CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 409-423

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.09

Existence and uniqueness of solution for a class of superlinear Kirchhoff-type equations on the real half-line

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ABSTRACT. This paper is concerned with a class of Kirchhoff-type equations on the real half-line. By using the Mountain Pass theorem, we show the existence and uniqueness of solution in the case where f is a superlinear function. Our results improve and extend a recently published ones.

1. INTRODUCTION

In this paper, we consider the following Kirchhoff-type equation

$$\begin{cases} -\left(a + b\int |u'(x)|^2 dx\right) u'' + pu = f(u), \quad x \in (0, +\infty), \\ u(0) = 0, \end{cases} \tag{1.1}$$

where $a > 0$, $b \geq 0$, $p > 0$ are constants, and $f \in C(\mathbb{R})$.

In [12], Kirchhoff proposed a model given by the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = f(x, u) \tag{1.2}$$

which extends the classical D'Alembert wave equations for free vibrations of elastic strings. Kirchhoff's model takes account the changes in length of string produced by transverse vibrations.

In (1.2), u denotes the displacement, $f(x, u)$ the external force and the parameters have the following meaning:

- L is the length of the string;
- h is the area of cross section;
- E is the Young modulus of the material;
- ρ is the mass density;
- ρ_0 is the initial tension.

We notice that problem (1.2) appears in other fields as biological systems, where u describes a process which depends on the average of density itself (for instance, population density). For more information on the physical background of problem (1.2), we refer to papers [1, 4, 15, 18] and the references therein.

In recent years, the following stationary Kirchhoff-type equation

$$-\left(a + b\int_\Omega |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x, u), \quad x \in \Omega, \tag{1.3}$$

have been widely investigated, and many interesting results on either a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$ or the whole space $\Omega = \mathbb{R}^N$ have been established by using variational methods, see for instance [3, 7, 9, 11, 16, 17, 19, 20, 21].

Received: 11.04.2024. In revised form: 20.09.2024. Accepted: 22.09.2024
2020 *Mathematics Subject Classification.* 35J20, 35B38, 34B40.
Key words and phrases. *Kirchhoff-type equation, superlinear, Mountain Pass theorem.*
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For the one-dimensional Kirchhoff-type equation on unbounded intervals, there are a few papers in the literature, see for instance [6, 10, 13]. In particular, in [6], by using variational methods, a monotonicity trick related to the mountain pass lemma, cut-off functional technique, and a Pohozaev type identity the authors obtained the existence of nontrivial non-negative solutions to the following problem

$$\begin{cases} \Big(a+\lambda\int\Big(|u'(x)|^2+bu(x)^2\Big)dx\Big)\Big(-u''(x)+bu(x)\Big)=f(u(x)), \quad x\in(0,+\infty), \\ u(0)=0. \end{cases} \tag{1.4}$$

where a and b are positive constants, $\lambda \geq 0$ and $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies the following assumptions:

(f_1) There exists $\theta > 1$ and non-negative constants α, β such that

$$f(\xi) \leq \alpha + \beta\xi^\theta \qquad \forall \xi \in \mathbb{R}^+;$$

(f_2) $\displaystyle\lim_{\xi\to 0}\frac{f(\xi)}{\xi} = 0;$

(f_3) $\displaystyle\lim_{\xi\to +\infty}\frac{f(\xi)}{\xi} = +\infty.$

Using variational methods to solve such a problem, needs to prove that the functional energy $I(u)$ corresponding to equation (1.4) satisfy the Palais–Smale compactness condition, that is, any sequence $\{u_n\}_{n\in\mathbb{N}} \subset H$ (H is the working space) such that

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } H^*, \tag{1.5}$$

has a strongly convergent subsequence. Since the embeddings $H_0^1(\mathbb{R}^+) \hookrightarrow L^p(\mathbb{R}^+)$, $p \geq 2$ and $H_0^1(\mathbb{R}^+) \hookrightarrow C(\mathbb{R}^+)$ are not compact, it is so difficult to prove a such convergent criteria. To this end, the authors in [6] have considered the following weighted Banach space

$$C_p = \Big\{u \in C(\mathbb{R}^+) : \lim_{x\to\infty} p(x)u(x) = 0\Big\},$$

equipped with the norm $\|u\|_{\infty,p} = \sup_{x\in\mathbb{R}^+} p(x)|u(x)|$, where $p : \mathbb{R}^+ \to (0,\infty)$ is a continuous function such that

$$\lim_{x\to\infty}\sqrt{x}p(x) = 0 \quad \text{and} \quad p^{-(\theta+1)} \in L^1(\mathbb{R}^+). \tag{1.6}$$

They proved that the embedding $H_0^1(\mathbb{R}^+) \hookrightarrow C_p(\mathbb{R}^+)$ is compact, and then they can easily proved the strongly convergence of the (PS)-sequence (i.e., a sequence that satisfies (1.5)). However, the condition (1.6) contains a contradiction. Indeed, from (1.6), there exists $A > 0$ such that for all $x \geq A$, one has

$$\sqrt{x}p(x) \leq 1,$$

and then

$$\int_A^{+\infty}\frac{1}{(p(x))^{\theta+1}}dx \geq \int_A^{+\infty} x^{\frac{\theta+1}{2}}dx = +\infty,$$

since $\theta > 1$. Therefore, $p^{-(\theta+1)} \notin L^1(\mathbb{R}^+)$ and also $p^{-1} \notin L^1(\mathbb{R}^+)$. Thus, the two conditions in (1.6) will never be satisfied at the same time, and the argument in [6] is not correct.

In [13], the author studied the following class of Kirchhoff equation on the whole $\mathbb{R}$

$$-\left(1+\int_{\mathbb{R}}|u'(x)|^2dx\right)u'' + p(x)u = l(x)u^3 + f(x,u), \quad x \in \mathbb{R}, \tag{1.7}$$

where $p, l \in C(\mathbb{R})$ are $1-$periodic in x and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is $1-$periodic in x. Under some suitable assumptions on p, l and f, the existence of nontrivial ground solutions is obtained by needs of the Non-Nehari manifold method in combination with the Mountain Pass Theorem.

Meng and Zeng [16] obtained nontrivial solutions for the following Kirchhoff-type equation in $\mathbb{R}^2$

$$-\Big(1+b\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)\Delta u+V(x)u=\beta u^3,\qquad x\in\mathbb{R}^2, \tag{1.8}$$

where $b,\beta>0$ and V is periodic and satisfies some further assumptions. By minimizing the following energy functional

$$E^b_\beta(u)=\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx+\frac{b}{4}\Big(\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)^2+\frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2dx-\frac{\beta}{4}\int_{\mathbb{R}^2}|u|^4dx,$$

the authors prove the existence of β^* such that for all $\beta>\beta^*$ the problem (1.8) have a nontrivial non-negative solution. Moreover, the asymptotically behavior of the obtained solution is discussed as $b\longrightarrow 0$.

In [17], by using the fixed point principle of Banach and Schaefer, the authors established the existence of solution to the following class of Kirchhoff equation with reaction term

$$\begin{cases}-\Big(a+b\int_{\mathbb{R}^2}|\nabla u|^2dx\Big)\nabla u=f+g(x,u,\nabla u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.9}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $f\in H^{-1}(\Omega)$ and $g:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfies the Carathéodory conditions.

Motivated by the aforementioned facts, in this paper we will give an existence results for equation (1.1). First, inspired by [23], using a suitable change of variables, we transform the non-local Kirchhoff-type equation (1.1) into an equivalent system of a semilinear equation of unknown function u and algebraic equation of unknown $\lambda>0$ (see (1.10)). Then, by using the mountain pass theorem that the semilinear equation has at least one solution under suitable assumptions on the nonlinear term f, and if f is non-negative on $\mathbb{R}$ the obtained solution is non-negative. Finally, we show that if f is k-Lipschitz with $0<k<p$, then the obtained solution is unique. We believe that the approach of the present paper is much simpler and transparent than that in [6].

Now, we introduce the equivalent system with respect to (u,λ) for problem (1.1).

Lemma 1.1. *The problem* (1.1) *is equivalent to the following system*

$$\begin{cases}-v''(y)+pv(y)=f(v), & y\in(0,+\infty),\\ \lambda=a+\dfrac{b}{\sqrt{\lambda}}\int|v'(y)|^2dy,\end{cases} \tag{1.10}$$

where $(\lambda,v)\in(0,+\infty)\times H^1_0(\mathbb{R}^+)$.

Proof. We consider the following change of variable

$$x=\sqrt{\lambda}y\qquad\text{and}\qquad v(y)=u(\sqrt{\lambda}y). \tag{1.11}$$

By using the chain rule we get

$$v'(y)=\sqrt{\lambda}u'(\sqrt{\lambda}y)\qquad\text{and}\qquad v''(y)=\lambda u''(\sqrt{\lambda}y). \tag{1.12}$$

From (1.11) and (1.12) by straightforward calculation, we get

$$\int|u'(x)|^2dx=\frac{1}{\sqrt{\lambda}}\int|v'(y)|^2dy. \tag{1.13}$$

By substituting (1.11), (1.12) and (1.13) into (1.1), one has

$$-\frac{1}{\lambda}\Big(a+\frac{b}{\sqrt{\lambda}}\int|v'(y)|^2dy\Big)v''(y)+pv(y)=f\big(v(y)\big).$$

and if we take

$$\frac{1}{\lambda}\Big(a+\frac{b}{\sqrt{\lambda}}\int|v'(y)|^2dy\Big)=1,$$

then we get the following equation

$$-v''(y)+pv(y)=f\big(v(y)\big).$$

Finally, since we have made a bijective change of variable, the problem (1.1) is equivalent to the system (1.10) and the proof is completed. □

For a given $v\in H_0^1(\mathbb{R}^+)$, by the intermediate value theorem, it is easy to check that the second equation of system (1.10) has a unique solution $\lambda_v\in(0,+\infty)$. Consequently, that system (1.10) can be transformed into a nonlinear equation which permits to use variational methods. Therefore, the pair $(\lambda,v)\in(0,+\infty)\times H_0^1(\mathbb{R}^+)$ is a solution of (1.10) if and only if $\lambda=\lambda_v$ and $v\in H_0^1(\mathbb{R}^+)$ is a weak solution of the problem

$$(\mathcal{P})\qquad \begin{cases}-v''+pv=f(v), & x\in(0,+\infty),\\ v(0)=0.\end{cases}$$

To establish the main results of this paper, we make the following assumptions

(F_1) $f\in C(\mathbb{R})$ and there exist $\theta>2$, $0<\alpha<p$ and $\beta>0$ such that

$$|f(s)|\le\alpha|s|+\beta|s|^{\theta-1}\qquad\forall s\in\mathbb{R}.$$

(F_2) $\lim\limits_{s\to+\infty}\dfrac{F(s)}{s^2}=+\infty$, where $F(s)=\int_0^s f(t)dt$.

(F_3) There exist $\mu>2$, $L>0$ and $c_1>0$ such that

$$0\le\mu F(s)\le f(s)s+c_1s^2,\qquad\forall|s|\ge L.$$

Remark 1.1. *The present paper has two major difficulties which are the lack of compactness and the boundedness of the (PS)-sequence, respectively. In the one hand, since $\mathbb{R}^+$ is unbounded then the injection $H_0^1(\mathbb{R}^+)\hookrightarrow L^2(\mathbb{R}^+)$ is not compact, therefore it is not easy to prove the convergence of the (PS)-sequence. Compared with* [6], *our result ensures the existence of a non-trivial solution without requiring any compactness conditions.*

On the other hand, when using variational method to study the semi-linear equation of system (1.10), *the Ambrosetti–Rabinowitz condition (see* [2]*)*

$$(AR)\qquad \text{there exists }\ \mu>2\ \text{ such that }\ 0<\mu F(u)\le uf(u)\ \text{ for all }\ u\in\mathbb{R},$$

is usually used to guarantee that every (PS)-sequence is bounded. Obviously, condition (F_3) is weaker than (AR). So, the proof of the boundedness of (PS)-sequence needs a very careful analysis (see Lemma 2.5).

The outline of the rest of the paper is as follows. In section 2, we introduce the variational framework associated with problem $(\mathcal{P})$ and we prove some technical lemmas. Section 3 is devoted to the proof of Theorem 3.1. and Theorem 3.2.

2. VARIATIONAL FRAMEWORK AND TECHNICAL LEMMAS

In this section, we start by establishing the variational setting corresponding to the problem (1.1). Firstly, we introduce the following notations:

- We denote by $\|.\|_r$ the usual L^r-norm for $r \in [1, +\infty]$;
- $\langle ., . \rangle$ denotes action of the dual;
- $\rightharpoonup$ denotes the weak convergence in X;
- C, C_i and c_i denote various positive constants, which may vary from line to line;
- $\mathcal{D}(\mathbb{R}^+) := C_c^\infty(\mathbb{R}^+)$ denotes the space of infinitely differentiable functions $\varphi : \mathbb{R}^+ \to \mathbb{R}$ with compact support in $\mathbb{R}^+$;
- $\mu(\cdot)$ denotes the Lebesgue measure;
- $o_n(1) \to 0$ as $n \to \infty$.

We consider the Sobolev space

$$H_0^1(\mathbb{R}^+) = \left\{ u \in L^2(\mathbb{R}^+) \ : u' \in L^2(\mathbb{R}^+), u(0) = 0 \right\},$$

equipped with the inner product

$$(u, v) = \int (u'v' + puv)\, dx, \quad u, v \in H_0^1(\mathbb{R}^+),$$

and the norm

$$\|u\|^2 = \int_0^{+\infty} \left(|u'|^2 + pu^2 \right) dx, \quad u \in H_0^1(\mathbb{R}^+).$$

We denote by $H^{-1}(\mathbb{R}^+)$ the topological dual space of $H_0^1(\mathbb{R}^+)$.
It is well known that the embedding $H_0^1(\mathbb{R}^+) \hookrightarrow L^r(\mathbb{R}^+)$ for $r \in [2, +\infty]$ is continuous (see [5]). Then there exists $\mu_r > 0$ such that

$$\|u\|_r \leqslant \mu_r \|u\|, \quad \forall u \in H_0^1(\mathbb{R}^+). \tag{2.14}$$

We notice that if $u \in H_0^1(\mathbb{R}^+)$ then $\lim\limits_{x \to +\infty} u(x) = 0$.

For the problem $(\mathcal{P})$, the associated energy functional is defined on $H_0^1(\mathbb{R}^+)$ as follows

$$I(v) = \frac{1}{2}\|v\|^2 - \int_0^{+\infty} F(v) dx, \tag{2.15}$$

where (and in the sequel) $F(v) = \displaystyle\int_0^v f(s) ds$.
We have the following result.

Lemma 2.2. *The functional I is of class C^1 on $H_0^1(\mathbb{R}^+)$, and*

$$\langle I'(v), w \rangle = \int_0^{+\infty} (v'w' + pvw)\, dx - \int_0^{+\infty} f(v) w dx, \tag{2.16}$$

for all $v, w \in H_0^1(\mathbb{R}^+)$

Proof. We consider the functional J defined on $H_0^1(\mathbb{R}^+)$ by

$$J(v) = \int_0^{+\infty} F(v) dx, \qquad \forall v \in H_0^1(\mathbb{R}^+).$$

From (F_1) it follows that

$$|F(s)| \leq \frac{\alpha}{2}|s|^2 + \frac{\beta}{\theta}|s|^\theta, \qquad \forall s \in \mathbb{R}, \tag{2.17}$$

and then by (2.14) and the fact that $\theta > 2$, one has

$$\int_0^{+\infty} |F(v)|dx \leq \frac{\alpha}{2}\int_0^{+\infty} |v(x)|^2 dx + \frac{\beta}{\theta}\int_0^{+\infty} |v(x)|^\theta dx$$
$$\leq \frac{\alpha\mu_2^2}{2}\|v\|^2 + \frac{\beta\mu_\theta^\theta}{\theta}\|v\|^\theta, \tag{2.18}$$

which implies that J is well defined.
To prove that I is of class C^1 on $H_0^1(\mathbb{R}^+)$, it is sufficient to prove this property only for J. To this aim, firstly we prove that J is Gâteaux differentiable, and then we show that J'_G is continuous.

Claim 1. J is Gâteaux differentiable.
It is obvious that, for all $v, w \in H_0^1(\mathbb{R}^+)$ and almost every $x \in (0, +\infty)$

$$\lim_{\tau \longrightarrow 0} \frac{F(v(x)+\tau w(x)) - F(v(x))}{\tau} = \int_0^{+\infty} f(v(x))w(x)dx.$$

By the Lagrange Theorem there exists a real number $0 < \theta_\tau < |\tau|$ with $|\tau| \leq 1$ such that

$$F\big(v(x)+\tau w(x)\big) - F\big(v(x)\big) = \tau f\big(v(x)+\theta_\tau w(x)\big)w(x).$$

By using (F_2) and the inequality $|a+b|^r \leq C_r(|a|^r + |b|^r),\ a; b \in \mathbb{R}$ we get

$$\left|\frac{F\big(v(x)+\tau w(x)\big) - F\big(v(x)\big)}{\tau}\right| = \Big|f\big(v(x)+\theta_\tau w(x)\big)w(x)\Big|$$
$$\leq C\bigg(|v(x)||w(x)| + |w(x)|^2 + |w(x)||v(x)|^{\theta-1} + |w(x)|^\theta\bigg).$$

As the function $|v||w| + |w|^2 + |w||v|^{\theta-1} + |w|^\theta \in L^1(\mathbb{R}^+)$, by the Lebesgue dominated convergence theorem we have

$$\lim_{\tau \longrightarrow 0}\int_0^{+\infty} \frac{F(v+\tau w) - F(v)}{\tau}dx = \int_0^{+\infty} f(v)wdx.$$

Since the right-hand side, as a function of w, is a continuous and linear functional on $H_0^1(\mathbb{R}^+)$, it is the Gâteaux differential J'_G of J.

Claim 2. J'_G is continuous.
We complete the proof by checking that the function J'_G is continuous on $H^{-1}(\mathbb{R}^+)$. To this purpose, let take $\{v_n\}$ in $H_0^1(\mathbb{R}^+)$ such that $v_n \longrightarrow v$ as $n \longrightarrow +\infty$. Therefore, it follows from (2.14) that

(1) $v_n \longrightarrow v$ in $L^r(0, +\infty), \forall r \in [2, +\infty]$;
(2) $v_n(x) \longrightarrow v(x)$ a.e in $(0, +\infty)$;

We have for all $w \in H_0^1(\mathbb{R}^+)$

$$\Big|\langle J'_G(v_n) - J'_G(v), w\rangle\Big| \leq \int_0^{+\infty} |f(v_n) - f(v)||w|dx.$$

By using (F_1), it is easy to show that $f(v_n) - f(v) \in L^2(\mathbb{R}^+)$, and then by (2.14) and the Hölder's inequality we obtain for all $w \in H_0^1(\mathbb{R}^+)$

$$\Big|\langle J'_G(v_n) - J'_G(v), w\rangle\Big| \leq \int_0^{+\infty} |f(v_n) - f(v)||w|dx$$
$$\leq \mu_2\|f(v_n) - f(v)\|_2\ \|w\|.$$

Hence,

$$\|J'_G(v_n) - J'_G(v)\|_{H^{-1}(\mathbb{R}^+)} \leq \mu_2\|f(v_n) - f(v)\|_2. \tag{2.19}$$

Since f is continuous and by using once again the Lebesgue dominated convergence theorem, we get that

$$\|f(v_n) - f(v)\|_2 \longrightarrow 0, \qquad \text{as } n \longrightarrow +\infty.$$

By passing to the limit in (2.19) as $n \longrightarrow +\infty$, we conclude that

$$\|J'_G(v_n) - J'_G(v)\|_{H^{-1}(\mathbb{R}^+)} = 0,$$

which implies the continuity of J'_G. The proof is completed. □

Remark 2.2. *From the previous Lemma, we deduce that the critical points of I correspond to the weak solutions of the problem $(\mathcal{P})$.*

Definition 2.1. *The functional I satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted by $(PS)_c$ if every sequence $\{v_n\} \subset H_0^1(\mathbb{R}^+)$ satisfies*

$$I(v_n) \longrightarrow c \qquad \text{and} \qquad I'(v_n) \longrightarrow 0,\ n \longrightarrow +\infty, \tag{2.20}$$

possesses a strongly convergent subsequence.

Remark 2.3. *If I satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, then we say that I satisfies the (PS) condition.*

Our main tool is the following Mountain Pass theorem.

Proposition 2.1. *(*[22, Theorem 1.15], *Mountain Pass theorem) Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the (PS) condition, $I(0) = 0$ and*

(1) There exist $\rho, \alpha > 0$ such that $I(v) \geq \alpha$ whenever $\|v\| = \rho$.
(2) There exists $e \in X$ with $\|e\| > \rho$ such that $I(e) \leq 0$.

Then, I has at least a critical value $c_0 \geq \alpha$, which is characterized by

$$c_0 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I\big(\gamma(t)\big), \tag{2.21}$$

where

$$\Gamma = \big\{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\big\}.$$

In what follow, we will prove some technical lemmas which will be used for proving Theorem 3.1 and Theorem 3.2.

Lemma 2.3. *Assume that (F_1) holds. Then, there exist $\rho, \alpha_* > 0$ such that $I(v) \geq \alpha_*$ whenever $\|v\| = \rho$.*

Proof. From (2.14) and (2.17) we get

$$\begin{aligned}
I(v) &= \frac{1}{2}\|v\|^2 - \int_0^{+\infty} F(v)dx \\
&\geq \frac{1}{2}\Big(\|v'\|_2^2 + p\|v\|_2^2\Big) - \frac{\alpha}{2}\|v\|_2^2 - \frac{\beta}{\theta}\|v\|_\theta^\theta \\
&\geq \frac{1}{2}\|v'\|_2^2 + \frac{p-\alpha}{2}\|v\|_2^2 - \frac{\beta\mu_\theta^\theta}{\theta}\|v\|^\theta \\
&\geq C_1\|v\|^2 - C_2\|v\|^\theta, \quad \text{where } C_1 = \min\left\{\frac{1}{2}, \frac{p-\alpha}{2p}\right\} > 0 \text{ and } C_2 = \frac{\beta\mu_\theta^\theta}{\theta} > 0.
\end{aligned}$$

It follows for all $v \in H_0^1(\mathbb{R}^+)$ with $\|v\| = \rho$ that

$$I(v) \geq \rho^2(C_1 - C_2\rho^{\theta-2}).$$

Since $\theta > 2$, and by choosing $\rho > 0$ sufficiently small, we conclude that there exists a constant $\alpha_* > 0$ such that

$$I(v) \geq \alpha_*, \qquad \forall v \in H_0^1(\mathbb{R}^+) \text{ with } \|v\| = \rho,$$

and this finishes the proof. □

Lemma 2.4. *Assume that (F_1) and (F_2) hold. Then, there exists a function $e \in H_0^1(\mathbb{R}^+)$ with $\|e\| > \rho$ such that $I(e) \leq 0$.*

Proof. From (2.17) and (F_2), for all $M >> 1$ there exists $C_M >> 1$ such that

$$F(v) \geq M|v|^2 - C_M|v|, \qquad \forall v \in \mathbb{R}. \tag{2.22}$$

Let $v \in \mathcal{D}(\mathbb{R}^+)$ and $t \in (0, +\infty)$. Then by (2.15) and (2.22), one has

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|^2 - \int_0^{+\infty} F(tv)dx \\ &\leq \frac{t^2}{2}\|v\|^2 + tC_M \int_0^{+\infty} |v|\, dx - t^2 M \int_0^{+\infty} |v|^2\, dx \\ &\leq t^2\left(\frac{\|v\|^2}{2} - M\|v\|_2^2\right) + C_M t\|v\|_1. \end{aligned} \tag{2.23}$$

Choosing $M > 0$ such that $\|v\|^2 - 2M\|v\|_2^2 > 0$, then it follows from (2.23) that $I(tv) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. Therefore, there exists $t_1 > 0$ so large such that $\|t_1 v\| > \rho$ and $I(t_1 v) < 0$. Thus, we complete the proof by taking $e = t_1 v \in H_0^1(\mathbb{R}^+)$. □

Lemma 2.5. *Assume that (F_1) and (F_3) hold. Then, the functional I satisfies the (PS) condition.*

Proof. Let $\{v_n\} \subset H_0^1(\mathbb{R}^+)$ be a Palais-Smale sequence at level $c \in \mathbb{R}$. From (2.20), we easily see that there exists $C > 0$ such that

$$|I(v_n)| \leq C \qquad \text{and} \qquad |\langle I'(v_n), v_n\rangle| \leq C\|v_n\|, \tag{2.24}$$

for every $n \in \mathbb{N}$.

Step 1. $\{v_n\}$ is bounded in $H_0^1(\mathbb{R}^+)$.

We follow the argument [14], let set $w_n = \dfrac{v_n}{\|v_n\|}$ and we assume that $\{v_n\}$ is unbounded in $H_0^1(\mathbb{R}^+)$.

Under the condition (F_1) and from (2.17), for $x \in (0, +\infty)$ with $|v(x)| \leq L$, it follows that

$$\begin{aligned} |f(v)v - \mu F(v)| &\leq |f(v)v| + \mu|F(v)| \\ &\leq \left(\alpha|v|^2 + \beta|v|^\theta\right) + \mu\left(\frac{\alpha}{2}|v|^2 + \frac{\beta}{\theta}|v|^\theta\right) \\ &\leq \left(\alpha + \frac{\mu\alpha}{2} + \beta L^{\theta-2} + \frac{\mu\beta}{\theta}L^{\theta-2}\right)|v|^2 := c_2|v|^2, \qquad c_2 > 0. \end{aligned} \tag{2.25}$$

Combining (2.25) with (F_3) we get

$$f(v)v - \mu F(v) \geq -c_3|v|^2, \qquad \forall v \in \mathbb{R}. \tag{2.26}$$

By (2.15), (2.16), (2.24) and (2.26) we obtain

$$\begin{aligned} C\Big(1+\frac{1}{\mu}\|v_n\|\Big) &\geq I(v_n)-\frac{1}{\mu}\langle I'(v_n),v_n\rangle \\ &= \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|v_n\|^2+\frac{1}{\mu}\int_0^{+\infty}\Big(f(v_n)v_n-\mu F(v_n)\Big)dx \\ &\geq \Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|v_n\|^2-\frac{c_3}{\mu}\|v_n\|_2^2, \end{aligned}$$

then

$$\frac{\|v_n\|_2^2}{\|v_n\|^2}\geq\frac{\mu}{c_3}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)-\frac{\mu C}{c_3\|v_n\|^2}\Big(1+\frac{1}{\mu}\|v_n\|\Big).$$

Thus, by taking n sufficiently large such that

$$\frac{1}{\|v_n\|^2}\Big(1+\frac{1}{\mu}\|v_n\|\Big)\leq\frac{1}{2C}\Big(\frac{1}{2}-\frac{1}{\mu}\Big),$$

we get

$$\frac{\|v_n\|_2^2}{\|v_n\|^2}\geq\frac{\mu}{2c_3}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)>0,$$

since $\mu>2$. Consequently, we deduce that

$$\|w_n\|_2>0. \tag{2.27}$$

We set

$\Omega_n=\big\{x\in(0,+\infty):\ |v_n(x)|\leq L\big\}$, $\Omega_n'=(0,+\infty)\backslash\Omega_n$ and $A_n=\big\{x\in(0,+\infty):\ w_n(x)\neq 0\big\}$.

It follows from (2.27) that $\operatorname{meas}(A_n)>0$. Moreover, since $\|v_n\|\longrightarrow+\infty$ as $n\longrightarrow+\infty$, we obtain

$$|v_n(x)|\longrightarrow+\infty \text{ as } n\longrightarrow+\infty \text{ for } x\in A_n.$$

Hence, $A_n\subset\Omega_n'$ for n sufficiently large.

Thus, by (2.15), (2.17), (2.24) and Fatou's lemma, one has

$$\begin{aligned} 0 &= \lim_{n\longrightarrow+\infty}\frac{I(v_n)}{\|v_n\|^2} \\ &= \lim_{n\longrightarrow+\infty}\Big(\frac{1}{2}-\int_0^{+\infty}\frac{F(v_n)}{\|v_n\|^2}\,dx\Big) \\ &\leq \frac{1}{2}-\lim_{n\longrightarrow+\infty}\Big(\int_{\Omega_n}\frac{F(v_n)}{v_n^2}w_n^2\,dx+\int_{\Omega_n'}\frac{F(v_n)}{v_n^2}w_n^2\,dx\Big) \\ &\leq \frac{1}{2}-\lim_{n\longrightarrow+\infty}\Big[\int_{\Omega_n}\Big(\frac{\alpha}{2}+\frac{\beta}{\theta}L^{\theta-2}\Big)w_n^2\,dx+\int_{\Omega_n'}\frac{F(v_n)}{v_n^2}w_n^2\,dx\Big] \\ &\leq \frac{1}{2}-\lim_{n\longrightarrow+\infty}\Big[\Big(\frac{\alpha}{2}+\frac{\beta}{\theta}L^{\theta-2}\Big)\int_{\Omega_n}w_n^2\,dx+\int_{\Omega_n'}\frac{F(v_n)}{v_n^2}w_n^2\,dx\Big] \\ &\leq \frac{1}{2}+c_4-\liminf_{n\longrightarrow+\infty}\int_{\Omega_n'}\frac{F(v_n)}{v_n^2}w_n^2\,dx \\ &\leq \frac{1}{2}+c_4-\liminf_{n\longrightarrow+\infty}\int_{A_n}\frac{F(v_n)}{v_n^2}w_n^2\,dx \\ &\leq \frac{1}{2}+c_4-\int_0^{+\infty}\liminf_{n\longrightarrow+\infty}\frac{F(v_n)}{v_n^2}\Big[\chi_{A_n}(x)\Big]w_n^2\,dx=-\infty \end{aligned}$$

which is a contradiction. Hence $\{v_n\}$ is bounded in $H_0^1(\mathbb{R}^+)$.

Step 2. $\{v_n\}$ converges strongly in $H_0^1(\mathbb{R}^+)$.
First of all, following [8], we will prove that for any $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$\limsup_{n\to\infty} \int_{R_1}^{+\infty} \left(|v_n'|^2 + pv_n^2\right) dx \leq \varepsilon. \tag{2.28}$$

Let $R > 0$ be a constant and $\psi_R \in C^1(\mathbb{R}^+)$ be a non-decreasing cut-off function such that $0 \leq \psi_R \leq 1$,

$$\psi_R(x) = \begin{cases} 1, & \text{for } x \geq R, \\ 0, & \text{for } 0 \leq x \leq R/2, \end{cases}$$

and

$$|\psi_R'(x)| \leq \frac{C}{R}, \qquad \forall x \in (0, +\infty). \tag{2.29}$$

It follows from the definition of a (PS) sequence that

$$\langle I'(v_n), \psi_R v_n \rangle = o_n(1).$$

Therefore, by (2.16) we have

$$\begin{aligned} o_n(1) &= \langle I'(v_n), \psi_R v_n \rangle \\ &= \int_0^{+\infty} \Big(v_n'(\psi_R v_n)' + pv_n\psi_R v_n\Big)dx - \int_0^{+\infty} f(v_n)\psi_R v_n dx \\ &= \int_0^{+\infty} v_n' v_n \psi_R' dx + \int_0^{+\infty} \Big(|v_n'|^2 + pv_n^2\Big)\psi_R dx - \int_0^{+\infty} f(v_n)\psi_R v_n dx \\ &\geq \int_0^{+\infty} v_n' v_n \psi_R' dx + \int_R^{+\infty} \Big(|v_n'|^2 + pv_n^2\Big) dx - \int_0^{+\infty} f(v_n)\psi_R v_n dx. \end{aligned} \tag{2.30}$$

From (2.14), (2.29), the Hölder inequality and the boundedness of $\{v_n\}$ in $H_0^1(\mathbb{R}^+)$, one has

$$\Big|\int_0^{+\infty} v_n' v_n \psi_R' dx\Big| \leq \frac{C}{R}\|v_n'\|_2\|v_n\|_2 \leq \frac{C_1}{R}. \tag{2.31}$$

On the other hand, it follows from (F_1) that

$$\int_0^{+\infty} f(v_n)v_n\psi_R dx \leq \int_{R/2}^{+\infty} \Big(\alpha|v_n|^2 + \beta|v_n|^\theta\Big)dx \leq C_2\int_{R/2}^{+\infty} \Big(|v_n|^2 + |v_n|^\theta\Big)dx.$$

Since $\{v_n\}$ is bounded in $H_0^1(\mathbb{R}^+)$, by (2.14), for any $\varepsilon > 0$ we can choose $R_1 > 0$ so that

$$\int_{R_1}^{+\infty} |v_n|^r dx \leq \frac{\varepsilon}{2C_2}, \qquad \forall r \in [2, +\infty]. \tag{2.32}$$

Hence, introducing (2.31), (2.32) and (2.32) in (2.30) we get for all $n \in \mathbb{N}$ and $R > 2R_1$

$$\int_{R_1}^{+\infty} \Big(|v_n'|^2 + pv_n^2\Big)dx \leq \int_R^{+\infty} \Big(|v_n'|^2 + pv_n^2\Big)dx$$

$$\leq \int_0^{+\infty} f(v_n)\psi_R v_n dx - \int_0^{+\infty} v_n' v_n \psi_R' dx + o_n(1) \leq \varepsilon + \frac{C_1}{R} + o_n(1).$$

Consequently, for $R > 0$ sufficiently large, we easily obtain (2.28).

Next, we shall prove that $\{v_n\}$ converges strongly in $H_0^1(\mathbb{R}^+)$. The sequence $\{v_n\}$ is bounded in $H_0^1(\mathbb{R}^+)$, then passing to a subsequence if necessary, there exists $v_0 \in H_0^1(\mathbb{R}^+)$ such that

$$\begin{aligned} &v_n \rightharpoonup v_0 \text{ weakly in } H_0^1(\mathbb{R}^+), \\ &v_n \to v_0 \text{ strongly in } L^2(0,R), \\ &v_n(x) \to v_0(x) \qquad \text{a.e in } (0,+\infty). \end{aligned} \tag{2.33}$$

It is easy to check that

$$\|v_n - v_0\|^2 = \langle I'(v_n) - I'(v_0), v_n - v_0\rangle + \int_0^{+\infty} \Big(f(v_n) - f(v_0)\Big)(v_n - v_0)dx. \tag{2.34}$$

Since $\{v_n\}$ is a (PS) sequence of I and $v_n \rightharpoonup v_0$ in $H_0^1(\mathbb{R}^+)$, one has

$$\langle I'(v_n) - I'(v_0), v_n - v_0\rangle \longrightarrow 0, \qquad \text{as } n \longrightarrow +\infty. \tag{2.35}$$

On one hand, by using (F_1) and (2.14) we can prove that

$$\begin{aligned} \int_0^{+\infty} |f(v_n)|^2 dx &\leq C\Big[\int_0^{+\infty} |v_n|^2 dx + \int_0^{+\infty} |v_n|^{2(\theta-1)} dx\Big] \\ &\leq C\Big(\|v_n\|^2 + \|v_n\|^{2(\theta-1)}\Big), \end{aligned}$$

and taking into account that $\{v_n\}$ is bounded in $H_0^1(\mathbb{R}^+)$ it follows from the inequality above that $\{f(v_n)\}$ is bounded in $L^2(\mathbb{R}^+)$.
Thus, from (2.33) and the Hölder inequality we obtain

$$\int_0^{R_1} \Big(f(v_n) - f(v_0)\Big)(v_n - v_0)dx \longrightarrow 0, \qquad \text{as } n \longrightarrow +\infty. \tag{2.36}$$

On the other hand, from (2.28), the Hölder inequality, the weakly lower semi-continuity of the norm and the fact that $\{f(v_n)\}$ is bounded in $L^2(\mathbb{R}^+)$, one has

$$\begin{aligned} \mathcal{F}_n &= \int_{R_1}^{+\infty} \Big(f(v_n) - f(v_0)\Big)(v_n - v_0)dx \\ &\leq \Big(\int_{R_1}^{+\infty} |f(v_n) - f(v_0)|^2\Big)^{\frac{1}{2}} \Big(\int_{R_1}^{+\infty} |v_n - v_0|^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\|f(v_n)\|_2 + \|f(v_0)\|_2\Big)\Bigg[\Big(\int_{R_1}^{+\infty} v_n^2\, dx\Big)^{\frac{1}{2}} + \Big(\int_{R_1}^{+\infty} v_0^2\, dx\Big)^{\frac{1}{2}}\Bigg] \\ &\leq C\Bigg[\Big(\int_{R_1}^{+\infty} |v_n'|^2 + pv_n^2\, dx\Big)^{\frac{1}{2}} + \Big(\int_{R_1}^{+\infty} |v_0'|^2 + pv_0^2\, dx\Big)^{\frac{1}{2}}\Bigg] \\ &\leq C\Bigg[\Big(\int_{R_1}^{+\infty} |v_n'|^2 + pv_n^2\, dx\Big)^{\frac{1}{2}} + \Big(\liminf_{n\longrightarrow+\infty} \int_{R_1}^{+\infty} |v_n'|^2 + pv_n^2\, dx\Big)^{\frac{1}{2}}\Bigg] \\ &\leq C\Big(\limsup_{n\longrightarrow+\infty} \int_{R_1}^{+\infty} |v_n'|^2 + pv_n^2\, dx\Big)^{\frac{1}{2}} \\ &\leq C\sqrt{\varepsilon}. \end{aligned} \tag{2.37}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\int_{R_1}^{+\infty} \Big(f(v_n) - f(v_0)\Big)(v_n - v_0)dx \longrightarrow 0, \qquad \text{as } n \longrightarrow +\infty. \tag{2.38}$$

Consequently, from (2.35), (2.36) and (2.38), and by passing to the limit in (2.34) as $n \longrightarrow +\infty$, we deduce that

$$\|v_n - v_0\| \longrightarrow 0,$$

which means that $v_n \to v_0$ in $H_0^1(\mathbb{R}^+)$. This completes the proof. □

3. Main results

Now, we are ready to state our main results of this paper as follows.

Theorem 3.1. *Suppose that* $(F_1) - (F_3)$ *are satisfied. Then problem* (1.1) *has at least one nontrivial solution. Moreover, if* $f(u) \geq 0$ *for all* $u \in \mathbb{R}$ *then the solution is non-negative.*

Proof of Theorem 3.1. We have $I \in C\big(H_0^1(\mathbb{R}^+), \mathbb{R}\big)$ and $I(0) = 0$. By Lemmas 2.3 and 2.4, the functional I satisfies the geometric property of the Mountain Pass theorem. Lemma 2.5 implies that the functional I satisfies the (PS) condition. Therefore, applying Proposition 2.1, we deduce that there exists $v_0 \in H_0^1(\mathbb{R}^+)$ such that

$$I(v_0) = c_0 \geq \alpha_* > 0 \text{ and } I'(v_0) = 0.$$

Thus, the pair (λ_{v_0}, v_0) is a nontrivial solution for system (1.10).

Suppose that the function f is non-negative on $\mathbb{R}$.
For a function u, we will use the following decomposition

$$u = u^+ + u^-,$$

where

$$u^+ = \max\{u, 0\} \qquad \text{and} \qquad u^- = \min\{u, 0\},$$

and it follows that

$$u^+ u^- = (u^+)'(u^-)' = 0. \tag{3.39}$$

Since the problem ($\mathcal{P}$) admits at least one nontrivial solution $v_0 \in H_0^1(\mathbb{R}^+)$, that is

$$(v_0, w) - \int_0^{+\infty} f(v_0) w dx = 0, \qquad \forall w \in H_0^1(\mathbb{R}^+). \tag{3.40}$$

Replacing w by v_0^- in (3.40) and taking into account (3.39) we obtain

$$\|v_0^-\|^2 = \int_0^{+\infty} f(v_0) v_0^- dx.$$

Since f is non-negative and $v_0^- \leq 0$, it follows necessarily that

$$\|v_0^-\|^2 = 0,$$

and then

$$v_0^- = 0, \qquad \text{a.e in } (0, +\infty).$$

Consequently

$$v_0 = v_0^+ + v_0^- = v^+ \geq 0.$$

The proof is ended. □

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. If* f *is* k*-lipschitz in* $(0, +\infty)$ *with* $0 < k < p$*, then problem* (1.1) *has exactly one solution.*

Proof. From Theorem 3.1, the problem (1.1) has at least one nontrivial solution. Let v_1 and v_2 be two solutions of our problem, and we put $v = v_1 - v_2$.
It is easy to check that v satisfies the following problem

$$\begin{cases} -v'' + pv = f(v_1) - f(v_2), & x \in (0, +\infty), \\ v(0) = 0, \end{cases} \tag{3.41}$$

and then by multiplying the first equation of (3.41) by v and integrating over the interval $(0, +\infty)$, we obtain

$$\|v\|^2 = \int_0^{+\infty} \Big(f(v_1) - f(v_2) \Big) v dx. \tag{3.42}$$

Since f is k-lipschitz in $(0, +\infty)$ with $0 < k < p$ it follows that

$$|f(x) - f(y)| \leq k|x - y|, \qquad \forall x, y \in (0, +\infty),$$

and then

$$\int_0^{+\infty} \Big(f(v_1) - f(v_2) \Big) v dx \leq \frac{k}{p} \|v\|^2.$$

Consequently, we get

$$\left(1 - \frac{k}{p}\right) \|v\|^2 \leq 0$$

and since $0 < k < p$ we deduce that $v \equiv 0$ and the proof is completed. □

4. Example

We finish by giving an example in which the hypotheses of Theorem 3.1 are satisfied. We consider the following function

$$f(t) = \begin{cases} \frac{1}{2} t^{3/2} + \frac{1}{2} t, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0, \end{cases}$$

and simple computation yields

$$F(t) = \begin{cases} \frac{1}{5} t^{5/2} + \frac{1}{4} t^2, & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

It is clear that $f \in C(\mathbb{R}, \mathbb{R}^+)$ and

- $|f(t)| \leq \dfrac{1}{2}|t|^{3/2} + \dfrac{1}{2}|t|$, for all $t \in \mathbb{R}$, then $\alpha = \beta = \dfrac{1}{2}$ and $\theta = \dfrac{5}{2} > 2$.
- $\displaystyle\lim_{t \longrightarrow +\infty} \frac{F(t)}{t^2} = \lim_{t \longrightarrow +\infty} \left(\frac{1}{5} t^{3/2} + \frac{1}{4} \right) = +\infty.$
- $F(t) \geq 0$ and $f(t)t + \dfrac{5}{8}t^2 - \dfrac{5}{2}F(t) = \dfrac{1}{2}t^2 \geq 0$, for all $|t| \geq 1$, then $\mu = \dfrac{5}{2} > 2$, $c_1 = \dfrac{5}{8}$ and $L = 1$.

Thus, the function f satisfies the conditions (F_1), (F_2) and (F_3) and then by Theorem 3.1 the problem

$$\begin{cases} -\left(a + b \displaystyle\int_0^{+\infty} |u'(x)|^2 dx \right) u'' + pu = f(u), & x \in (0, +\infty), \\ u(0) = 0. \end{cases}$$

has at least one non-negative solution for all $a > 0$, $b \geq 0$ and $p > \dfrac{1}{2}$.
But the function f does not satisfy the condition (f_2) in [6], therefore, our results extend and improve those obtained in [6].

5. Conclusions

In this work, by using variational methods and critical points theory, we have established the existence and uniqueness of nontrivial solution for a class of one-dimensional Kirchhoff-type equations involving a superlinear nonlinearity. Our study was motivated by the work of [6], which provided valuable insights into our work. However, upon careful examination, we discovered a contradiction in the original hypothesis presented in [6]. This discrepancy prompted us to reevaluate the problem under different conditions.
The main steps of our work can be outlined as follows: First, by using an appropriate change of variables, we have transformed the problem into an equivalent system in which there appears to be a semilinear equation. Next, we have proved that the functional energy corresponding to the problem satisfies the mountain pass geometry. Finally, without any compactness criteria for the embedding of the working space, and by using the Mountain Pass theorem, we were able to show the existence of nontrivial solution to our problem. Moreover, the uniqueness and positiveness of the obtained solutions are also discussed.

Acknowledgments

The authors thank an anonymous referee for a careful reading and some helpful comments, which greatly improve the manuscript.

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