Q}_i) - f(x_i)\mu(\mathcal{Q}_i)| < \mu(\mathcal{U})$, for each δ-fine partial partition P of $\mathcal{Q}$.*

Proof. The proof is of similar to [1, Lemma 2.4.1]. □

Next, we see Lebesgue integrable function on each cell $\mathcal{Q}$ is THK integrable and the two integrals coincide. Let us denote $(L) \int_{\mathcal{Q}} f d\mu$ be Lebesgue integrable functions with respect to μ.

Theorem 3.6. *Let $f : \mathcal{Q} \to \mathbb{R}$ be a function. If f is Lebesgue integrable on $\mathcal{Q}$, with respect to μ, then f is THK integrable on $\mathcal{Q}$ and $(L) \int_{\mathcal{Q}} f d\mu = \int_{\mathcal{Q}} f d\mu$.*

Proof. The proof is similar to [1, Theorem 2.5.2]. □

The following counter example shows that the converse of Theorem 3.6 is always not true.

Example 3.3. *Let $\mathcal{C}$ be Cantor topological subspace of $[0,1]$, where $\mathcal{C} = \cap \mathcal{C}_n$, and $\mathcal{C}_n$ is open intervals from $\mathcal{C}_{n-1}$ where $\mathcal{C}_0 = [0,1]$. It is easy to see the Cantor set $\mathcal{C}$ with the interval components of the space $\mathcal{C}_n$ form the basis for the topology on $\mathcal{C}$. It is well known that Hausdorff measure H^s is a non-trivial radon measure on $\mathbb{R}^n$ if and only if $s = n$. Let μ be $log_3 2$ dimensional Hausdorff measure and $f : \mathcal{C} \to \mathbb{R}$ be the function by*

$$f(x) = \begin{cases} \frac{(-1)^n 3^n}{n}, \ x \in \left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right] \cap \mathcal{C}, \ n = 1,2,3... \\ \\ 0, \ x = 0 \end{cases} \tag{3.1}$$

Let $\mathcal{U}$ be any θ-nbd and let $N \in \mathbb{N}$ be such that $\mu(\mathcal{U})N \geq 2$ and $\left| \sum_{j=n+1}^{\infty} \frac{(-1)^{j+1}}{j} \right| < \frac{\mu(\mathcal{U})}{2}$ for each $n \geq N$. Let us construct a gauge δ on $\mathcal{C}$ such that

$$\delta(x) = \begin{cases} K, \ x \in (x - \delta(x), x + \delta(x)) \cap C \\ \\ \frac{1}{3^{n-1}}, \ x = 0 \end{cases}$$

Next consider a δ-fine partition $P = \Big\{ (x_1, \mathcal{Q}_1), (x_2, \mathcal{Q}_2), ..., (x_m, \mathcal{Q}_m) \Big\}$ of $\mathcal{C}$ such that $\mathcal{Q}_1 = [0,c] \cap \mathcal{C}$ where $[0,c] \subset [0,1]$. Then $x_1 = 0$ and $c < \frac{1}{3^{n-1}}$. Next, if we consider $n \in \mathbb{N}$ such that $\frac{1}{3^n} < c < \frac{1}{3^{n-1}}$ then $n \geq N$. In this situation

$$\bigcup_{j=2}^{m} \mathcal{Q}_j = \begin{cases} \Big([c, \frac{1}{3^{n-1}}] \cup [\frac{2}{3^{n-1}}, 1] \Big) \cap \mathcal{C} \text{ if } c > \frac{2}{3^n} \\ \\ ([\frac{2}{3^n}, 1] \cap \mathcal{C}) \text{ if } c < \frac{2}{3^n} \end{cases}$$

Then

$$\Big| S(f,P) - log2 \Big| = \begin{cases} \mu(\mathcal{U}) \text{ if } c \geq \frac{2}{3^n} \\ \\ \frac{\mu(\mathcal{U})}{2} \text{ if } c < \frac{2}{3^n} \end{cases}$$

Hence f is THK on $\mathcal{C}$. But $(L) \int_{\mathcal{C}} |f| d\mu = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. Hence f is not Lebesgue integrable on $\mathcal{C}$.

It is known that, any function f is Lebesgue integrable if and only if both f and $|f|$ are Henstock-Kurzweil integrable. We shall investigate this for THK integrable functions defined on $\mathcal{Q}$ with respect to μ.

Definition 3.8. *(1) Let M be the collection of all subcells of $\mathcal{Q}$. An additive cell function $\pi : M \to \mathbb{R}$ is said to be absolutely continuous with respect to μ if for each $\mathcal{U}$, there exists a constant $\nu > 0$ such that $\sum_{i=1}^{m} |\pi(\mathcal{Q}_i)| < \mu(\mathcal{U})$ whenever $\{\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_m\}$ is a collection of non-overlapping subcells of $\mathcal{Q}$ with $\sum_{i=1}^{m} \mu(\mathcal{Q}_i) < \nu$. We denote it by AC_T.*

(2) We say π is ACG_T on $\mathcal{Q}$ if there exists a countable sequence of closed subcells $\mathcal{Q}_i$ such that $\bigcup_i \mathcal{Q}_i = \mathcal{Q}$ and π is AC_T on $\mathcal{Q}_i$ for each $i \in \mathbb{N}$.

Theorem 3.7. *Let $\mathcal{Q}$ be a cell. A function $f : \mathcal{Q} \to \mathbb{R}$ is THK integrable on $\mathcal{Q}$ if and only if there exists an additive cell function F which is ACG_T on $\mathcal{Q}$ such that $F'_T(x) = f(x)$ μ-a.e. in $\mathcal{Q}$.*

Proof. The proof is similar to [1, Theorem 2.6.1]. □

Definition 3.9. *Let $\mathcal{Q}$ be a cell. We say that a function $f : \mathcal{Q} \to \mathbb{R}$ is absolutely THK integrable on $\mathcal{Q}$ if $|f|$ is THK integrable on $\mathcal{Q}$.*

Now, we define topological derivative at $x \in \mathcal{X}$ as follows.

Definition 3.10. *Let $x \in \mathcal{X}$ and $A \in \mathcal{F}$. Let F be a cell function defined on $\mathcal{F}$.*

(1) We define the upper topological derivative of F at x, with respect to μ as $U_T F(x) = \lim\limits_{A\to x} \sup \frac{F(A)}{\mu(A)}$ whenever the limit superior is taken over all sequences of cells A such that $x \in A$ and $\partial A \to 0$.

(2) We define the lower topological derivative of F at x, with respect to μ as $L_T F(x) = \lim\limits_{A\to x} \inf \frac{F(A)}{\mu(A)}$ whenever the limit inferior is taken over all sequences of cells A such that $x \in A$ and $\partial A \to 0$.

If $U_T F(x) = L_T F(x) < \infty$, then we call F is topological differentiable at x and their common value is called the topological derivative of F at x and we denote it by $F'_T(x)$.

Theorem 3.8. *If f is a non-negative THK integrable function on a cell $\mathcal{Q}$ and F is its indefinite THK integral, then F is topological differentiable μ-a.e. in $\mathcal{Q}$ and $F'_T(x) = f(x)$ a.e. in $\mathcal{Q}$,*

Proof. We have, for all $x \in \mathcal{Q}$, $L_T F(x) \le U_T F(x)$. In order to prove $F'_T(x) = f(x)$ a.e. on $\mathcal{Q}$, we need to prove $U_T F(x) \le f(x) \le L_T F(x)$ μ-a.e. in $\mathcal{Q}$. Let for $\alpha, \beta \in \mathbb{Q},\ \alpha < \beta.$,

$$\kappa_{\alpha,\beta} = \Big\{ x \in \mathcal{Q} :\ U_T F(x) > \beta > \alpha > f(x) \Big\}.$$

For a given $\mathcal{U}$, by Lemma 3.2, there exists a gauge δ on $\mathcal{Q}$ such that

$$\sum_{j=1}^{m} \Big| F(\mathcal{Q}_j) - f(x_j)\mu(\mathcal{Q}_j) \Big| < \mu(\mathcal{U}),$$

for each δ-fine partition $\Big\{ (x_j, \mathcal{Q}_j) \Big\}_{j=1}^{m}$ of $\mathcal{Q}$. Consider $\daleth$ be the system of all cells $C \subset \mathcal{Q}$ such that $F(C) > \beta\mu(C)$ and there exists $x \in C \cap \kappa_{\alpha,\beta}$ with $\mu\Big(\mathcal{U}(C)\Big) < \delta(x)$. Clearly the system $\daleth$ is a fine cover of $\kappa_{\alpha,\beta}$. Therefore $\daleth$ be a μ-filter family. Then there exists a system of pairwise non-overlapping cells $\Big\{ C_j \Big\}_{j=1}^{m} \subset \daleth$ such that $\mu(\kappa_{\alpha,\beta}) \le \sum_{j=1}^{m} \mu(C_j)$. For $j = 1, 2, ..., m$, let $x_j \in C_j \cap \kappa_{\alpha,\beta}$ such that $\mu(C_j) < \delta(x_j)$. Since $\Big\{ (x_j, C_j) \Big\}_{j=1}^{m}$ is a δ-fine partial partition of $\mathcal{Q}$, we have

$$\begin{aligned} \beta \sum_{j=1}^{m} \mu(C_j) &< \sum_{j=1}^{m} F(C_j) \\ &\le \sum_{j=1}^{m} |F(C_j) - f(x_j)\mu(C_j)| + \sum_{j=1}^{m} f(x_j)\mu(C_j) \\ &< \mu(\mathcal{U}) + \alpha \sum_{j=1}^{m} \mu(C_j). \end{aligned}$$

So, $(\beta - \alpha) \sum_{j=1^m} \mu(C_j) < \mu(\mathcal{U})$. Consequently, if $\mu(\mathcal{U}) \to 0$, then $\sum_{j=1}^{m} \mu(C_j) = 0$ and hence $\mu(\kappa_{\alpha,\beta}) = 0$. Hence $U_T F(x) \le f(x)$ μ-a.e. on $\mathcal{Q}$. Similar way we can find $L_T F(x) \ge f(x)$ μ-a.e. on $\mathcal{Q}$. So, $U_T F(x) \le f(x) \le L_T F(x)$ μ-a.e. in $\mathcal{Q}$. □

For analyzing the power of an integration theory, convergence results are crucial. The result we got for the THK integral is shown below.

Theorem 3.9. *Let $f, f_n : \mathcal{Q} \to \mathbb{R}$, $n = 1, 2, ...$ be given where the integral $\int_{\mathcal{Q}} f_n$ exists for all $n \in \mathbb{N}$. Let $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in \mathcal{Q}$. Further, Let for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ such that*

$$|S(f_n, P) - \int_{\mathcal{Q}} f_n| < \mu(\mathcal{U}) \tag{3.2}$$

for every δ-fine partition P of $\mathcal{Q}$ and for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \int_{\mathcal{Q}} f_n = \int_{\mathcal{Q}} f$.

Proof. Let $\mathcal{U}$ be given. By (3.2), we can find a gauge δ on $\mathcal{Q}$ such that for every δ-fine partition P of $\mathcal{Q}$, we have $|S(f_n, P) - \int_{\mathcal{Q}} f_n d\mu| < \frac{\mu(\mathcal{U})}{2}$ for $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f_n = f$, we have, for every fixed δ-fine partition P, there exists a positive integer n_0 such that for $n \geq n_0$, we have

$$\left|S(f_n, P) - S(f, P)\right| = \left|\sum_{j=1}^{k} f_n(\mathcal{Q}_j)\mu(\mathcal{Q}_j)\right| < \frac{\mu(\mathcal{U})}{2}.$$

This gives, $\lim_{n\to\infty} S(f_n, P) = S(f, P)$. Hence for any δ-fine partition P of $\mathcal{Q}$, there is a positive integer n_0 such that for $n > n_0$,

$$|S(f, P) - \int_{\mathcal{Q}} f_n d\mu| < \mu(\mathcal{U}). \tag{3.3}$$

From (3.3), for all positive integers, $n, l > n_0$, $\left|\int_{\mathcal{Q}} f_n d\mu - \int_{\mathcal{Q}} f_l d\mu\right| < \mu(\mathcal{U})$ holds. Clearly, $\left(\int_{\mathcal{Q}} f_n d\mu\right)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore,

$$\lim_{n\to\infty} \int_{\mathcal{Q}} f_n d\mu = \tau \in \mathbb{R}. \tag{3.4}$$

Now, By (3.3),

$$\begin{aligned}\left|S(f, P) - \tau\right| &\leq \left|S(f, P) - \int_{\mathcal{Q}} f_n d\mu\right| + \left|\int_{\mathcal{Q}} f_n d\mu - \tau\right| \\ &< \mu(\mathcal{U}) + \left|\int_{\mathcal{Q}} f_n d\mu - \tau\right| \text{ for } n > n_0.\end{aligned}$$

By (3.4), we obtain for every δ-fine partition P of $\mathcal{Q}$, $|S(f, P) - \tau| < \mu(\mathcal{U})$ for $n > n_0$. Hence $\int_{\mathcal{Q}} f d\mu$ exists and (3.2) satisfies. □

Definition 3.11. *A sequence $(f_n)_{n=1}^{\infty}$ in $THK(\mathcal{Q})$ is said to be THK equi-integrable on $\mathcal{Q}$ if for each $\mathcal{U}$, there exists a gauge δ on $\mathcal{Q}$ such that $\sup_{n\in\mathbb{N}} |S(f_n, P) - \int_{\mathcal{Q}} f_n| < \mu(\mathcal{U})$ whenever $P = \left\{(x_i, \mathcal{Q}_i)\right\}_{i=1}^{m}$ is δ-fine partition on $\mathcal{Q}$.*

Theorem 3.9 gives a sufficient condition for a sequence of THK integrable functions tends to an integrable limit and for the integrals of the members of the sequence tends to the integral of the limit function. The convergence $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in \mathcal{Q}$ is pointwise convergence and the sufficient condition is the equi-integrability of the sequence of THK integrable functions (f_n). Using equi-integrable sequence of THK integrable functions $f_n : \mathcal{Q} \to \mathbb{R}$, Theorem 3.9 can be reformulated as follows.

Theorem 3.10. *Let $f : \mathcal{Q} \to \mathbb{R}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of THK equi-integrable function on $\mathcal{Q}$. If $f_n \to f$ pointwise on $\mathcal{Q}$, then $f \in THK(\mathcal{Q})$ and $\lim_{n\to\infty} \int_{\mathcal{Q}} |f_n - f| d\mu = 0$.*

Proof. Let $\mathcal{U}$ be given. Since $(f_n)_{n=1}^{\infty}$ is a sequence of THK equi-integrable functions on $\mathcal{Q}$. By Saks-Henstock Lemma, choose a gauge δ, independent of n on $\mathcal{Q}$ such that

$$\sum_{(x_i,\mathcal{Q}_i)\in P} |f_n(x_i) - f_p(x_i)| < \mu(\mathcal{U}) \tag{3.5}$$

for each δ-fine partial partition P of $\mathcal{Q}$. According to Cousin's Lemma, we fix a δ-fine partition P' of $\mathcal{Q}$. Since $f_n \to f$ pointwise on $\mathcal{Q}$, we may choose $N \in \mathbb{N}$ so that

$$\sum_{(x,\mathcal{Q})\in P'} |f_n(x) - f_p(x)| < \mu(\mathcal{U}) \tag{3.6}$$

for all $n, p \geq N$. Now from (3.5) and (3.6),

$$\begin{aligned} |f_n - f_p| &\leq \sum_{x,\mathcal{Q}\in P'} |f_n(x) - f_p(x)| + \sum_{x,\mathcal{Q}\in P'} |f_n(x) - f(x)| + \sum_{x,\mathcal{Q}\in P'} |f_p(x) - f(x)| \\ &< \mu(\mathcal{U}) \end{aligned}$$

for all $n, p \geq N$. By completeness of $\mathbb{R}$, there exists a function $F : \mathcal{Q} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathcal{Q}} | \int_{\mathcal{Q}} f_n(x) d\mu - F(\mathcal{Q})| = 0. \tag{3.7}$$

By (3.5), $\sum_{(x,\mathcal{Q})\in P} |f(x)\mu(\mathcal{Q}) - F(\mathcal{Q})| < \mu(\mathcal{U})$ for every δ-fine partial partition P of $\mathcal{Q}$. So, $f \in THK(\mathcal{Q})$ and $F(x) = \int_{\mathcal{Q}} f(x) dx$ for every $x \in \mathcal{Q}$. Hence $\lim_{n\to\infty} |f_n - f| = 0$. □

Proposition 3.3. *If f is a non-negative THK integrable on a cell $\mathcal{Q}$ and F is its indefinite THK integral, then f is μ-measurable.*

Proof. Let f_n be μ-simple function defined by $f_n(x) = \sum_{(x,\mathcal{Q})\in P_n} \frac{F(\mathcal{Q})}{\mu(\mathcal{Q})}$. Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \bigcup_{\mathcal{Q}\in P_k} \mu(\mathcal{U}(\mathcal{Q}))$ and $\mathcal{D} = \left\{ x \in \mathcal{Q} : \ F_T'(x) \text{ does not exists or } F_T'(x) \text{ exists nad } F_T' \neq f(x) \right\}$. By last condition of Definition 2.4 and Theorem 3.8, $\mu(\mathcal{C} \cup \mathcal{D}) = \emptyset$. Let $\Xi = \mu(\mathcal{C} \cup \mathcal{D})$. Let $x \in \mathcal{Q} \setminus \Xi$. Then for each $N \in \mathbb{N}$, there exists $\mathcal{Q}_{n,x} \in \mathcal{F}$ such that $(x, \mathcal{Q}_{n,x}) \in P_n$, $\mu(\mathcal{Q}_{n,x}) < \frac{1}{n}$ and $f_n(x) = \frac{F(\mathcal{Q}_{n,x})}{\mu(\mathcal{Q}_{n,x}))}$. By $F_T'(x) = f(x)$, $f_n(x) \to f(x)$. s f_n is μ-measurability for $n \in \mathbb{N}$, so $f(x)$ is μ-measurable. □

Next, we prove monotone convergence type theorem for THK integrable function.

Theorem 3.11. *Let $\{f_n\}$ be a non-decreasing sequence of THK integrable function on a cell $\mathcal{Q}$ and let $f = \lim_n f_n$. If $\lim_{n\to\infty} \int_{\mathcal{Q}} f_n d\mu < \infty$ then f is THK integrable on $\mathcal{Q}$ and $\int_{\mathcal{Q}} f d\mu = \lim_{n\to\infty} \int_{\mathcal{Q}} f_n d\mu$.*

Proof. Let (f_n) be a non-decreasing sequence of THK integrable function on a cell $\mathcal{Q}$ and since $\{\int_{\mathcal{Q}} f_n d\mu\}$ is bounded on $\mathcal{Q}$. So, $\{\int_{\mathcal{Q}} f_n d\mu\}$ converges to a real number $\rho \in \mathbb{R}$. Then, given $\mathcal{U}$ there exists $n \in \mathbb{N}$ such that $n \geq N$ we have $0 \leq \rho - \int_{\mathcal{Q}} f_n d\mu \leq \mu(\mathcal{U})$. From Lemma 3.3 there exists an additive function π on the subcells of $\mathcal{Q}$ such that for all $\mathcal{U}$, there exists a gauge δ_n on $\mathcal{Q}$ with

$$\sum_{(x_i,\mathcal{Q}_i)\in P} \left| \pi(\mathcal{Q}_i) - f_n(x_i)\mu(\mathcal{Q}_i) \right| < \frac{\mu(\mathcal{U})}{2^n},$$

for each δ_n-fine partition P of $\mathcal{Q}$ and $\pi(\mathcal{Q}_i) = \int_{\mathcal{Q}_i} f_n d\mu$. Since $f = \lim_n f_n$, so for each $x \in \mathcal{Q}$ there exists a natural number $n(x) \geq N$ such that

$$|f(x) - f_n(x)| < \mu(\mathcal{U}), \tag{3.8}$$

whenever $n \geq n(x) \geq N$. Suppose $\delta(x) = \delta_{n(x)}$ for $x \in \mathcal{Q}$, then δ is a gauge on $\mathcal{Q}$ which is δ-fine. If $P = \{(\mathcal{Q}_1, x_1), (\mathcal{Q}_2, x_2), ..., (\mathcal{Q}_n, x_n)\}$ be a δ-fine partition of $\mathcal{Q}$. Then from (3.8),

$$\sum_{i=1}^{n} |f(x_i) - f_{n(x_i)}|\mu(\mathcal{Q}_i) < \mu(\mathcal{U}(\mathcal{Q})). \tag{3.9}$$

Also,

$$\Big|\sum_{i=1}^{n} f_{n(x_i)}(x_i)\mu(\mathcal{Q}_i) - \sum_{i=1}^{n} f_{n(x_i)} d\mu\Big| \leq \sum_{i=1}^{n} \Big| f_{n(x_i)}(x_i)\mu(\mathcal{U}(\mathcal{Q}_i)) - \int_{\mathcal{Q}_i} f_{n(x_i)} d\mu \Big| < \mu(\mathcal{U}).$$

Again, from the hypothesis $f = \lim_{n\to\infty} f_n$. So, (f_n) is a point-wise bounded sequence of functions. Indeed,

$$\begin{aligned}\int_{\mathcal{Q}} f_n d\mu &= \sum_{i=1}^{n} \int_{\mathcal{Q}_i} f_n d\mu \\ &\leq \sum_{i=1}^{n} f_{n(x_i)} d\mu.\end{aligned}$$

Moreover,

$$0 \leq \rho - \sum_{i=1}^{n} \int_{\mathcal{Q}_i} f_{n(x_i)} d\mu \leq \rho - \int_{\mathcal{Q}} f_n d\mu < \mu(\mathcal{U}).$$

Now we have

$$\begin{aligned}|S(f, P) - \rho| &\leq \Big|\sum_{i=1}^{n} f(x_i)\mu(\mathcal{Q}_i) - \sum_{i=1}^{n} f_{n(x_i)}(x_i)\mu(\mathcal{Q}_i)\Big| \\ &+ \Big|\sum_{i=1}^{n} f_{n(x_i)}(x_i)\mu(\mathcal{Q}_i) - \sum_{i=1}^{n} \int_{\mathcal{Q}_i} f_{n(x_i)} d\mu\Big| \\ &+ \Big|\sum_{i=1}^{n} \int_{\mathcal{Q}_i} f_{n(x_i)} d\mu - \rho\Big| \\ &< \mu(\mathcal{Q}) + \mu(\mathcal{U}) + \mu(\mathcal{U}).\end{aligned}$$

Since $\mu(\mathcal{U})$ is arbitrary, f is THK integrable on $\mathcal{Q}$ and $\rho = \int_{\mathcal{Q}} f d\mu$. □

Finally, we prove every absolute THK integrable functions are Lebesgue integrable on $\mathcal{Q}$.

Theorem 3.12. *If f is absolutely THK integrable on a cell $\mathcal{Q}$, then f is Lebesgue integrable on $\mathcal{Q}$.*

Proof. Let us consider for $n \in \mathbb{N}$, $f_n(x) = \min\{|f(x)|, n\}$ for all $x \in \mathcal{Q}$. By Theorem 3.3, $|f|$ is Lebesgue measurable. Therefore f_n is Lebesgue measurable and bounded. By Theorem 3.6, f_n is THK integrable on $\mathcal{Q}$. As (f_n) is an increasing sequence of non-negative function convergent to $|f|$, by Theorem 3.11,

$$(L)\int_{\mathcal{Q}}|f|d\mu = \lim_{n\to\infty}\int_{\mathcal{Q}} f_n d\mu$$
$$= \int_{\mathcal{Q}}|f|d\mu < \infty.$$

Hence f is Lebesgue integrable on $\mathcal{Q}$. □

4. Statistical convergence

In this Section, we introduce statistical convergence of topological Henstock-Kurzweil integrals on a topological vector space. We prove that every convergence for topological Henstock-Kurzweil integrable function is also statistically convergent. Further, we introduce statistically equi-integrability for topological Henstock-Kurzweil integrable functions to prove statistically Cauchy on $THK(\mathcal{Q})$.

Let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. Let $A(n) = \{k \in A : \ k \le n\}$. Then the lower and upper asymptotic density of A are

$$\underline{d}(A) = \lim_{n\to\infty} \inf \frac{|A(n)|}{n}$$

and

$$\overline{d}(A) = \lim_{n\to\infty} \sup \frac{|A(n)|}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, then $d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$. Clearly $d(\mathbb{N}\setminus A) = 1 - d(A)$ for $A \subset \mathbb{N}$.

Definition 4.12. [9] *A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space $\mathcal{X}$ is said to converge statistically to $x \in \mathcal{X}$ if for every $\mathcal{U}$ of x,*

$$d\Big(\{n \in \mathbb{N} : \ x_n \notin \mathcal{U}\}\Big) = 0.$$

It is denoted by $\text{st-}\lim_n x_n = x$.

Theorem 4.13. [9] *The limit of a statistically convergent sequence is uniquely determined in Hausdorff topological spaces.*

Definition 4.13. *Let $\mathcal{Q} \in Bo(\mathcal{X})$. A sequence of THK integrable functions $f_n : \mathcal{Q} \to \mathbb{R}$ is said to be statistically convergent to f if for any θ-nbd $\mathcal{U}$ and a δ-fine partition P, $d\Bigg(\Bigg\{n \in \mathbb{N} : \ \Big|S(f_n, P) - \int_{\mathcal{Q}} f\Big| \ge \mu(\mathcal{U})\Bigg\}\Bigg) = 0.$*

We denote $\text{st-}\lim_n f_n = f$. It is not hard to see if st-limit is unique then the following result is true.

Theorem 4.14. *If a sequence of THK integrable functions (f_n) of statistically convergent, then its st-$\lim_n$ is unique.*

Proof. Suppose $\text{st-}\lim_n f_n = f$ and $\text{st-}\lim_n f_n = g$. Given $\epsilon > 0$, define

$$K_1(\mu(\mathcal{U})) = \left\{n \in \mathbb{N} : \ \left|S(f_n, P) - \int_{\mathcal{Q}} f\right| \ge \mu(\mathcal{U})\right\}$$

and

$$K_2(\mu(\mathcal{U})) = \left\{n \in \mathbb{N} : \ \left|S(f_n, P) - \int_{\mathcal{Q}} g\right| \ge \mu(\mathcal{U})\right\}.$$

Clearly $d\big(K_1(\mu(\mathcal{U})\big) = 0$ and $d\big(K_2(\mu(\mathcal{U})\big) = 0$. Let $K\big(\mu(\mathcal{U})\big) = K_1\big(\mu(\mathcal{U})\big) \bigcup K_2\big(\mu(\mathcal{U})\big)$. Then $d\Bigg(K(\mu(\mathcal{U})\Bigg) = 0$, gives $\mathbb{N} \setminus d\big(K(\mu(\mathcal{U})\big) = 1$. Next, if $k \in \mathbb{N} \setminus K(\mu(\mathcal{U}))$, we have

$$\Big|\int_{\mathcal{Q}} f - \int_{\mathcal{Q}} g\Big| \leq |\int_{\mathcal{Q}} f - S(f_k, P)| + |S(f_k, P) - \int_{\mathcal{Q}} g|$$
$$< \frac{\mu(\mathcal{U})}{2} + \frac{\mu(\mathcal{U})}{2} = \mu(\mathcal{U}).$$

Since $\mu(\mathcal{U})$ is arbitrary, we can find $\Bigg|\int_{\mathcal{Q}}(f-g)\Bigg| = 0$. Hence $\int_{\mathcal{Q}}(f-g) = 0$ gives $f = g$. □

Definition 4.14. *We say that (f_n) of THK integrable functions is statistically Cauchy if for any θ-nbd $\mathcal{U}$, and a δ-fine partition P there exists $M \in \mathbb{N}$ such that*

$$d\Bigg(\{n \in \mathbb{N} : \ |S(f_n, P) - S(f_M, P)| \geq \mu(\mathcal{U})\}\Bigg) = 0.$$

Next, we state several fundamental properties below.

Theorem 4.15. *Let (f_n) and $g_n)$ are in $THK([a,b], X)$ with $f = st\text{-}\lim_n f_n$ and $g = st\text{-}\lim_n g_n$. Then following holds:*

(1) $st\text{-}\lim_n(f_n + g_n) = f + g$.
(2) $st\text{-}\lim_n(\alpha f_n) = \alpha(st\text{-}\lim_n f_n) = \alpha f$.
(3) $st\text{-}\lim_n f_n g_n = fg$.

Proof. For (1): Let $f_n : \mathcal{Q} \to \mathbb{R}$ and $g_n : \mathcal{Q} \to \mathbb{R}$ are sequences of THK integrable functions. Let st-$\lim_n f_n = f$ and st-$\lim_n g_n = g$. Then for any θ-nbd $\mathcal{U}$, and δ-fine partition P, we have

$$d\Bigg(\Bigg\{k \in \mathbb{N} : \ |S(f_k, P) - \int_{\mathcal{Q}} f| < \mu(\mathcal{U})\Bigg\}\Bigg) = 1$$

and

$$d\Bigg(\Bigg\{k \in \mathbb{N} : \ |S(g_k, P) - \int_{\mathcal{Q}} g| < \mu(\mathcal{U})\Bigg\}\Bigg) = 1.$$

It is easy to see,

$$d\Bigg(\big\{k \in \mathbb{N} : \ |S(f_k, P) - \int_{\mathcal{Q}} f| \geq \mu(\mathcal{U})\big\} \cup \big\{k \in \mathbb{N} : \ |S(g_k, P) - \int_{\mathcal{Q}} g| \geq \mu(\mathcal{U})\big\}\Bigg) = 1.$$

Let $\mathcal{P} = \Bigg\{k \in \mathbb{N} : \ |S(f_k, P) - \int_{\mathcal{Q}} f| < \mu(\mathcal{U})\Bigg\} \cup \Bigg\{k \in \mathbb{N} : \ |S(g_k, P) - \int_{\mathcal{Q}} g| < \mu(\mathcal{U})\Bigg\}$. Then for any $k \in \mathcal{P}$, we have $|S(f_n, P) + S(g_n, P) - \int_{\mathcal{Q}}(f+g)| < 2\mu(\mathcal{U})$. Thus

$$d(\mathcal{P}) = 1 \leq d\Bigg(\big\{k \in \mathbb{N} : \ |S(f_n, P) + S(g_n, P) - \int_{\mathcal{Q}}(f+g)| < 2\mu(\mathcal{U})\Bigg) \leq 1.$$

Since $\mu(\mathcal{U})$ is arbitrary, we get st-$\lim_n(f_n + g_n) = f + g$.
Proof of (2) and (3) is straightforward, so ommitted. □

Next, we show that every convergent sequence of THK integrable functions is statistically convergent.

Theorem 4.16. *If a sequence (f_n) of THK integrable functions in $THK(\mathcal{Q}, \mathcal{X})$ converges to $f \in \mathcal{X}$, then (f_n) is statistically convergent to f.*

Proof. Let $\mathcal{U}$ be a θ-nbd. Since (f_n) is a sequence of THK integrable functions on $THK([a,b],\mathcal{X})$ converges to f, so, there exists $\mathcal{N} \subset \mathbb{N}$ with $\delta(\mathcal{N}) = 1$ and $n_0 = n_0(\mathcal{U})$ such that $n \geq n_o$ and $n \in \mathcal{N}$ implies $|S(f_n, P) - \int_{\mathcal{Q}} f| < \mu(\mathcal{U})$ whenever P is a free tagged partition of $\mathcal{Q}$. Clearly $|f_n| < \mu(\mathcal{U})$. Again,

$$\left\{n \in \mathbb{N} : \ |S(f_n, P) - \int_{\mathcal{Q}} f| \geq \mu(\mathcal{U})\right\} \subset \left\{1, 2, ..., n_o\right\} \cup (\mathbb{N} \setminus \mathcal{N}).$$

Since $\delta\left\{1, 2, ..., n_0\right\} \cup (\mathbb{N} \setminus \mathcal{N}) = 0$, it follows $f = s\text{-}\lim_n f_n$. □

The following example shows that converse of Theorem 4.16 does not hold.

Example 4.4. *Consider a sequence (f_n) of THK integrable functions whose terms are*

$$f_n = \begin{cases} n & \text{if } n = i^2, \ i = 1, 2, ..., \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

It is easy to see the sequence (f_n) is divergence. Let $K = \{i^2 : \ i = 1, 2, ...\}$, then $\delta(K) = 0$, it follows $0 = s\text{-}\lim_n f_n$.

We introduce statistical equi-integrability for THK integrable function as follows:

Definition 4.15. *A sequence of statistical THK integrable functions $f_n : \mathcal{Q} \to \mathbb{R}$ is said to be statistically equi-integrable if for any θ-nbd $\mathcal{U}$ and a δ-fine partition P, we have $d\left(\left\{n \in \mathbb{N} : \left|S(f_n, P) - \int_{\mathcal{Q}} f_n\right| \geq \mu(\mathcal{U})\right\}\right) = 0$.*

Proposition 4.4. *Let (f_n) be statistically convergent sequence of THK integrable functions on $\mathcal{Q}$. If (f_n) is statistically equi-integrable then (f_n) is statistically Cauchy on $THK(\mathcal{Q})$.*

Proof. Let st-$\lim_n f_n = f$. Consider $\mathcal{U}_n$ be a sequence of nested base of θ-nbd. Let $W^{(j)} = \left\{w \in \mathbb{N} : \ w \leq n, |S(f_n, P) - \int_{\mathcal{Q}} f| \geq \mu(\mathcal{U}_j)\right\}$ for any positive integer j. Clearly for each , $W^{(j+1)} \subset W^{(j)} < \lim_{n\to\infty} \frac{1}{n}|W^{j)}| = 1$. Let us choose $m \in \mathbb{N}$ such that $n > m$. Then $\frac{1}{n}|W^{(j)}| > 0$. This shows that $W^{(1)} \neq \emptyset$. In general we can find natural numbers $m(p+1) > m(p)$ such that we can a positive number $r > m(p+1)$ implies $W^{(p+1)} \neq \emptyset$. Further By Lemma 2.1, we have for every θ-nbd $\mathcal{U}$, there is a symmetric θ-nbd $\mathcal{V}$ such that $\mathcal{V} + \mathcal{V} \subseteq \mathcal{U}$. Let us consider (f_n) be a statistically equi-integrable THK integrable functions on $THK(\mathcal{Q})$. Then by definition of equi-integrable THK integrable function, for any θ-nbd $\mathcal{U}$ and a δ-fine partition P such that $d\left(\left\{n \in \mathbb{N} : \left|S(f_n, P) - \int_{\mathcal{Q}} f_n\right| \geq \mu(\mathcal{U})\right\}\right) = 0$. So,

$$d\left(\left\{n \in \mathbb{N} : \ |S(f_n, P) - \int_{\mathcal{Q}} f_n| \geq \mu(\mathcal{V})\right\}\right) = 0 \text{ and} \tag{4.10}$$

$$d\left(\left\{n \in \mathbb{N} : \ |S(f_m, P) - \int_{\mathcal{Q}} f_m| \geq \mu(\mathcal{V})\right\}\right) = 0 \tag{4.11}$$

Thus we have,

$$d\Bigg(\Bigg\{k \le n:\ |S(f_n,P) - S(f_m,P)| \ge \mu(\mathcal{U})\Bigg\}\Bigg) \le d\Bigg(\Bigg\{n \in \mathbb{N}:\ |S(f_n,P) - \int_{\mathcal{Q}} f_n| \ge \mu(\mathcal{V})\Bigg\}\Bigg) + d\Bigg(\Bigg\{n \in \mathbb{N}:\ |S(f_m,P) - \int_{\mathcal{Q}} f_m| \ge \mu(\mathcal{V})\Bigg\}\Bigg) \to 0 \text{ using (4.10), (11).}$$

Hence (f_n) is statistically Cauchy on $THK(\mathcal{Q})$. □

5. Conclusions

In this work, topological Henstock-Kurzweil integral on a μ-cell $\mathcal{Q}$ of a topological vector space $\mathcal{X}$ has been discussed. Several properties are discussed in this regard. Also, we have proved that every Lebesgue integrable function is topological Henstock-Kurzweil integrable. We introduce equi-integrability for topological Henstock-Kurzweil integrable functions and several convergence theorems are also proved. Finally, we extend the usual convergence of topological Henstock-Kurzweil integrable functions to statistical convergence and relationship between statistically equi-integrable topological Henstock-Kurzweil integral and statistical Cauchy convergence has been established. O. Dumanet al in [2, Theorem 3.1] proved that in association with finitely additive measure λ, any λ-statistically uniformly sequence of functions (f_n) on $[a,b]$ converges to function f and each f_n is integrable on $[a,b]$ with st-λ-$\lim\limits_n \int_a^b f_n(x)dx = \int_a^b$st-$\lambda$-$\lim\limits_n f_n(x)dx = \int_a^b f(x)dx$. We conclude this article with the following possibility: Is the following theorem true for topological Henstock-Kurzweil integrable functions?

Theorem 5.17. *A sequence (f_n) of statistical convergent THK integrable functions converges to f. The limit function f is THK integrable if and only if (f_n) is statistical equi-integrable THK function.*

Acknowledgments

The authors thank to the Journal editors and reviewers for their constructive suggestions for better presentation of the article.

References

[1] Corrao, G. *An Henstock-Kurzweil type integral on a measure metric space*. Doctoral Thesis, Universita Degli Studi Di Palermo, Palermo, 2013.

[2] Duman, O.; Orhan, C. μ-statistically convergent function sequences. *Czec. Math. J.* **54** (2004), no. 2, 413–422.

[3] Fast, H. Sur la convergence statistique. *Colloq. Math.* **2** (1951), no. 3, 241–244.

[4] Fridy, J. A. On statistical convergence. *Analysis.* **5** (1985), no. 4, 301–313.

[5] Fridy, J. A. Statistical limit points. *Proc. Amer. Math. Soc.* **118** (1993), no. 4, 1187–1192.

[6] Fridy, J. A.; Miller, H. I. A matrix characterization of statistical convergence. *Analysis.* **11** (1991), no. 1, 59–66.

[7] Fridy, J. A.; Orhan, C. Lacunary statistical convergence. *Pacific J. Math.* **160** (1993), no. 1, 43–51.

[8] Gardon, R. G. *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*. American Mathematical Society, 1994.

[9] Giuseppe, D. M.; Kocinac, L. D. R. Statistical convergence in topology. *Topol Appl.* **156** (2008), no. 1, 28–45.

[10] Henstock, R. *The general theory of integration*. Oxford University Press, Oxford, UK, 1991.

[11] Kalita, H.; Hazarika, B. A convergence theorem for ap-Henstock-Kurzweil integral and its relation to topology. *Filomat.* **36** (2022), no. 20, 1–10.

[12] Kalita, H; Hazarika, B.; Becerra, T. P. On AP-Henstock-Kurzweil Integrals and Non-Atomic Radon Measure. *Mathematics.* **11** (2023), no. 6, 1–16.

[13] Kalita, H.; Agarwal, R. P.; Hazarika, B. Convergence of ap-Henstock-Kurzweil integral on locally compact spaces. *Czec. Math. J.* (2023), https://doi.org/10.21136/CMJ.2023.0450-22.

[14] Klein, Ch.; Rolewicz, S. On Riemann integration of functions with values in topological linear spaces. *Studia Mathematica.* **80** (1984), no. 4, 109–118.

[15] Paluga, R.; Canoy Jr, S. The Henstock Integral in topological vector spaces. *Mat. Matematika.* **24** (2001), no. 3, 34–47.

[16] Sokolowski, J.; Zochowski, A. On the Topological Derivatives in Shape Optmization. *SIAM. J. Control and Optimization.* **37** (1999), no. 4, 1–35.

[17] Srivastava, H. M.; Jena, B. B.; Paikray, S. K. Statistical Riemann and Lebesgue Integrable Sequence of Functions with Korovkin Type Approximation Theorems. *Axioms.* **10** (2021), no. 10, 1–16.

[18] Yeong, L. T. *Henstock-Kurzweil integration on Euclidean spaces*. Series in Real Analysis, no. 12, World Scientific, 2011.

[1] Mathematics Division, School of Advanced Sciences and Languages, VIT-Bhopal University, Kothri-kalan, Bhopal-Indore Highway, Madhya Pradesh-466114, India
Email address: hemanta30kalita@gmail.com; hemantakalita@vitbhopal.ac.in

[2] Mathematics Division, School of Advanced Sciences and Languages, VIT-Bhopal University, Kothri-kalan, Bhopal-Indore Highway, Madhya Pradesh-466114, India
Email address: somkakdwip@gmail.com

[3] Department of Mathematics, Gauhati University, Guwahati 781014, Assam, India
Email address: bh_rgu@yahoo.co.in