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Positive solutions for multipoint boundary value problem of fractional differential equation with parameter

ZHI LIU AND WEIBING WANG

ABSTRACT. We discuss the existence of positive solutions for a fractional multipoint boundary value problem with parameter. By applying the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem, we obtain sufficient conditions for the existence of at least one or two positive solutions. Our main results highlight the influence of the parameter in different ranges on the existence of positive solutions.

1. INTRODUCTION

In this paper, we are concerned with the following nonlinear fractional multipoint boundary value problem

$$\begin{cases} -u^{(m)}(t) + M\,{}^{C}D^{\alpha}u(t) = f(t,u(t)), & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, & u(1) - \sum_{i=1}^{q} \gamma_i u(\eta_i) = b, \end{cases} \tag{1.1}$$

where $m-1 < \alpha < m$, $m \geq 2, q \geq 1$ are integers, $b \geq 0$ is a parameter, and ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of α order

$${}^{C}D^{\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} u^{(m)}(s)ds.$$

Throughout this paper, we always suppose that:

(H_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous,

(H_2) $\gamma_i > 0 (1 \leq i \leq q)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_q < 1$, $0 < \psi := \sum_{i=1}^{q} \gamma_i \eta_i^{m-1} < 1$ and $M > 0$.

Fractional differential equations have gained importance due to their broad application in various fields of science and engineering, such as control theory, physics, chemistry, etc. For more details, see [10, 16, 18] and its references. Recently, many scholars pay attention to the existence and multiplicity of solutions or positive solutions of nonlinear fractional differential equations. The main tools used are techniques of nonlinear analysis including fixed point theorems [2, 4, 23, 25], Leray-Shauder theory [1, 8, 22], monotone iterative method [5, 12, 24], etc.

In [9], Jia and Zhang studied the following fractional multipoint boundary value problem with changing sign nonlinearity

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u^{(i)}(1) = \sum_{j=0}^{m-2} \eta_j u'(\xi_j), \end{cases} \tag{1.2}$$

where $n-1 < \alpha \leq n$, D_{0+}^{α} is the Riemann-Liouville derivative, $1 \leq i \leq n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, λ is a parameter. By means of the Guo-Krasnosel'skii fixed point

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Corresponding author: Zhi Liu; zliu67@163.com

theorem, the authors obtained an interval of λ such that (1.2) has at least one positive solution for any λ lying in the interval.

In [21], Xu and Zhang discussed the following fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{cases} \tag{1.3}$$

where $1 < \alpha < 2$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ with $\sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} < 1$. By the Leray-Schauder alternative principle and Guo-Krasnosel'skii fixed point theorem, the authors obtained the existence of multiple positive solutions for (1.3). For other works about multipoint boundary value problems for fractional differential equations, we refer the reader to [11, 14, 19].

Recently, the differential equations with mixing ordinary derivative and fractional derivative have been confirmed to be applicable for describing specific physical phenomena. In modelling the motion of a rigid body immersed in a Newtonian fluid, Torvik and Bagley [17] established the following equation

$$Au''(t) + BD^{\frac{3}{2}}u(t) + Cu(t) = f(t), \tag{1.4}$$

where A, B, C are real numbers, $f(t)$ is the known function, which is called Bagley-Torvik fractional differential equation. After that, there are many important results related to this equation. For example, Staněk [15] considered the following Bagley-Torvik equation

$$\begin{cases} u''(t) + A\,{}^{C}D^{\alpha}u(t) = f(t, u(t), {}^{C}D^{\mu}u(t), u'(t)), \quad t \in [0,T], \\ u'(0) = 0, \ u(T) + au'(T) = 0, \end{cases} \tag{1.5}$$

where $A \in \mathbb{R}\backslash\{0\}$, $\alpha \in (1,2)$, $\mu \in (0,1)$. The author obtained the existence of solutions to the problem (1.5) by the Leray-Schauder alternative principle. For more information on Bagley-Torvik equations we refer to [6, 13, 20].

Fazli et al. [5] studied the following fractional differential equation with nonlinear boundary conditions

$$\begin{cases} u^{(m)}(t) + M\,{}^{C}D^{\alpha}u(t) = f(t, u(t)), \quad t \in [0,T], \\ g_k(u^{(k)}(t_0), u^{(k)}(t_1), \cdots, u^{(k)}(t_r)) = 0, \quad k = 0, 1, \cdots, m-1, \end{cases} \tag{1.6}$$

where $m - 1 < \alpha < m$, $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_r = T$. Under appropriate conditions, the authors showed the existence of extremal solutions for (1.6) by the upper and lower solutions method and monotone iterative technique.

As far as we know, few papers have considered the existence of positive solutions for higher-order fractional differential equations which contain ordinary derivative and fractional derivative. Motivated by the above works, we investigate the existence, nonexistence and multiplicity of positive solutions for (1.1). Here, a function $u \in C^{m-1}[0,1] \cap C^m(0,1)$ is said to be a positive solution of (1.1) if u satisfies (1.1) and $u > 0$ on $(0,1]$.

The organization of this paper is as follows. In Section 2, we give the properties of Green's function of the corresponding linear problem and some required Lemmas that will be used in the sequel. The main results are given in Section 3. Finally, an example is presented to demonstrate the applicability of our main results.

2. Preliminaries

Definition 2.1. *([7]). The two-parameter Mittag-Leffler function $E_{n_1,n_2}(\xi)$ is defined by*

$$E_{n_1,n_2}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(kn_1 + n_2)}, \quad n_1, n_2 > 0, \ \xi \in \mathbb{R}.$$

Lemma 2.1. *([7]). If $n_1, n_2 > 0$, $E_{n_1,n_2}(\xi)$ is convergent for $\xi \in \mathbb{R}$ and $E_{n_1,n_2}(\xi) > 0$ for $\xi \geq 0$.*

Lemma 2.2. *Let $M, \omega > 0$ and λ be a positive integer. Then it holds*

$$\frac{d}{dt} t^{\lambda} E_{\omega,\lambda+1}(Mt^{\omega}) = t^{\lambda-1} E_{\omega,\lambda}(Mt^{\omega}), \quad \forall t > 0.$$

Proof. By a direct calculation, we get

$$\begin{aligned}\frac{d}{dt} t^{\lambda} E_{\omega,\lambda+1}(Mt^{\omega}) =& \frac{d}{dt} t^{\lambda} \left(\frac{1}{\Gamma(\lambda+1)} + \frac{Mt^{\omega}}{\Gamma(\omega+\lambda+1)} + \frac{M^2 t^{2\omega}}{\Gamma(2\omega+\lambda+1)} + \cdots \right) \\ =& \frac{d}{dt} \left(\frac{t^{\lambda}}{\Gamma(\lambda+1)} + \frac{Mt^{\omega+\lambda}}{\Gamma(\omega+\lambda+1)} + \frac{M^2 t^{2\omega+\lambda}}{\Gamma(2\omega+\lambda+1)} + \cdots \right) \\ =& t^{\lambda-1} \left(\frac{1}{\Gamma(\lambda)} + \frac{Mt^{\omega}}{\Gamma(\omega+\lambda)} + \frac{M^2 t^{2\omega}}{\Gamma(2\omega+\lambda)} + \cdots \right) \\ =& t^{\lambda-1} E_{\omega,\lambda}(Mt^{\omega}),\end{aligned}$$

which completes the proof. □

Lemma 2.3. *Let $h \in C[0,1]$, then the following boundary value problem*

$$\begin{cases} -u^{(m)}(t) + M\,{}^{C}D^{\alpha}u(t) = h(t), & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, & u(1) - \sum_{i=1}^{q} \gamma_i u(\eta_i) = b \end{cases} \tag{2.7}$$

has a unique solution $u \in C^{m-1}[0,1] \cap C^{m}(0,1)$ given by

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{t^{m-1}}{1-\psi} \sum_{i=1}^{q} \gamma_i \int_0^1 G(\eta_i, s)h(s)ds + \frac{bt^{m-1}}{1-\psi}, \tag{2.8}$$

where

$$G(t,s) = \begin{cases} (t-ts)^{m-1} E_{m-\alpha,m}(M(1-s)^{m-\alpha}) \\ \quad -(t-s)^{m-1} E_{m-\alpha,m}(M(t-s)^{m-\alpha}), & 0 \le s \le t \le 1, \\ (t-ts)^{m-1} E_{m-\alpha,m}(M(1-s)^{m-\alpha}), & 0 \le t \le s \le 1. \end{cases} \tag{2.9}$$

Proof. We first show that (2.7) has at most a solution. Let u_1, u_2 be two solutions of (2.7) and $v = u_1 - u_2$. Clearly,

$$\begin{cases} -v^{(m)}(t) + M\,{}^{C}D^{\alpha}v(t) = 0, & t \in (0,1), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v(1) - \sum_{i=1}^{q} \gamma_i v(\eta_i) = 0. \end{cases} \tag{2.10}$$

From the Laplace transform formula of ordinary derivative and Lemma 2.9 of [7], we have

$$L[v^{(m)}](s) = s^m V(s) - \sum_{k=0}^{m-1} s^{m-k-1} v^{(k)}(0),$$

$$L[{}^{C}D^{\alpha}v](s) = s^{\alpha} V(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} v^{(k)}(0), \tag{2.11}$$

where L denotes the Laplace transform operator, V denotes the Laplace transform of v. Doing Laplace transform to (2.10), we get

$$-(s^m V(s) - v^{(m-1)}(0)) + M(s^{\alpha} V(s) - s^{\alpha-m} v^{(m-1)}(0)) = 0,$$

$$V(s) = \frac{v^{(m-1)}(0) - Ms^{\alpha-m} v^{(m-1)}(0)}{s^m - Ms^{\alpha}} = \frac{v^{(m-1)}(0)}{s^m}.$$

Using the following formula (see [3])

$$\int_0^{\infty} e^{-pt} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^{\alpha}) dt = \frac{k! p^{\alpha-\beta}}{(p^{\alpha} \mp a)^{k+1}}, \tag{2.12}$$

by the inverse Laplace transform, we obtain

$$v(t)=\frac{v^{(m-1)}(0)}{\Gamma(m)}t^{m-1}.$$

It follows from $v(1)-\sum_{i=1}^{q}\gamma_i v(\eta_i)=0$ and (H_2) that $v=0$. Hence, (2.7) has at most a solution.

Next, we show that (2.8) is a solution of (2.7). Substituting (2.9) into (2.8), we get

$$\begin{aligned}u(t)=&-\int_0^t (t-s)^{m-1}E_{m-\alpha,m}(M(t-s)^{m-\alpha})h(s)ds\\&+t^{m-1}\int_0^1(1-s)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha})h(s)ds\\&+\frac{\psi t^{m-1}}{1-\psi}\int_0^1(1-s)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha})h(s)ds\\&-\frac{t^{m-1}}{1-\psi}\sum_{i=1}^{q}\gamma_i\int_0^{\eta_i}(\eta_i-s)^{m-1}E_{m-\alpha,m}(M(\eta_i-s)^{m-\alpha})h(s)ds+\frac{bt^{m-1}}{1-\psi}\\=&-\int_0^t(t-s)^{m-1}E_{m-\alpha,m}(M(t-s)^{m-\alpha})h(s)ds+\frac{Ct^{m-1}}{\Gamma(m)},\end{aligned}\tag{2.13}$$

where

$$\begin{aligned}C=&\frac{\Gamma(m)}{1-\psi}\left(\int_0^1(1-s)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha})h(s)ds\right.\\&\left.-\sum_{i=1}^{q}\gamma_i\int_0^{\eta_i}(\eta_i-s)^{m-1}E_{m-\alpha,m}(M(\eta_i-s)^{m-\alpha})h(s)ds+b\right).\end{aligned}$$

Differentiating (2.13), by Lemma 2.2, we have

$$u'(t)=-\int_0^t(t-s)^{m-2}E_{m-\alpha,m-1}(M(t-s)^{m-\alpha})h(s)ds+\frac{Ct^{m-2}}{\Gamma(m-1)},$$

$$\vdots$$

$$u^{(m-2)}(t)=-\int_0^t(t-s)E_{m-\alpha,2}(M(t-s)^{m-\alpha})h(s)ds+Ct,$$

$$u^{(m-1)}(t)=-\int_0^t E_{m-\alpha,1}(M(t-s)^{m-\alpha})h(s)ds+C,$$

$$\begin{aligned}u^{(m)}(t)=&-M(m-\alpha)\int_0^t(t-s)^{m-\alpha-1}E^{(1)}_{m-\alpha,1}(M(t-s)^{m-\alpha})h(s)ds-h(t)\\=&-M\int_0^t(t-s)^{m-\alpha-1}E_{m-\alpha,m-\alpha}(M(t-s)^{m-\alpha})h(s)ds-h(t),\end{aligned}$$

where $E^{(1)}_{m-\alpha,1}(M(t-s)^{m-\alpha})=\sum_{k=1}^{\infty}\frac{k(M(t-s)^{m-\alpha})^{k-1}}{\Gamma(k(m-\alpha)+1)}$. According to the continuity of $E_{m-\alpha,j}(j=1,2,\cdots,m,m-\alpha)$ and h, we can check that

$$u\in C^{m-1}[0,1]\cap C^m(0,1),$$

$$u(0)=u'(0)=\cdots=u^{(m-2)}(0)=0,\ \ u^{(m-1)}(0)=C.$$

Doing Laplace transform to (2.13), from (2.12) we obtain

$$U(s)=\frac{-H(s)}{s^m-Ms^{\alpha}}+\frac{C}{s^m},$$

where U, H denote the Laplace transform of u, h. It follows from (2.11) that

$$L[^C D^\alpha u](s) = s^\alpha U(s) - s^{\alpha-m}u^{(m-1)}(0) = \frac{-H(s)}{s^{m-\alpha} - M}.$$

By doing inverse Laplace transform to above equality, we have

$$^C D^\alpha u(t) = -\int_0^t (t-s)^{m-\alpha-1} E_{m-\alpha,m-\alpha}(M(t-s)^{m-\alpha})h(s)ds.$$

Hence,

$$-u^{(m)}(t) + M\, ^C D^\alpha u(t) = h(t).$$

It is easily verified that $u(1) - \sum_{i=1}^q \gamma_i u(\eta_i) = b$. Therefore, (2.8) is a solution of (2.7). This completes the proof. □

Lemma 2.4. *(1) For any $t, s \in [0,1]$,*

$$0 \le G(t,s) \le N(s) := (1-s)^{m-1} E_{m-\alpha,m}(M(1-s)^{m-\alpha}).$$

(2) For any $\theta \in (0, \frac{1}{2})$, $s \in [0,1]$,

$$\min_{t\in[\theta,1-\theta]} G(t,s) \ge K_\theta N(s), \quad K_\theta := \frac{M^{k_0}\theta^{k_0(m-\alpha)+2m-2}}{\Gamma(k_0(m-\alpha)+m)E_{m-\alpha,m}(M)} \in (0,1),$$

where $k_0 = [\frac{1}{m-\alpha}] + 1$, $[x]$ denotes the integer part of the number x.

Proof. (1) is obvious. Here we only prove (2). For $0 \le s \le t \le 1$, we get

$$\begin{aligned} G(t,s) =& (t-ts)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha}) - (t-s)^{m-1}E_{m-\alpha,m}(M(t-s)^{m-\alpha}) \\ =& (t-ts)^{m-1}\left(\frac{1}{\Gamma(m)} + \frac{M(1-s)^{m-\alpha}}{\Gamma(2m-\alpha)} + \cdots + \frac{(M(1-s)^{m-\alpha})^{k_0}}{\Gamma(k_0(m-\alpha)+m)} + \cdots\right) \\ &- (t-s)^{m-1}\left(\frac{1}{\Gamma(m)} + \frac{M(t-s)^{m-\alpha}}{\Gamma(2m-\alpha)} + \cdots + \frac{(M(t-s)^{m-\alpha})^{k_0}}{\Gamma(k_0(m-\alpha)+m)} + \cdots\right) \\ \ge& \frac{M^{k_0}}{\Gamma(k_0(m-\alpha)+m)}\left(t^{m-1}(1-s)^{k_0(m-\alpha)+m-1} - (t-s)^{k_0(m-\alpha)+m-1}\right). \end{aligned}$$

Setting $f(x) = d^{m-1}(1-x)^{k_0(m-\alpha)+m-1} - (d-x)^{k_0(m-\alpha)+m-1}$, $x \in [0,d]$, where $0 < d \le 1$ is a constant, we obtain

$$\begin{aligned} f'(x) =& (k_0(m-\alpha)+m-1)\left((d-x)^{k_0(m-\alpha)+m-2} - d^{m-1}(1-x)^{k_0(m-\alpha)+m-2}\right) \\ =& (k_0(m-\alpha)+m-1)\left((d-x)^{m-1}(d-x)^{k_0(m-\alpha)-1} - (d-dx)^{m-1}(1-x)^{k_0(m-\alpha)-1}\right). \end{aligned}$$

It is easy to check that $f'(x) \le 0$ for any $x \in [0,d]$, which implies that

$$\min_{x\in[0,d]} f(x) = f(d) = d^{m-1}(1-d)^{k_0(m-\alpha)+m-1}.$$

Hence, for any $t \in [\theta, 1-\theta]$, we get

$$\begin{aligned} \frac{G(t,s)}{N(s)} \ge& \frac{M^{k_0}t^{m-1}(1-t)^{k_0(m-\alpha)+m-1}}{\Gamma(k_0(m-\alpha)+m)(1-s)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha})} \\ \ge& \frac{M^{k_0}\theta^{k_0(m-\alpha)+2m-2}}{\Gamma(k_0(m-\alpha)+m)E_{m-\alpha,m}(M)} = K_\theta. \end{aligned}$$

It follows from the definition of $E_{m-\alpha,m}$ that $\Gamma(k_0(m-\alpha)+m)E_{m-\alpha,m}(M) \ge M^{k_0}$, which implies that $K_\theta \in (0,1)$ for any $\theta \in (0, \frac{1}{2})$.

In addition, for $0 \le t \le s \le 1$, $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} G(t,s) =& t^{m-1}(1-s)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha}) \\ =& t^{m-1}N(s) \ge \theta^{m-1}N(s) \ge K_\theta N(s). \end{aligned}$$

This completes the proof. □

Lemma 2.5. *If* $h \in C^+[0,1] := \{x \in C[0,1] : x(t) \ge 0, t \in [0,1]\}$, *then the unique solution* u *of* (2.7) *is nonnegative and satisfies*

$$\min_{t\in[\theta,1-\theta]} u(t) \ge K_\theta \|u\|.$$

Proof. Clearly, $u \ge 0$. From (2.8) and Lemma 2.4, we obtain that for any $t \in [0,1]$,

$$\begin{aligned} u(t) =& \int_0^1 G(t,s)h(s)ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)h(s)ds + \frac{bt^{m-1}}{1-\psi} \\ \le& \int_0^1 N(s)h(s)ds + \frac{1}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)h(s)ds + \frac{b}{1-\psi}, \end{aligned}$$

and so

$$\|u\| \le \int_0^1 N(s)h(s)ds + \frac{1}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)h(s)ds + \frac{b}{1-\psi}.$$

On the other hand, for any $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} u(t) =& \int_0^1 G(t,s)h(s)ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)h(s)ds + \frac{bt^{m-1}}{1-\psi} \\ \ge& K_\theta \int_0^1 N(s)h(s)ds + \frac{\theta^{m-1}}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)h(s)ds + \frac{b\theta^{m-1}}{1-\psi} \\ \ge& K_\theta \left(\int_0^1 N(s)h(s)ds + \frac{1}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)h(s)ds + \frac{b}{1-\psi} \right). \end{aligned}$$

Consequently, the conclusion is obviously true. □

3. Main results

Let $E = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$. Define the cone $P \subset E$ by

$$P = \left\{ u \in C^+[0,1] : \min_{t\in[\theta_0,1-\theta_0]} u(t) \ge K_{\theta_0}\|u\| \right\}$$

and the operator $A : E \to E$ by

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{bt^{m-1}}{1-\psi}, \tag{3.14}$$

where $\theta_0 \in (0, \frac{1}{2})$ such that $\forall \eta_i \in [\theta_0, 1-\theta_0]$ and $K_{\theta_0} = \frac{M^{k_0}\theta_0^{k_0(m-\alpha)+2m-2}}{\Gamma(k_0(m-\alpha)+m)E_{m-\alpha,m}(M)}$. From Lemma 2.3, u is a solution of (1.1) if u is a fixed point of the operator A.

Lemma 3.6. *The operator* $A : P \to P$ *is completely continuous.*

Proof. It follows from Lemma 2.5 that $A(P) \subseteq P$. Due to the continuity of G and f, the operator $A : P \to P$ is continuous. Let $X \subset P$ be bounded, that is, there exists a constant $L > 0$ such that $\|u\| \leq L$ for all $u \in X$. Let $Q = \max_{0\leq t\leq 1, 0\leq u\leq L} |f(t,u)| + 1$. From Lemma 2.4, we deduce that for any $u \in X$,

$$|Au(t)| \leq Q\int_0^1 N(s)ds + \frac{Q\sum_{i=1}^q \gamma_i}{1-\psi}\int_0^1 N(s)ds + \frac{b}{1-\psi} := W_1.$$

Hence, the set $A(X)$ is bounded in E.

Denote

$$W_2 = \frac{Q\sum_{i=1}^q \gamma_i}{1-\psi}\int_0^1 N(s)ds + \frac{b}{1-\psi}.$$

For any $t_1, t_2 \in [0,1], t_1 < t_2$, we have

$$\begin{aligned}
&|(Au)(t_2) - (Au)(t_1)| \\
\leq & Q\left|\int_0^1 (G(t_2,s) - G(t_1,s))ds\right| + W_2(t_2^{m-1} - t_1^{m-1}) \\
\leq & Q\left|\int_0^{t_2} (t_2-s)^{m-1}E_{m-\alpha,m}(M(t_2-s)^{m-\alpha})ds - \int_0^{t_1}(t_1-s)^{m-1}E_{m-\alpha,m}(M(t_1-s)^{m-\alpha})ds\right| \\
&+ Q\left|\int_0^1 (t_2^{m-1} - t_1^{m-1})(1-s)^{m-1}E_{m-\alpha,m}(M(1-s)^{m-\alpha})ds\right| + W_2(t_2^{m-1} - t_1^{m-1}) \\
\leq & Q\left|\int_0^{t_1} ((t_2-s)^{m-1}E_{m-\alpha,m}(M(t_2-s)^{m-\alpha}) - (t_1-s)^{m-1}E_{m-\alpha,m}(M(t_1-s)^{m-\alpha}))ds\right| \\
&+ Q\int_{t_1}^{t_2}(t_2-s)^{m-1}E_{m-\alpha,m}(M(t_2-s)^{m-\alpha})ds + W_1(t_2^{m-1} - t_1^{m-1}) \\
\leq & Q\left|\int_0^{t_1} ((t_2-s)^{m-1}E_{m-\alpha,m}(M(t_2-s)^{m-\alpha}) - (t_1-s)^{m-1}E_{m-\alpha,m}(M(t_1-s)^{m-\alpha}))ds\right| \\
&+ \frac{QE_{m-\alpha,m}(M)}{m}(t_2-t_1)^m + W_1(t_2^{m-1} - t_1^{m-1}).
\end{aligned}$$

Since $t^{m-1}E_{m-\alpha,m}(Mt^{m-\alpha}) \in C[0,1]$, it holds that

$$|(Au)(t_2) - (Au)(t_1)| \to 0 \quad as \quad |t_1 - t_2| \to 0.$$

Therefore, A is equicontinuous. By the Arzela-Ascoli theorem, it follows that the operator $A : P \to P$ is completely continuous. This completes the proof. □

Theorem 3.1. *([2]). Let P be a cone in Banach space E. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let A: $P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that, either*

(1) $\|Au\| \leq \|u\|$, if $u \in P \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, if $u \in P \cap \partial\Omega_2$, or

(2) $\|Au\| \geq \|u\|$, if $u \in P \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, if $u \in P \cap \partial\Omega_2$.

Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Define

$$F_0 = \limsup_{u\to 0^+}\frac{f(t,u)}{u}, \quad F_\infty = \limsup_{u\to\infty}\frac{f(t,u)}{u},$$

$$f_0 = \liminf_{u\to 0^+}\frac{f(t,u)}{u}, \quad f_\infty = \liminf_{u\to\infty}\frac{f(t,u)}{u}.$$

In the rest of this section, $\Omega_l = \{u \in P : \|u\| < l\}$, where l is a positive constant.

Theorem 3.2. *Assume that the following conditions are satisfied:*

(H_3) $F_0 = F_\infty = 0$,

(H_4) *There exists a constant* $c_1 > 0$ *such that for any* $u \in [K_{\theta_0} c_1, c_1]$,

$$f(t,u) \geq c_1 m_1, \qquad m_1 > \frac{1}{K_{\theta_0}} \left(\frac{1-\psi+\theta_0^{m-1}\sum_{i=1}^{q}\gamma_i}{1-\psi} \int_{\theta_0}^{1-\theta_0} N(s)ds \right)^{-1}.$$

Then (1.1) *has at least two positive solutions* u_1, u_2 *with* $0 < \|u_1\| < c_1 < \|u_2\|$ *for* b *small enough and at least one positive solution for any* $b \in [0, \infty)$.

Proof. Since $F_0 = 0$, there exists $\rho_1 \in (0, c_1)$ such that $f(t,u) \leq R_1 u$ for $\forall u \in [0, \rho_1]$, where R_1 satisfies

$$0 < R_1 < \frac{1}{2} \left(\frac{1-\psi+\sum_{i=1}^{q}\gamma_i}{1-\psi} \int_0^1 N(s)ds \right)^{-1}.$$

Let b satisfy

$$0 \leq b \leq \frac{(1-\psi)\rho_1}{2}.$$

Using Lemma 2.4, we obtain that for any $u \in \partial\Omega_{\rho_1}$,

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^{q}\gamma_i \int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{bt^{m-1}}{1-\psi} \\ &\leq \int_0^1 N(s)f(s,u(s))ds + \frac{\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_0^1 N(s)f(s,u(s))ds + \frac{b}{1-\psi} \\ &\leq R_1 \frac{1-\psi+\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_0^1 N(s)ds\|u\| + \frac{\rho_1}{2} < \frac{\rho_1}{2} + \frac{\rho_1}{2} = \|u\|, \end{aligned} \tag{3.15}$$

which implies that

$$\|Au\| < \|u\|, \quad u \in \partial\Omega_{\rho_1}. \tag{3.16}$$

Define the function $\tilde{f}(t,u) = \max_{z\in[0,u]}\{f(t,z)\}$, which is nondecreasing on $[0,\infty)$ with respect to u. By $F_\infty = 0$, it is deduced that

$$\limsup_{u\to\infty} \frac{\tilde{f}(t,u)}{u} = 0.$$

Hence, there exists $\rho_2 \geq \max\{2c_1, \frac{2b}{1-\psi}\}$ such that $\tilde{f}(t,u) \leq R_1 u$ for $\forall u \geq \rho_2$. For any $u \in \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^{q}\gamma_i \int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{bt^{m-1}}{1-\psi} \\ &\leq \int_0^1 N(s)\tilde{f}(s,\|u\|)ds + \frac{\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_0^1 N(s)\tilde{f}(s,\|u\|)ds + \frac{b}{1-\psi} \\ &\leq R_1 \frac{1-\psi+\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_0^1 N(s)ds\|u\| + \frac{\rho_2}{2} < \frac{\rho_2}{2} + \frac{\rho_2}{2} = \|u\|. \end{aligned}$$

This shows that

$$\|Au\| < \|u\|, \quad u \in \partial\Omega_{\rho_2}. \tag{3.17}$$

Finally, by Lemma 2.5, we have $\min_{t\in[\theta_0,1-\theta_0]} u(t) \geq K_{\theta_0}\|u\| = K_{\theta_0}c_1$ for any $u \in \partial\Omega_{c_1}$. It follows from Lemma 2.4 that for any $u \in \partial\Omega_{c_1}$,

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)f(s,u(s))ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^{q}\gamma_i\int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{bt^{m-1}}{1-\psi}\right\}\\
&\geq \min_{t\in[\theta_0,1-\theta_0]}\left\{\int_0^1 G(t,s)f(s,u(s))ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^{q}\gamma_i\int_0^1 G(\eta_i,s)f(s,u(s))ds + \frac{bt^{m-1}}{1-\psi}\right\}\\
&\geq K_{\theta_0}\int_{\theta_0}^{1-\theta_0} N(s)f(s,u(s))ds + \frac{K_{\theta_0}\theta_0^{m-1}\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)f(s,u(s))ds \qquad (3.18)\\
&\geq c_1 m_1 K_{\theta_0}\left(\frac{1-\psi+\theta_0^{m-1}\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)ds\right) > c_1 = \|u\|,
\end{aligned}$$

which implies that

$$\|Au\| > \|u\|, \quad u \in \partial\Omega_{c_1}. \qquad (3.19)$$

By (3.16), (3.17), (3.19) and Theorem 3.1, the operator A has at least two fixed points $u_1 \in \overline{\Omega}_{c_1}\setminus\Omega_{\rho_1}$, $u_2 \in \overline{\Omega}_{\rho_2}\setminus\Omega_{c_1}$ with $0 < \|u_1\| < c_1 < \|u_2\|$ for b small enough. For any $t_* \in (0,1)$, there exists $\theta_* \in (0,\frac{1}{2})$ such that $t_* \in [\theta_*, 1-\theta_*]$. By Lemma 2.5, we have

$$u_j(t_*) \geq K_{\theta_*}\|u_j\| > 0,$$

where $j = 1,2$. Hence $u_j > 0$ on $(0,1)$. Moreover, $u_j(1) = \sum_{i=1}^{q}\gamma_i u_j(\eta_i) + b > 0$. Thus u_1, u_2 are two positive solutions of (1.1). In addition, from (3.17) and (3.19), we obtain that (1.1) has at least one positive solution for any $b \in [0,\infty)$. This completes the proof. □

Theorem 3.3. *Assume that the following conditions are satisfied:*

(H_5) $f_0 = f_\infty = \infty$,

(H_6) *There exists a constant $c_2 > 0$ such that for any $u \in [0, c_2]$,*

$$f(t,u) \leq c_2 m_2, \quad 0 < m_2 < \frac{1}{2}\left(\frac{1-\psi+\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_0^1 N(s)ds\right)^{-1}.$$

Then (1.1) has at least two positive solutions u_3, u_4 with $0 < \|u_3\| < c_2 < \|u_4\|$ for b small enough and no positive solution for b large enough.

Proof. (1) We prove that (1.1) has at least two positive solutions for sufficiently small b.

Since $f_0 = \infty$, there exists $\nu_1 \in (0, c_2)$ such that $f(t,u) \geq R_2 u$ for $\forall u \in [0,\nu_1]$, where R_2 satisfies

$$R_2 > \frac{1}{K_{\theta_0}^2}\left(\frac{1-\psi+\theta_0^{m-1}\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)ds\right)^{-1}.$$

From (3.18) and Lemma 2.5, we obtain that for any $u \in \partial\Omega_{\nu_1}$,

$$\begin{aligned}
\|Au\| &\geq K_{\theta_0}\int_{\theta_0}^{1-\theta_0} N(s)f(s,u(s))ds + \frac{K_{\theta_0}\theta_0^{m-1}\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)f(s,u(s))ds\\
&\geq R_2 K_{\theta_0}^2\left(\frac{1-\psi+\theta_0^{m-1}\sum_{i=1}^{q}\gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)ds\right)\|u\| > \|u\|.
\end{aligned}$$

It implies that

$$\|Au\| > \|u\|, \quad u \in \partial\Omega_{\nu_1}. \qquad (3.20)$$

By $f_\infty = \infty$, there exists $\omega_1 > c_2$ such that $f(t,u) \geq R_2 u$ for $\forall u \geq \omega_1$. Choosing a positive constant $\nu_2 \geq \frac{\omega_1}{K_{\theta_0}}$, we have $\min_{t\in[\theta_0,1-\theta_0]} u(t) \geq K_{\theta_0}\|u\| \geq \omega_1$ for any $u \in \partial\Omega_{\nu_2}$.

Similar to the proof of (3.20), it follows that

$$\|Au\| > \|u\|, \quad u \in \partial\Omega_{\nu_2}. \tag{3.21}$$

Finally, let b satisfy

$$0 \le b \le \frac{(1-\psi)c_2}{2}.$$

Using (3.15), we get that for any $u \in \Omega_{c_2}$,

$$\begin{aligned} Au(t) &\le \int_0^1 N(s)f(s,u(s))ds + \frac{\sum_{i=1}^q \gamma_i}{1-\psi}\int_0^1 N(s)f(s,u(s))ds + \frac{b}{1-\psi} \\ &\le c_2 m_2 \left(\frac{1-\psi+\sum_{i=1}^q \gamma_i}{1-\psi}\int_0^1 N(s)ds\right) + \frac{c_2}{2} < \frac{c_2}{2} + \frac{c_2}{2} = \|u\|, \end{aligned}$$

which means that

$$\|Au\| < \|u\|, \quad u \in \partial\Omega_{c_2}. \tag{3.22}$$

By (3.20), (3.21), (3.22) and Theorem 3.1, the operator A has at least two fixed points $u_3 \in \overline{\Omega}_{c_2} \setminus \Omega_{\nu_1}$, $u_4 \in \overline{\Omega}_{\nu_2} \setminus \Omega_{c_2}$ with $0 < \|u_3\| < c_2 < \|u_4\|$ for b small enough. Similar to the proof of the positivity of u_1, u_2, we get that u_3, u_4 are two positive solutions of (1.1).

(2) We verify that (1.1) has no positive solution for b large enough.

Suppose that there exists a sequence $\{b_n\}$ satisfying $0 < b_1 < b_2 < \cdots < b_n < \cdots$ and $\lim_{n\to\infty} b_n = \infty$, such that for any positive integer n, the boundary value problem

$$\begin{cases} -u^{(m)}(t) + M\, {}^C D^\alpha u(t) = f(t,u(t)), & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, & u(1) - \sum_{i=1}^q \gamma_i u(\eta_i) = b_n \end{cases}$$

has a positive solution u_n. By (3.14), we have

$$\begin{aligned} u_n(1) &= \int_0^1 G(1,s)f(s,u_n(s))ds + \frac{1}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)f(s,u_n(s))ds + \frac{b_n}{1-\psi} \\ &\ge \frac{b_n}{1-\psi} \to \infty, \ (n \to \infty). \end{aligned}$$

Hence $\|u_n\| \to \infty$ as $n \to \infty$. Since $f_\infty = \infty$, there exists $\omega_2 > 0$ such that $f(t,u) \ge 2R_2 u$ for $\forall u \ge \omega_2$. Let n be large enough such that $\|u_n\| \ge \frac{\omega_2}{K_{\theta_0}}$. Then

$$\begin{aligned} \|u_n\| &\ge K_{\theta_0}\int_{\theta_0}^{1-\theta_0} N(s)f(s,u_n(s))ds + \frac{K_{\theta_0}\theta_0^{m-1}\sum_{i=1}^q \gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)f(s,u_n(s))ds \\ &\ge 2R_2K_{\theta_0}^2\left(\frac{1-\psi+\theta_0^{m-1}\sum_{i=1}^q \gamma_i}{1-\psi}\int_{\theta_0}^{1-\theta_0} N(s)ds\right)\|u_n\| > 2\|u_n\|, \end{aligned}$$

a contradiction. This completes the proof. □

Theorem 3.4. *Assume that $f_\infty = \infty$ and condition (H_6) holds. If $f(t,u)$ is nondecreasing with respect to u, then there exists a positive constant b^* such that* (1.1) *has at least one positive solution for $b \in [0, b^*]$ and no positive solution for $b \in (b^*, \infty)$.*

Proof. Let $B = \{b \ge 0 :$ (1.1) has at least one positive solution$\}$ and $b^* = \sup B$. From (3.21), (3.22), we obtain that (1.1) has at least one positive solution for b small enough, which means that we only need to consider the case $b > 0$ in what follows. By the second part of the proof of Theorem 3.3, we can deduce $b^* < \infty$. Thus $0 < b^* < \infty$. From the definition of b^*, we know that for any $b \in (0, b^*)$, there exists a constant $c \in B : c > b$ such that boundary value problem

$$\left\{ \begin{array}{ll} -u^{(m)}(t) + M\, {}^C D^\alpha u(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, & u(1) - \sum_{i=1}^q \gamma_i u(\eta_i) = c \end{array} \right. \tag{3.23}$$

has a positive solution u_c. Next, we consider the following boundary value problem

$$\left\{ \begin{array}{ll} -u^{(m)}(t) + M\, {}^C D^\alpha u(t) = F(t, u(t)), & t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, & u(1) - \sum_{i=1}^q \gamma_i u(\eta_i) = b, \end{array} \right. \tag{3.24}$$

where

$$F(t, u(t)) = \begin{cases} f(t, u_c(t)), & u(t) > u_c(t), \\ f(t, u(t)), & 0 \le u(t) \le u_c(t), \\ f(t, 0), & u(t) < 0. \end{cases}$$

Define the operator $\tilde{A} : E \to E$ as follows:

$$\tilde{A}u(t) = \int_0^1 G(t,s)F(s,u(s))ds + \frac{t^{m-1}}{1-\psi} \sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i, s)F(s, u(s))ds + \frac{bt^{m-1}}{1-\psi}.$$

Since the function F is continuous and bounded, there is a constant $D > 0$ such that $\|\tilde{A}u\| \le D$ for any $u \in E$. Let $\Omega = \{u \in E : \|u\| \le D\}$, it is clear that $\tilde{A}(\Omega) \subseteq \Omega$. Similar to Lemma 3.6, we obtain that $\tilde{A} : \Omega \to \Omega$ is completely continuous. By the Schauder's fixed point theorem, (3.24) has a solution $u_b \in \Omega$. Obviously, $u_b > 0$ on $(0, 1]$ since $F \ge 0$. Setting $z = u_c - u_b$, from (3.23) and (3.24), we have

$$\left\{ \begin{array}{ll} -z^{(m)}(t) + M\, {}^C D^\alpha z(t) = f(t, u_c(t)) - F(t, u_b(t)) \ge 0, & t \in (0,1), \\ z(0) = z'(0) = \cdots = z^{(m-2)}(0) = 0, & z(1) - \sum_{i=1}^q \gamma_i z(\eta_i) = c - b \ge 0. \end{array} \right.$$

Denote $p(t) = f(t, u_c(t)) - F(t, u_b(t))$. By the continuity of f, F, u_c, u_b, we easily get $p \in C^+[0,1]$. From Lemma 2.3, we have

$$z(t) = \int_0^1 G(t,s)p(s)ds + \frac{t^{m-1}}{1-\psi} \sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i, s)p(s)ds + \frac{(c-b)t^{m-1}}{1-\psi}.$$

Obviously, $z \ge 0$, i.e., $u_b \le u_c$. As a consequence, u_b is a positive solution of (1.1).

Finally, we show that $b^* \in B$. Let $0 < b_1 < b_2 < \cdots < b_n \to b^*$ and x_n be the positive solution of (1.1) with $b = b_n$. Then

$$x_n = \int_0^1 G(t,s)f(s, x_n(s))ds + \frac{t^{m-1}}{1-\psi} \sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i, s)f(s, x_n(s))ds + \frac{b_n t^{m-1}}{1-\psi}.$$

Since $f_\infty = \infty$, there exists $\varsigma > 0$ such that $f(t,x) \ge Rx$ for $\forall x \ge \varsigma$, where $R = \frac{1}{K_{\theta_0}^2 \int_{\theta_0}^{1-\theta_0} N(s)ds} + 1$. Suppose that $\|x_n\| \to \infty$, there exists $N > 0$ such that for $n > N$, $\min_{t \in [\theta_0, 1-\theta_0]} x_n \ge K_{\theta_0} \|x_n\| \ge \varsigma$. Hence,

$$\begin{aligned} x_n &\ge \int_0^1 G(t,s)f(s, x_n(s))ds \ge K_{\theta_0} \int_{\theta_0}^{1-\theta_0} N(s)Rx_n(s)ds \\ &\ge RK_{\theta_0}^2 \int_{\theta_0}^{1-\theta_0} N(s)ds\|x_n\| > \|x_n\|, \end{aligned}$$

a contradiction, which implies that $\|x_n\| \le C_0$ for some $C_0 > 0$. Therefore, the sequence $\{x_n\}$ is uniformly bounded. On the other hand, by Lemma 2.2 and (2.13), we have

$$\begin{aligned}|x_n'| \le &\int_0^t (t-s)^{m-2} E_{m-\alpha,m-1}(M(t-s)^{m-\alpha}) f(s,x_n(s))ds + \frac{b_n(m-1)t^{m-2}}{1-\psi} \\ &+ \frac{(m-1)t^{m-2}}{1-\psi}\int_0^1 (1-s)^{m-1} E_{m-\alpha,m}(M(1-s)^{m-\alpha}) f(s,x_n(s))ds \\ &+ \frac{(m-1)t^{m-2}}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^{\eta_i} (\eta_i - s)^{m-1} E_{m-\alpha,m}(M(\eta_i-s)^{m-\alpha}) f(s,x_n(s))ds \\ \le& C_1\int_0^1 f(s,x_n(s))ds + C_2 \le C_1 \int_0^1 f(s,C_0)ds + C_2 \le C_3\end{aligned}$$

for some positive constants C_1, C_2, C_3. For any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \to 0$, it holds that

$$|x_n(t_1) - x_n(t_2)| = |x_n'(\xi)(t_1-t_2)| \le C_3|t_1-t_2| \to 0,$$

where ξ is between t_1 and t_2. Hence $\{x_n\}$ is equicontinuous. By the Arzela-Ascoli Theorem, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to $x^* \in C[0,1]$, i.e., $\lim_{k\to\infty} x_{n_k} = x^*$. It is easy to verify that

$$x^* = \int_0^1 G(t,s)f(s,x^*(s))ds + \frac{t^{m-1}}{1-\psi}\sum_{i=1}^q \gamma_i \int_0^1 G(\eta_i,s)f(s,x^*(s))ds + \frac{b^* t^{m-1}}{1-\psi},$$

that is, x^* is the positive solution of (1.1) with $b = b^*$. This completes the proof. □

Example 3.1. *Consider the following fractional differential equation*

$$\begin{cases} -u'''(t) + \frac{1}{2}\,{}^C D^{\frac{5}{2}} u(t) = \frac{1}{2}(u^{\frac{1}{2}}(t) + u^{\frac{3}{2}}(t) + t), & t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) - \frac{1}{6}u(\frac{1}{3}) - \frac{1}{5}u(\frac{2}{3}) = b. \end{cases} \tag{3.25}$$

In fact, $m = 3, M = \frac{1}{2}, \alpha = \frac{5}{2}, f(t,x) = \frac{1}{2}(\sqrt{x} + x^{\frac{3}{2}} + t), \gamma_1 = \frac{1}{6}, \gamma_2 = \frac{1}{5}, \eta_1 = \frac{1}{3}, \eta_2 = \frac{2}{3}$. It is easy to show that the conditions $(H_1), (H_2), (H_5)$ hold. Moreover,

$$\begin{aligned} H :=& \frac{1-\psi+\sum_{i=1}^q \gamma_i}{1-\psi}\int_0^1 N(s)ds \\ =& \frac{1-\sum_{i=1}^2\gamma_i\eta_i^2 + \sum_{i=1}^2 \gamma_i}{1-\sum_{i=1}^2 \gamma_i\eta_i^2}\int_0^1 (1-s)^2 E_{\frac{1}{2},3}\left(\frac{1}{2}(1-s)^{\frac{1}{2}}\right)ds \\ \le& \frac{E_{\frac{1}{2},3}(\frac{1}{2})(1-\sum_{i=1}^2\gamma_i\eta_i^2 + \sum_{i=1}^2\gamma_i)}{1-\sum_{i=1}^2\gamma_i\eta_i^2}\int_0^1 (1-s)^2 ds \approx 0.332. \end{aligned}$$

Taking $m_2 = 1.5$, we have $m_2 < \frac{1}{2\times 0.332} = 1.506 \le \frac{1}{2H}$. Let $c_2 = 4$, for any $(t,x) \in [0,1]\times[0,4]$, we obtain

$$f(t,x) = \frac{1}{2}(\sqrt{x} + x^{\frac{3}{2}} + t) \le 5.5 \le 6 = c_2 m_2.$$

By Theorem 3.3 and Theorem 3.4, there exist positive constants $b_1 \le b_2$ such that the problem (3.25) has at least two positive solutions for $b \in [0, b_1)$, one positive solution for $b \in [b_1, b_2]$, and no positive solution for $b \in (b_2, \infty)$.

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DEPARTMENT OF MATHEMATICS
HUNAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
XIANGTAN 411201, CHINA
Email address: zliu67@163.com
Email address: wwbing2013@126.com