CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 479-502

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.14

Unified Convergence Analysis of Certain At Least Fifth Order Methods

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ABSTRACT. A class of iterative methods was developed by Xiao and Yin in 2015 and obtained convergence order five using Taylor expansion. They had imposed the conditions on the derivatives of the involved operator of order at least up to four. In this paper, the order of convergence is achieved by imposing conditions only on the first two derivatives of the operator involved. The assumptions under consideration are weaker and the analysis is done in the more general setting of Banach spaces without using Taylor series expansion. The semi-local convergence analysis is also given. Further, the theory is justified by numerical examples.

1. INTRODUCTION

Numerous problems in applied scientific branches and numerical analysis prominently focus on solving a nonlinear equation given by

$$\wp(x) = 0 \tag{1.1}$$

defined on suitable spaces i.e., to find a solution ϱ^* of (1.1) [3, 5, 6]. Unlike linear cases, direct methods do not hold good in finding solutions of nonlinear equations and thus require iterative methods.

Number of iterations a method takes to reach the root within the desired precision, characterized by its 'convergence order' is an important feature of any iterative method. Recall [1, 5, 6, 16, 19] that an iterative process is of convergence order at least $k > 0$ if

$$\|x_{n+1} - \varrho^*\| \leq c_k \|x_n - \varrho^*\|^k,$$

where c_k is called the rate of convergence or asymptotic error constant. One of the most used methods is the Newton-Raphson's method (also known as the Newton's method) of quadratic convergence. It is always of research interest to come up with efficient iterative techniques of better order of convergence at a justified computational cost.

Several authors [2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18] have come up with the construction of many new multi-point iterative methods with faster convergence rate compared to that of the one-point Newton's method. One such class of efficient iterative methods was developed by Xiao and Yin in 2015 [22]. This class gives rise to methods of order at least three and five, respectively, for different choices of $a \neq 0$ defined for $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - a\wp'(x_n)^{-1}\wp(x_n) \\ x_{n+1} &= x_n - \left[\frac{1}{2a}\wp'(y_n) + \left(1 - \frac{1}{2a}\right)\wp'(x_n)\right]^{-1}\wp(x_n) \end{aligned} \tag{1.2}$$

Received: 13.06.2024. In revised form: 05.11.2024. Accepted: 07.11.2024
2020 *Mathematics Subject Classification.* 49M15, 65J15, 65D99, 65G99, 47H99.
Key words and phrases. *Fréchet derivative, Iterative methods, Banach space, Order of convergence, Local convergence, Semi-local convergence.*
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and

$$
\begin{aligned}
y_n &= x_n - a\wp'(x_n)^{-1}\wp(x_n)\\
z_n &= x_n - \left[\frac{1}{2a}\wp'(y_n) + \left(1-\frac{1}{2a}\right)\wp'(x_n)\right]^{-1}\wp(x_n) \qquad (1.3)\\
x_{n+1} &= z_n - \left(2\left[\frac{1}{2a}\wp'(y_n) + \left(1-\frac{1}{2a}\right)\wp'(x_n)\right]^{-1} - \wp'(x_n)^{-1}\right)\wp(z_n).
\end{aligned}
$$

Some of the particular cases of (1.3), by taking $a = -1, \frac{1}{2}, \frac{2}{3}, 1$ give rise to the four well known extended Newton-like methods studied in ([18, 22]) (based on the predictor step), respectively, as given below:

Method in [22]**:**

$$
\begin{aligned}
y_n &= x_n + \wp'(x_n)^{-1}\wp(x_n)\\
z_n &= x_n + 2\left(\wp'(y_n) - 3\wp'(x_n)\right)^{-1}\wp(x_n) \qquad (1.4)\\
x_{n+1} &= z_n + \left(4\left[\wp'(y_n) - 3\wp'(x_n)\right]^{-1} + \wp'(x_n)^{-1}\right)\wp(z_n).
\end{aligned}
$$

Method in [18, 22]**:**

$$
\begin{aligned}
y_n &= x_n - \frac{1}{2}\wp'(x_n)^{-1}\wp(x_n)\\
z_n &= x_n - \wp'(y_n)^{-1}\wp(x_n) \qquad (1.5)\\
x_{n+1} &= z_n - [2\wp'(y_n)^{-1} - \wp'(x_n)^{-1}]\wp(z_n).
\end{aligned}
$$

Method in [22]**:**

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\wp'(x_n)^{-1}\wp(x_n)\\
z_n &= x_n - 4(\wp'(x_n) + 3\wp'(y_n))^{-1}\wp(x_n) \qquad (1.6)\\
x_{n+1} &= z_n - (8(\wp'(x_n) + 3\wp'(y_n))^{-1} - \wp'(x_n)^{-1})\wp(z_n).
\end{aligned}
$$

Method in [22]**:**

$$
\begin{aligned}
y_n &= x_n - \wp'(x_n)^{-1}\wp(x_n)\\
z_n &= x_n - 2\left(\wp'(y_n) + \wp'(x_n)\right)^{-1}\wp(x_n) \qquad (1.7)\\
x_{n+1} &= z_n - \left(4\left[\wp'(y_n) + \wp'(x_n)\right]^{-1} - \wp'(x_n)^{-1}\right)\wp(z_n).
\end{aligned}
$$

In this paper, we study the local convergence of the methods (1.2) and (1.3) in the Banach space setting, i.e., we consider $\wp : \mathcal{D} \subset X \to Y$ to be a Fréchet differentiable operator between Banach spaces X and Y, $\mathcal{D}$ is an open convex set. Also, the earlier studies [22] on local convergence is done based on the Taylor expansion for $X = Y = \mathbb{R}^j$ which requires the operator to have high order derivatives not on the method. This is not always possible, as evidenced below.

Consider $q : [-2, 2] \longrightarrow \mathbb{R}$ defined by

$$
q(x) = \begin{cases} \frac{1}{20}(x^4 log x^2 + x^6 - x^5) & if\ x \neq 0\\ 0 & if\ x = 0. \end{cases}
$$

Then,

$$\begin{aligned} q'(x) &= \frac{1}{20}(2x^3 + 4x^3logx^2 + 6x^5 - 5x^4), \\ q''(x) &= \frac{1}{20}(14x^2 + 12x^2logx^2 + 30x^4 - 20x^3), \\ q'''(x) &= \frac{1}{20}(52x + 24xlogx^2 + 120x^3 - 60x^2), \\ q^{IV}(x) &= \frac{1}{20}(24logx^2 + 360x^2 - 120x + 100). \end{aligned}$$

Note that, q^{IV} at $x = 0$ is unbounded. Thus, the results in [22] assuming the existence of q^{IV} cannot assure convergence of the method (1.3) to the solution $\rho^* = 1 \in [-2, 2]$. But (1.3) converges to ρ^* if, e.g. $x_0 = 1.1$. Thus, the results in [22] can be weakened. Other drawbacks of the Taylor series approach are, lack of apriori error estimates on the distances $\|x_n - \rho^*\|$ and uniqueness results for the solution. We are motivated by these problems. That is why we positively address all these problems in this paper.

Our study relaxes this condition and requires $\wp$ to be just two times differentiable, even though the order of convergence of the considered method (1.3) is five.

Section 2 of the paper deals with local convergence analysis of the method (1.2) of order three and method (1.3) of order five, without using Taylor expansion. The semi-local convergence analysis of the general class of methods (1.2) and (1.3) is discussed in Section 3. The theory is supported by numerical examples in Section 4, with concluding remarks in Section 5.

2. LOCAL CONVERGENCE OF (1.2) AND (1.3)

Let $\mathcal{S}(\varrho^*, r)$ be the ball with center ϱ^* and radius r and $\bar{\mathcal{S}}(\varrho^*, r)$ be the closure of $\mathcal{S}(\varrho^*, r)$. Take into account the assumptions listed below:

(A1) ϱ^* is a solution (simple) of (1.1) and $\wp'(\varrho^*)^{-1} \in \mathcal{L}(Y, X)$.
(A2) $\|\wp'(\varrho^*)^{-1}(\wp'(x) - \wp'(\varrho^*))\| \leq L_1\|x - \varrho^*\| \;\forall\, x \in \mathcal{D}$.
(A3) $\|\wp'(\varrho^*)^{-1}(\wp''(x) - \wp''(\varrho^*))\| \leq L_3\|x - \varrho^*\| \;\forall\, x \in \mathcal{D}$.
(A4) $\|\wp'(\varrho^*)^{-1}\wp''(x)\| \leq L_4 \;\forall\, x \in \mathcal{D}$ and $L_1, L_3, L_4 > 0$.

From (A2) we can obtain,

$$\|\wp'(\varrho^*)^{-1}\wp'(x)\| \leq 1 + L_1\|x - \varrho^*\|. \tag{2.8}$$

Consider the functions $\phi, \phi_1, h_1 : [0, \frac{1}{L_1}) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \phi(t) &= \frac{1}{1 - L_1 t}\left[|1-a| + \left(|1-a| + \frac{|a|}{2}\right)L_1 t\right], \\ \phi_1(t) &= \frac{L_1}{2|a|}\left[\phi(t) + |2a-1|\right] \end{aligned}$$

and

$$h_1(t) = \phi_1(t)t - 1.$$

Observe that, since $\frac{1}{1-L_1 t}$ is monotonic increasing in $[0, \frac{1}{L_1})$, we have ϕ, ϕ_1 and h_1 are continuous and non-decreasing function (C.N.D.F.) with $h_1(0) = -1 < 0$ and $lim_{t \longrightarrow \frac{1}{L_1}^-} h_1(t) = +\infty$. Thus, $\exists$ smallest $r_1 \in (0, \frac{1}{L_1})$ such that $h_1(r_1) = 0$.

Consider the functions $\phi_2, h_2 : [0, r_1) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}\phi_2(t) &= \frac{L_3}{3(1-L_1t)} + \frac{L_3(2+L_1t)^2}{8(1-\phi_1(t)t)(1-L_1t)^2}\left[1+\frac{|a|}{4}\frac{2+L_1t}{(1-L_1t)}\right] \\ &\quad + \frac{3L_4L_1(4+3L_1t)}{8(1-\phi_1(t)t)(1-L_1t)^2} + \frac{L_4^2(2+L_1t)}{8(1-\phi_1(t)t)(1-L_1t)^2}\end{aligned}$$

and

$$h_2(t) = \phi_2(t)t^2 - 1.$$

Again, since $\frac{1}{1-L_1t}$ and $\frac{1}{1-\phi_1(t)t}$ are monotonic increasing in $[0, r_1)$, ϕ_2 is monotonic increasing. Further, since ϕ_2 is the ratio of two nonzero polynomials, it is continuous, hence h_2 is C.N.D.F. with $h_2(0) = -1 < 0$ and $lim_{t \longrightarrow r_1^-} h_2(t) = +\infty$. Thus, $\exists$ smallest $\rho \in (0, r_1)$ such that $h_2(\rho) = 0$.

Next, we define the functions $\phi_3, h_3 : [0, r_1) \longrightarrow \mathbb{R}$ by

$$\begin{aligned}\phi_3(t) &= (1+|2a-1|)\frac{L_4(\phi_2(t))^2t}{4|a|(1-\phi_1(t)t)} + \frac{L_3\phi(t)\phi_2(t)}{4|a|(1-\phi_1(t)t)}\left(\frac{1}{2}\phi_2(t)t^2+\phi(t)\right) \\ &\quad + \frac{|2a-1|L_3\phi_2(t)}{4|a|(1-\phi_1(t)t)}\left(\frac{1}{2}\phi_2(t)t^2+1\right) \\ &\quad + \frac{L_3}{2(1-\phi_1(t)t)}\left(1+\frac{|a|}{4}\frac{2+L_1t}{(1-L_1t)}\right)\frac{2+L_1t}{2(1-L_1t)^2}t^2\left(1+\frac{L_1}{2}\phi_2(t)t\right)\phi_2(t) \\ &\quad + \frac{3L_4L_1\phi_2(t)}{2(1-\phi_1(t)t)(1-L_1t)} + \frac{L_4(2+L_1t)\phi_2(t)}{4(1-\phi_1(t)t)(1-L_1t)^2}\left(L_1+\frac{L_1}{2}\phi_2(t)t^2\right)\end{aligned}$$

and

$$h_3(t) = \phi_3(t)t^4 - 1.$$

As above, one can prove that, h_3 is C.N.D.F. with $h_3(0) = -1 < 0$ and $lim_{t \longrightarrow r_1^-} h_3(t) = +\infty$. Thus, $\exists$ smallest $\rho_1 \in (0, r_1)$ such that $h_3(\rho_1) = 0$.

Let

$$r = \min\{\rho, \rho_1, 1\}. \tag{2.9}$$

Then,

$$\begin{aligned} 0 &\leq \phi_1(t)t < 1, \\ 0 &\leq \phi_2(t)t^2 < 1 \end{aligned}$$

and

$$0 \leq \phi_3(t)t^4 < 1,$$

$\forall\, t \in [0, r)$.

For simplicity, we write $\|x_n - \varrho^*\| = e_{x_n}$, $\|y_n - \varrho^*\| = e_{y_n}$, $\|z_n - \varrho^*\| = e_{z_n}$ and $\mathcal{A}_n = (\wp'(y_n) + (2a-1)\wp'(x_n))$.

THEOREM 2.1. *Let (A1)-(A4) hold and sequence $\{x_n\}$ be as in (1.2) with $x_0 \in \mathcal{S}(\varrho^*, r) - \{\varrho^*\}$. Then, $x_n \in \bar{\mathcal{S}}(\varrho^*, r)$ is well defined $\forall n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_n = \rho^*$. Furthermore, we can estimate*

$$e_{x_{n+1}} \leq \phi_2(r)e_{x_n}^3. \tag{2.10}$$

Proof. We proceed to prove (2.10) inductively. Let $x \in \mathcal{S}(\varrho^*, r)$. Then by assumption (A2), we have

$$\|\wp'(\varrho^*)^{-1}(\wp'(x) - \wp'(\varrho^*))\| \leq L_1\|x - \varrho^*\| \leq L_1 r < 1,$$

so by Banach Lemma on invertible operators (B.L.I.O.) [1], we have

$$\|\wp'(x)^{-1}\wp'(\varrho^*)\| \leq \frac{1}{1 - L_1\|x - \varrho^*\|}. \tag{2.11}$$

By the Mean Value Theorem (M.V.T.),

$$\wp(x) = \wp(x) - \wp(\varrho^*) = \int_0^1 \wp'(\varrho^* + t(x - \varrho^*))dt(x - \varrho^*). \tag{2.12}$$

Consider the sub-step 1 of (1.2). One can observe that

$$\begin{aligned} & y_0 - \varrho^* \\ = \; & x_0 - \varrho^* - a\wp'(x_0)^{-1}\int_0^1 \wp'(\varrho^* + t(x_0 - \varrho^*))dt(x_0 - \varrho^*) \\ = \; & \wp'(x_0)^{-1}\left[\wp'(x_0) - a\int_0^1 \wp'(\varrho^* + t(x_0 - \varrho^*))dt\right](x_0 - \varrho^*) \\ = \; & \wp'(x_0)^{-1}\wp'(\varrho^*) \\ & \times\left[(1-a)\wp'(\varrho^*)^{-1}\wp'(x_0) + a\int_0^1 \wp'(\varrho^*)^{-1}[\wp'(x_0) - \wp'(\varrho^* + t(x_0 - \varrho^*))]dt\right](x_0 - \varrho^*). \end{aligned}$$

Then, by using (2.8) and (2.11), we get

$$\begin{aligned} e_{y_0} & \leq \frac{1}{1 - L_1 e_{x_0}}[|1-a|(1 + L_1 e_{x_0}) + \frac{|a|L_1}{2}e_{x_0}]e_{x_0} \\ & \leq \phi(e_{x_0})e_{x_0}. \end{aligned} \tag{2.13}$$

We now prove the invertibility of $\mathcal{A}_0 = (\wp'(y_0) + (2a-1)\wp'(x_0))$.
Consider

$$\begin{aligned} & \|(2a\wp'(\varrho^*))^{-1}[\wp'(y_0) + (2a-1)\wp'(x_0) - 2a\wp'(\varrho^*)]\| \\ \leq \; & \frac{1}{2|a|}\left[\|\wp'(\varrho^*)^{-1}[\wp'(y_0) - \wp'(\varrho^*)]\| + |2a-1|\|\wp'(\varrho^*)^{-1}(\wp'(x_0) - \wp'(\varrho^*))\|\right] \\ \leq \; & \frac{1}{2|a|}\left[L_1 e_{y_0} + |2a-1|L_1 e_{x_0}\right] \leq \frac{L_1}{2|a|}\left[\phi(e_{x_0}) + |2a-1|\right]e_{x_0} \\ \leq \; & \phi_1(e_{x_0})e_{x_0} < 1. \end{aligned} \tag{2.14}$$

Thus, $\mathcal{A}_0^{-1}$ exists by B.L.I.O [1] and

$$\|\mathcal{A}_0^{-1}\wp'(\varrho^*)\| \leq \frac{1}{2|a|(1 - \phi_1(e_{x_0})e_{x_0})}. \tag{2.15}$$

Consider

$$\begin{aligned} x_1 - \varrho^* & = x_0 - \varrho^* - 2a\mathcal{A}_0^{-1}\wp(x_0) \\ & = x_0 - \varrho^* - \wp'(x_0)^{-1}\wp(x_0) + (\wp'(x_0)^{-1} - 2a\mathcal{A}_0^{-1})\wp(x_0) \\ & = \wp'(x_0)^{-1}\int_0^1 [\wp'(x_0) - \wp'(\varrho^* + t(x_0 - \varrho^*))]dt(x_0 - \varrho^*) \\ & \quad + \mathcal{A}_0^{-1}[\wp'(y_0) + (2a-1)\wp'(x_0) - 2a\wp'(x_0)]\wp'(x_0)^{-1}\wp(x_0). \end{aligned}$$

By the M.V.T., we get

$$\begin{aligned} x_1-\varrho^* &= \wp'(x_0)^{-1}\int_0^1\int_0^1 \wp''(\varrho^*+(t+\theta(1-t))(x_0-\varrho^*))d\theta(1-t)dt(x_0-\varrho^*)^2 \\ &\quad +\mathcal{A}_0^{-1}[\wp'(y_0)-\wp'(x_0)]\wp'(x_0)^{-1}\wp(x_0). \end{aligned}$$

Again by applying the M.V.T. on the second term and adding and substracting $\wp''(\varrho^*)$ to the first term, we get,

$$\begin{aligned} & x_1-\varrho^* \\ = \; & \wp'(x_0)^{-1}\int_0^1\int_0^1 [\wp''(\varrho^*+(t+\theta(1-t))(x_0-\varrho^*))-\wp''(\varrho^*)]d\theta(1-t)dt(x_0-\varrho^*)^2 \\ & +\wp'(x_0)^{-1}\int_0^1\int_0^1 \wp''(\varrho^*)d\theta(1-t)dt(x_0-\varrho^*)^2 \\ & +\mathcal{A}_0^{-1}\int_0^1 \wp''(x_0+\theta(y_0-x_0))d\theta(y_0-x_0)\wp'(x_0)^{-1}\wp(x_0). \end{aligned}$$

Then, by the sub-step 1 of method (1.2), we obtain

$$\begin{aligned} x_1-\varrho^* &= H_1+\frac{1}{2}\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2 \\ &\quad -a\mathcal{A}_0^{-1}\int_0^1 \wp''(x_0+\theta(y_0-x_0))d\theta(\wp'(x_0)^{-1}\wp(x_0))^2, \end{aligned} \tag{2.16}$$

where

$$H_1=\wp'(x_0)^{-1}\int_0^1\int_0^1 [\wp''(\varrho^*+(t+\theta(1-t))(x_0-\varrho^*))-\wp''(\varrho^*)]d\theta(1-t)dt(x_0-\varrho^*)^2.$$

Next, by adding and substracting $\wp''(\varrho^*)$ to the third term of (2.16) we get,

$$\begin{aligned} x_1-\varrho^* &= H_1+\frac{1}{2}\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2 \\ &\quad -a\mathcal{A}_0^{-1}\int_0^1 [\wp''(x_0+\theta(y_0-x_0))-\wp''(\varrho^*)]d\theta\left(\wp'(x_0)^{-1}\wp(x_0)\right)^2 \\ &\quad -a\mathcal{A}_0^{-1}\wp''(\varrho^*)(\wp'(x_0)^{-1}\wp(x_0))^2. \end{aligned}$$

Adding and substracting $(x_0-\varrho^*)^2$ to the fourth term, we get

$$\begin{aligned} x_1-\varrho^* &= H_1+\frac{1}{2}\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2 \\ &\quad -a\mathcal{A}_0^{-1}\int_0^1 [\wp''(x_0+\theta(y_0-x_0))-\wp''(\varrho^*)]d\theta(\wp'(x_0)^{-1}\wp(x_0))^2 \\ &\quad -a\mathcal{A}_0^{-1}\wp''(\varrho^*)[(\wp'(x_0)^{-1}\wp(x_0))^2-(x_0-\varrho^*)^2] \\ &\quad -a\mathcal{A}_0^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2. \end{aligned}$$

Combine the second and fifth term and taking

$$(\wp'(x_0)^{-1}\wp(x_0))^2-(x_0-\varrho^*)^2=(\wp'(x_0)^{-1}\wp(x_0)-(x_0-\varrho^*))(\wp'(x_0)^{-1}\wp(x_0)+(x_0-\varrho^*))$$

to get,

$$\begin{aligned}
& x_1-\varrho^* \\
= \; & H_1 - a\mathcal{A}_0^{-1}\int_0^1 [\wp''(x_0+\theta(y_0-x_0))-\wp''(\varrho^*)]d\theta(\wp'(x_0)^{-1}\wp(x_0))^2 \\
& -a\mathcal{A}_0^{-1}\wp''(\varrho^*)[\wp'(x_0)^{-1}\wp(x_0)-(x_0-\varrho^*)][\wp'(x_0)^{-1}\wp(x_0)+(x_0-\varrho^*)] \\
& +\mathcal{A}_0^{-1}\left[\frac{1}{2}(\wp'(y_0)+(2a-1)\wp'(x_0))-a\wp'(x_0)\right]\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2 \\
= \; & H_1+H_2 \\
& +a\mathcal{A}_0^{-1}\wp''(\varrho^*)[x_0-\varrho^*-\wp'(x_0)^{-1}\wp(x_0)][x_0-\varrho^*+\wp'(x_0)^{-1}\wp(x_0)] \\
& +\frac{1}{2}\mathcal{A}_0^{-1}[\wp'(y_0)-\wp'(x_0)]\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2, \qquad (2.17)
\end{aligned}$$

where

$$H_2 = -a\mathcal{A}_0^{-1}\int_0^1 [\wp''(x_0+\theta(y_0-x_0))-\wp''(\varrho^*)]d\theta(\wp'(x_0)^{-1}\wp(x_0))^2.$$

Applying M.V.T. to the fourth term of (2.17), gives

$$\begin{aligned}
& x_1-\varrho^* \\
= \; & H_1+H_2+H_3 \\
& +\frac{1}{2}\mathcal{A}_0^{-1}\int_0^1 \wp''(x_0+\theta(y_0-x_0))d\theta(y_0-x_0)\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2, \qquad (2.18)
\end{aligned}$$

where

$$H_3 = a\mathcal{A}_0^{-1}\wp''(\varrho^*)[x_0-\varrho^*-\wp'(x_0)^{-1}\wp(x_0)][x_0-\varrho^*+\wp'(x_0)^{-1}\wp(x_0)].$$

Then by substituting the sub-step 1 of (1.2) in the fourth term of (2.18), we get

$$x_1-\varrho^* = H_1+H_2+H_3+H_4,$$

where

$$\begin{aligned}
H_4 = & -\frac{a}{2}\mathcal{A}_0^{-1}\int_0^1 \wp''(x_0+\theta(y_0-x_0))d\theta \\
& \times(\wp'(x_0)^{-1}\wp(x_0))\wp'(x_0)^{-1}\wp''(\varrho^*)(x_0-\varrho^*)^2.
\end{aligned}$$

We use the assumptions (A1)-(A4) in order to compute e_{x_1} and the following inequalities.

Observe that by applying M.V.T., we get

$$\begin{aligned}
\|\wp'(x_0)^{-1}\wp(x_0)\| & \leq \left\|\wp'(x_0)^{-1}\int_0^1 \wp'(\varrho^*+t(x_0-\varrho^*))dt\right\| e_{x_0} \\
& \leq \left\|\wp'(x_0)^{-1}\wp'(\varrho^*)\int_0^1 \wp'(\varrho^*)^{-1}\wp'(\varrho^*+t(x_0-\varrho^*))dt\right\| e_{x_0} \\
& \leq \frac{(2+L_1e_{x_0})}{2(1-L_1e_{x_0})}e_{x_0}, \qquad (2.19)
\end{aligned}$$

$$
\begin{aligned}
&\|x_0-\varrho^*+\wp'(x_0)^{-1}\wp(x_0)\|\\
\leq\ & \left\|x_0-\varrho^*+\wp'(x_0)^{-1}\int_0^1\wp'(\varrho^*+t(x_0-\varrho^*))dt(x_0-\varrho^*)\right\|\\
\leq\ & \|\wp'(x_0)^{-1}\wp'(\varrho^*)\|\left\|\wp'(\varrho^*)^{-1}\wp'(x_0)+\wp'(\varrho^*)^{-1}\int_0^1\wp'(\varrho^*+t(x_0-\varrho^*))dt\right\|e_{x_0}\\
\leq\ & \frac{1}{2(1-L_1e_{x_0})}(4+3L_1e_{x_0})e_{x_0} \qquad (2.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|x_0-\varrho^*-\wp'(x_0)^{-1}\wp(x_0)\|\\
=\ & \|x_0-\varrho^*-\wp'(x_0)^{-1}\int_0^1\wp'(\varrho^*+t(x_0-\varrho^*))dt(x_0-\varrho^*)\|\\
\leq\ & \|\wp'(x_0)^{-1}\wp'(\varrho^*)\|\\
&\times\int_0^1\|\wp'(\varrho^*)^{-1}(\wp'(x_0)-\wp'(\varrho^*)+\wp'(\varrho^*)-\wp'(\varrho^*+t(x_0-\varrho^*)))\|dte_{x_0}\\
\leq\ & \|\wp'(x_0)^{-1}\wp'(\varrho^*)\|\int_0^1\|\wp'(\varrho^*)^{-1}[\wp'(x_0)-\wp'(\varrho^*)]\|dte_{x_0}\\
&+\|\wp'(x_0)^{-1}\wp'(\varrho^*)\|\int_0^1\|\wp'(\varrho^*)^{-1}[\wp'(\varrho^*)-\wp'(\varrho^*+t(x_0-\varrho^*))]\|dte_{x_0}\\
\leq\ & \frac{3L_1}{2(1-L_1e_{x_0})}e_{x_0}^2. \qquad (2.21)
\end{aligned}
$$

Then by using assumption (A3), (2.11) and (2.15),

$$
\begin{aligned}
\|H_1\| \leq\ & \left\|\wp'(x_0)^{-1}\wp'(\varrho^*)\right\|\\
&\times\int_0^1\int_0^1\left\|\wp'(\varrho^*)^{-1}[\wp''(\varrho^*+(t+\theta(1-t))(x_0-\varrho^*))-\wp''(\varrho^*)]\right\||1-t|d\theta dte_{x_0}^2\\
\leq\ & \frac{L_3}{1-L_1e_{x_0}}\int_0^1\int_0^1|t(1-t)+\theta(1-t)^2|d\theta dte_{x_0}^3\\
\leq\ & \frac{L_3}{3(1-L_1e_{x_0})}e_{x_0}^3, \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
\|H_2\| \leq\ & |a|\left\|\mathcal{A}_0^{-1}\wp'(\varrho^*)\right\|\int_0^1\|F(\varrho^*)^{-1}\wp''(x_0+\theta(y_0-x_0))-\wp''(\varrho^*)\|d\theta\|\wp'(x_0)^{-1}\wp(x_0)\|^2\\
\leq\ & \frac{L_3}{2(1-\phi_1(e_{x_0})e_{x_0})}\left[e_{x_0}+\frac{|a|}{2}\|\wp'(x_0)^{-1}\wp(x_0)\|\right]\|\wp'(x_0)^{-1}\wp(x_0)\|^2.
\end{aligned}
$$

Then by using (2.19), we get

$$
\|H_2\| \leq \frac{L_3(2+L_1e_{x_0})^2}{8(1-\phi_1(e_{x_0})e_{x_0})(1-L_1e_{x_0})^2}\left[1+\frac{|a|}{4}\frac{2+L_1e_{x_0}}{(1-L_1e_{x_0})}\right]e_{x_0}^3, \qquad (2.23)
$$

$$
\begin{aligned}
\|H_3\| \leq\ & a\left\|\mathcal{A}_0^{-1}\wp'(\varrho^*)\right\|\left\|\wp'(\varrho^*)^{-1}\wp''(\varrho^*)\right\|\\
&\times\|x_0-\varrho^*-\wp'(x_0)^{-1}\wp(x_0)\|\|x_0-\varrho^*+\wp'(x_0)^{-1}\wp(x_0)\|.
\end{aligned}
$$

Again by the assumption (A4), (2.20) and (2.21), we have

$$
\|H_3\| \leq \frac{3L_4L_1(4+3L_1e_{x_0})}{8(1-\phi_1(e_{x_0})e_{x_0})(1-L_1e_{x_0})^2}e_{x_0}^3 \qquad (2.24)
$$

and

$$\begin{aligned}\|H_4\| &\leq \frac{|a|}{2}\|\mathcal{A}_0^{-1}\wp'(\varrho^*)\|\int_0^1 \|\wp'(\varrho^*)^{-1}\wp''(x_0+\theta(y_0-x_0))\|d\theta \\ &\quad \times\|\wp'(x_0)^{-1}\wp(x_0)\|\|\wp'(x_0)^{-1}\wp'(\varrho^*)\|\|\wp'(\varrho^*)^{-1}\wp''(\varrho^*)\|e_{x_0}^2 \\ &\leq \frac{L_4^2(2+L_1e_{x_0})}{8(1-\phi_1(e_{x_0})e_{x_0})(1-L_1e_{x_0})^2}e_{x_0}^3. \end{aligned} \tag{2.25}$$

Thus by (2.22), (2.23), (2.24) and (2.25), we get

$$\begin{aligned} e_{x_1} &\leq \|H_1\|+\|H_2\|+\|H_3\|+\|H_4\| \\ &\leq \phi_2(e_{x_0})e_{x_0}^3. \end{aligned} \tag{2.26}$$

Thus, since $\phi_2(e_{x_0})e_{x_0}^2<1$, we have $e_{x_1}<r$ and the iterate $x_1\in\mathcal{S}(\varrho^*,r)$.

Simply replace x_0,y_0,x_1 by x_n,y_n,x_{n+1} in the estimates above to complete the induction for (2.10). □

THEOREM 2.2. *Let (A1)-(A4) hold and sequence $\{x_n\}$ be as in (1.3) with $x_0\in\mathcal{S}(\varrho^*,r)-\{\varrho^*\}$. Then, $x_n\in\bar{\mathcal{S}}(\varrho^*,r)$ is well defined $\forall n\in\mathbb{N}\cup\{0\}$ and $\lim_{n\to\infty}x_n=\rho^*$. Further,*

$$e_{z_n}\leq\phi_2(r)e_{x_n}^3 \tag{2.27}$$

and

$$e_{x_{n+1}}\leq\phi_3(r)e_{x_n}^5. \tag{2.28}$$

Proof. Observe that, (2.27) holds as in Theorem 2.1 by taking $x_{n+1}=z_n$.

Next consider,

$$x_{n+1}-\varrho^* = z_n-\varrho^*-4a\mathcal{A}_n^{-1}\wp(z_n)+\wp'(x_n)^{-1}\wp(z_n).$$

Applying the M.V.T., we obtain

$$\begin{aligned} & x_{n+1}-\varrho^* \\ =\ & z_n-\varrho^*-4a\mathcal{A}_n^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*) \\ & +\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*) \\ =\ & \mathcal{A}_n^{-1}\left[\wp'(y_n)+(2a-1)\wp'(x_n)-4a\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt\right](z_n-\varrho^*) \\ & +\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*). \end{aligned}$$

Rearranging the first term, we get

$$\begin{aligned} x_{n+1}-\varrho^* &= \mathcal{A}_n^{-1}\left[\wp'(y_n)-\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt\right](z_n-\varrho^*) \\ &\quad +(2a-1)\mathcal{A}_n^{-1}\left[\wp'(x_n)-\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt\right](z_n-\varrho^*) \\ &\quad -2a\mathcal{A}_n^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*) \\ &\quad +\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*). \end{aligned}$$

Combining the last two terms,

$$\begin{aligned}
x_{n+1}-\varrho^* &= \mathcal{A}_n^{-1}\left[\wp'(y_n)-\int_0^1 \wp'(\varrho^*+t(z_n-\varrho^*))dt\right](z_n-\varrho^*)\\
&\quad +(2a-1)\mathcal{A}_n^{-1}\left[\wp'(x_n)-\int_0^1 \wp'(\varrho^*+t(z_n-\varrho^*))dt\right](z_n-\varrho^*)\\
&\quad -\mathcal{A}_n^{-1}[2a\wp'(x_n)-\wp'(y_n)-(2a-1)\wp'(x_n)]\\
&\quad \times\wp'(x_n)^{-1}\int_0^1 \wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*)\\
&= \mathcal{A}_n^{-1}\left[\int_0^1\int_0^1 \wp''(h(t,\theta,y_n,z_n))d\theta(y_n-\varrho^*-t(z_n-\varrho^*))dt\right](z_n-\varrho^*)\\
&\quad +(2a-1)\mathcal{A}_n^{-1}\\
&\quad \times\left[\int_0^1\int_0^1 \wp''(h(t,\theta,x_n,z_n))d\theta(x_n-\varrho^*-t(z_n-\varrho^*))dt\right](z_n-\varrho^*)\\
&\quad +\mathcal{A}_n^{-1}[\wp'(y_n)-\wp'(x_n)]\wp'(x_n)^{-1}\int_0^1 \wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*),
\end{aligned}$$

where the function $h(t,\theta,w,z):[0,1]\times[0,1]\times X\times X\to X$, is defined as

$$h(t,\theta,w,z)=\varrho^*+t(z-\varrho^*)+\theta(w-\varrho^*-t(z-\varrho^*)).$$

Then,

$$\begin{aligned}
x_{n+1}-\varrho^* &= \mathcal{A}_n^{-1}\int_0^1\int_0^1 \wp''(h(t,\theta,y_n,z_n))d\theta dt(y_n-\varrho^*)(z_n-\varrho^*)+I_1\\
&\quad +(2a-1)\mathcal{A}_n^{-1}\int_0^1\int_0^1 \wp''(h(t,\theta,x_n,z_n))d\theta dt(x_n-\varrho^*)(z_n-\varrho^*)+I_2\\
&\quad +\mathcal{A}_n^{-1}\int_0^1 \wp''(x_n+\theta(y_n-x_n))d\theta\left(-a\wp'(x_n)^{-1}\wp(x_n)\right)\\
&\quad \times\wp'(x_n)^{-1}\int_0^1 \wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*),
\end{aligned} \tag{2.29}$$

where

$$I_1 = -\mathcal{A}_n^{-1}\int_0^1\int_0^1 \wp''(h(t,\theta,y_n,z_n))d\theta t dt(z_n-\varrho^*)^2$$

and

$$I_2 = -(2a-1)\mathcal{A}_n^{-1}\int_0^1\int_0^1 \wp''(h(t,\theta,x_n,z_n))d\theta t dt(z_n-\varrho^*)^2.$$

Add and substract $\wp''(\varrho^*)$ in the first, third and fifth terms of (2.29) to get,

$$\begin{aligned} x_1-\varrho^* &= \mathcal{A}_n^{-1}\int_0^1\int_0^1[\wp''(h(t,\theta,y_n,z_n))-\wp''(\varrho^*)]d\theta dt(y_n-\varrho^*)(z_n-\varrho^*)\\ &+\mathcal{A}_n^{-1}\wp''(\varrho^*)(y_n-\varrho^*)(z_n-\varrho^*)+I_1\\ &+(2a-1)\mathcal{A}_n^{-1}\int_0^1\int_0^1[\wp''(h(t,\theta,x_n,z_n))-\wp''(\varrho^*)]d\theta dt(x_n-\varrho^*)(z_n-\varrho^*)\\ &+(2a-1)\mathcal{A}_n^{-1}\wp''(\varrho^*)(x_n-\varrho^*)(z_n-\varrho^*)+I_2\\ &-a\mathcal{A}_n^{-1}\int_0^1[\wp''(x_n+\theta(y_n-x_n))-\wp''(\varrho^*)]d\theta\wp'(x_n)^{-1}\wp(x_n)\\ &\times\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*)\\ &-a\mathcal{A}_n^{-1}\wp''(\varrho^*)\wp'(x_n)^{-1}\wp(x_n)\\ &\times\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*). \end{aligned} \tag{2.30}$$

Let

$$\begin{aligned} I_3 &= \mathcal{A}_n^{-1}\int_0^1\int_0^1[\wp''(h(t,\theta,y_n,z_n))-\wp''(\varrho^*)]d\theta dt(y_n-\varrho^*)(z_n-\varrho^*),\\ I_4 &= (2a-1)\mathcal{A}_n^{-1}\int_0^1\int_0^1[\wp''(h(t,\theta,x_n,z_n))-\wp''(\varrho^*)]d\theta dt(x_n-\varrho^*)(z_n-\varrho^*),\\ I_5 &= -a\mathcal{A}_n^{-1}\int_0^1[\wp''(x_n+\theta(y_n-x_n))-\wp''(\varrho^*)]d\theta\wp'(x_n)^{-1}\wp(x_n)\\ &\times\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt(z_n-\varrho^*). \end{aligned}$$

Combining the fourth, sixth and eighth terms in (2.30), we get

$$\begin{aligned} x_1-\varrho^* &= I_1+I_2+I_3+I_4+I_5+\mathcal{A}_n^{-1}\wp''(\varrho^*)\\ &\times\Big[(y_n-\varrho^*)+(2a-1)(x_n-\varrho^*)-a\wp'(x_n)^{-1}\wp(x_n)\\ &\times\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt\Big](z_n-\varrho^*). \end{aligned}$$

By the sub-step 1 of (1.2), we get

$$\begin{aligned} & x_{n+1}-\varrho^*\\ =\ & I_1+I_2+I_3+I_4+I_5+\mathcal{A}_n^{-1}\wp''(\varrho^*)\\ &\times\Big[(x_n-\varrho^*-a\wp'(x_n)^{-1}\wp(x_n))+(2a-1)(x_n-\varrho^*)-a\wp'(x_n)^{-1}\wp(x_n)\\ &\times\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt\Big](z_n-\varrho^*). \end{aligned}$$

On adding and substracting $a\wp'(x_n)^{-1}\wp(x_n)$ to the last term, we get

$$\begin{aligned}
& x_1-\varrho^* \\
= \;& I_1+I_2+I_3+I_4+I_5+\mathcal{A}_n^{-1}\wp''(\varrho^*) \\
& \times\Big[(x_n-\varrho^*-a\wp'(x_n)^{-1}\wp(x_n))+(2a-1)(x_n-\varrho^*)-a\wp'(x_n)^{-1}\wp(x_n) \\
& \times\wp'(x_n)^{-1}\int_0^1\wp'(\varrho^*+t(z_n-\varrho^*))dt+a\wp'(x_n)^{-1}\wp(x_n) \\
& -a\wp'(x_n)^{-1}\wp(x_n)\Big](z_n-\varrho^*) \\
= \;& I_1+I_2+I_3+I_4+I_5+\mathcal{A}_n^{-1}\wp''(\varrho^*) \\
& \times\Big[2a(x_n-\varrho^*-\wp'(x_n)^{-1}\wp(x_n))+a\wp'(x_n)^{-1}\wp(x_n) \\
& \times\wp'(x_n)^{-1}\int_0^1[\wp'(x_n)-\wp'(\varrho^*+t(z_n-\varrho^*))]dt\Big](z_n-\varrho^*) \\
= \;& I_1+I_2+I_3+I_4+I_5+I_6+I_7, \qquad (2.31)
\end{aligned}$$

where

$$I_6 = 2a\mathcal{A}_n^{-1}\wp''(\varrho^*)\left[x_n-\varrho^*-\wp'(x_n)^{-1}\wp(x_n)\right](z_n-\varrho^*)$$

and

$$\begin{aligned}
I_7 = \;& a\mathcal{A}_n^{-1}\wp''(\varrho^*) \\
& \times\Big[\wp'(x_n)^{-1}\wp(x_n)\wp'(x_n)^{-1}\int_0^1[\wp'(x_n)-\wp'(\varrho^*+t(z_n-\varrho^*))]dt\Big](z_n-\varrho^*).
\end{aligned}$$

We use the assumptions (A1)-(A4) in order to compute e_{z_n}. Consider

$$\begin{aligned}
\|I_1\| \le\;& \|\mathcal{A}_n^{-1}\wp'(\varrho^*)\|\int_0^1\int_0^1\|\wp'(\varrho^*)^{-1}\wp''(h(t,\theta,y_n,z_n))\||t|d\theta dt e_{z_n}^2 \\
\le\;& \frac{L_4(\phi_2(e_{x_n}))^2e_{x_n}}{4|a|(1-\phi_1(e_{x_n})e_{x_n})}e_{x_n}^5. \qquad (2.32)
\end{aligned}$$

Similarly,

$$\|I_2\| \le \frac{|2a-1|L_4(\phi_2(e_{x_n}))^2e_{x_n}}{4|a|(1-\phi_1(e_{x_n})e_{x_n})}e_{x_n}^5. \qquad (2.33)$$

Further,

$$\begin{aligned}
\|I_3\| \le\;& \|\mathcal{A}_n^{-1}\wp'(\varrho^*)\| \\
& \times\int_0^1\int_0^1\|\wp'(\varrho^*)^{-1}[\wp''(h(t,\theta,y_n,z_n))-\wp''(\varrho^*)]\|d\theta dt e_{y_n}e_{z_n} \\
\le\;& \frac{L_3}{2|a|(1-\phi_1(e_{x_n})e_{x_n})} \\
& \times\int_0^1\int_0^1\|t(z_n-\varrho^*)+\theta(y_n-\varrho^*-t(z_n-\varrho^*))\|d\theta dt e_{y_n}e_{z_n} \\
\le\;& \frac{L_3\phi(e_{x_n})\phi_2(e_{x_n})}{4|a|(1-\phi_1(e_{x_n})e_{x_n})}\left(\frac{1}{2}\phi_2(e_{x_n})e_{x_n}^2+\phi(e_{x_n})\right)e_{x_n}^5. \qquad (2.34)
\end{aligned}$$

Similarly considering (2.13) and (2.15), one can observe that

$$\begin{aligned} \|I_4\| &\leq |2a-1|\|\mathcal{A}_n^{-1}\wp'(\varrho^*)\| \\ &\quad \times \int_0^1\int_0^1 \|\wp'(\varrho^*)^{-1}[\wp''(h(t,\theta,x_n,z_n)) - \wp''(\varrho^*)]\|d\theta dt e_{x_n} e_{z_n} \\ &\leq \frac{|2a-1|L_3\phi_2(e_{x_n})}{4|a|(1-\phi_1(e_{x_n})e_{x_n})}\left(\frac{1}{2}\phi_2(e_{x_n})e_{x_n}^2 + 1\right)e_{x_n}^5. \end{aligned} \tag{2.35}$$

Using (2.19), (A3) and (2.11), we get

$$\begin{aligned} \|I_5\| &\leq |a|\|\mathcal{A}_n^{-1}\wp'(\varrho^*)\| \int_0^1 \|\wp'(\varrho^*)^{-1}[\wp''(x_n+\theta(y_n-x_n)) - \wp''(\varrho^*)]\|d\theta \\ &\quad \times \|\wp'(x_n)^{-1}\wp(x_n)\|\|\wp'(x_n)^{-1}\wp'(\varrho^*)\| \int_0^1 \|\wp'(\varrho^*)^{-1}\wp'(\varrho^*+t(z_n-\varrho^*))\|dt e_{z_n} \\ &\leq \frac{L_3}{2(1-\phi_1(e_{x_n})e_{x_n})}\int_0^1 \|x_n + \theta(-a\wp'(x_n)^{-1}\wp(x_n)) - \varrho^*\|d\theta \\ &\quad \times \left(\frac{2+L_1e_{x_n}}{2(1-L_1e_{x_n})}\right)e_{x_n}\frac{1}{1-L_1e_{x_n}}\left(1+\frac{L_1}{2}\phi_2(e_{x_n})e_{x_n}^3\right)\phi_2(e_{x_n})e_{x_n}^3 \\ &\leq \frac{L_3}{2(1-\phi_1(e_{x_n})e_{x_n})}\left(1+\frac{|a|}{4}\frac{2+L_1e_{x_n}}{(1-L_1e_{x_n})}\right)\frac{2+L_1e_{x_n}}{2(1-L_1e_{x_n})^2}e_{x_n}^2 \\ &\quad \times \left(1+\frac{L_1}{2}\phi_2(e_{x_n})e_{x_n}\right)\phi_2(e_{x_n})e_{x_n}^5, \end{aligned} \tag{2.36}$$

$$\begin{aligned} \|I_6\| &\leq 2|a|\|\mathcal{A}_n^{-1}\wp'(\varrho^*)\|\|\wp'(\varrho^*)^{-1}\wp''(\varrho^*)\|\|x_n - \varrho^* - \wp'(x_n)^{-1}\wp(x_n)\|e_{z_n} \\ &\leq \frac{3L_4L_1\phi_2(e_{x_n})}{2(1-\phi_1(e_{x_n})e_{x_n})(1-L_1e_{x_n})}e_{x_n}^5 \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \|I_7\| &\leq |a|\|\mathcal{A}_n^{-1}\wp'(\varrho^*)\|\|\wp'(\varrho^*)^{-1}\wp''(\varrho^*)\|\|\wp'(x_n)^{-1}\wp(x_n)\| \\ &\quad \times \|\wp'(x_n)^{-1}\wp'(\varrho^*)\| \int_0^1 \|\wp'(\varrho^*)^{-1}[\wp'(x_n) - \wp'(\varrho^*+t(z_n-\varrho^*))]\|dt e_{z_n} \\ &\leq |a|\|\mathcal{A}_n^{-1}\wp'(\varrho^*)\|\|\wp'(\varrho^*)^{-1}\wp''(\varrho^*)\|\|\wp'(x_n)^{-1}\wp(x_n)\| \\ &\quad \times \|\wp'(x_n)^{-1}\wp'(\varrho^*)\| \big(\|\wp'(\varrho^*)^{-1}[\wp'(x_n) - \wp'(\varrho^*)]\| \\ &\quad + \int_0^1 \|\wp'(\varrho^*)^{-1}[\wp'(\varrho^*+t(z_n-\varrho^*)) - \wp'(\varrho^*)]\|dt\bigg) e_{z_n} \\ &\leq \frac{L_4(2+L_1e_{x_n})\phi_2(e_{x_n})}{4(1-\phi_1(e_{x_n})e_{x_n})(1-L_1e_{x_n})^2}\left(L_1 + \frac{L_1}{2}\phi_2(e_{x_n})e_{x_n}^2\right)e_{x_n}^5. \end{aligned} \tag{2.38}$$

Combining (2.32), (2.33), (2.34), (2.35), (2.36), (2.37) and (2.38), we get

$$\begin{aligned} e_{x_{n+1}} &\leq \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\| + \|I_5\| + \|I_6\| + \|I_7\| \\ &\leq \phi_3(e_{x_n})e_{x_n}^5. \end{aligned}$$

□

We next give the uniqueness of ϱ^*.

PROPOSITION 2.3. *Suppose:*
(1) $\exists$ a solution (simple) $\varrho^* \in \mathcal{S}(\varrho^*, \nu)$ of (1.1) for some $\nu > 0$ and a parameter $K > 0$ such that

$$\|\wp'(\varrho^*)^{-1}(\wp'(\varrho^*) - \wp'(x))\| \leq K\|\varrho^* - x\| \tag{2.39}$$

for each $x \in \mathcal{S}(\varrho^*, \rho)$.
(2) $\exists\, \delta \geq \nu$ *such that*

$$\delta < \frac{2}{K}. \tag{2.40}$$

Set $B = \bar{\mathcal{S}}(\varrho^*, \delta) \cap \mathcal{D}$. *Then, (1.1) is uniquely solvable at* ϱ^* *in* B.

Proof. Let $\gamma \in B$ be a solution of the equation (1.1). A linear operator M is defined as $M = \int_0^1 \wp'(\varrho^* + \tau(\gamma - \varrho^*))d\tau$. By (2.39) and (2.40), we get

$$\begin{aligned} \|\wp'(\varrho^*)^{-1}(M - \wp'(\varrho^*))\| &\leq K \int_0^1 \tau \|\varrho^* - \gamma\| d\tau \\ &\leq \frac{K}{2}\delta < 1. \end{aligned}$$

i.e., M is invertible. But then, $\gamma - \varrho^* = M^{-1}(\wp(\gamma) - \wp(\varrho^*)) = M^{-1}(0) = 0$ implies $\gamma = \varrho^*$. □

3. Semi-local Convergence of (1.2) and (1.3)

The $\omega-$ continuity concepts and scalar majorizing sequences are employed to formulate the analysis of semi-local convergence [1, 2].

For the method (1.2), we define the scalar sequences $\{\mathcal{X}_n\}$ and $\{\mathcal{Y}_n\}$. Let $\omega_0 : [0, +\infty) \longrightarrow \mathbb{R}$ and $\omega : [0, +\infty) \longrightarrow \mathbb{R}$ be two C.N.D.F..

For $\mathcal{X}_0 = 0$ and $\mathcal{Y}_0 \geq 0$, let

$$\begin{aligned} s_n &= \frac{1}{2|a|}(\omega_0(\mathcal{Y}_n) + |2a-1|\omega_0(\mathcal{X}_n)), \\ \bar{\omega}_n &= \begin{cases} \omega_n^1 = \omega(\mathcal{Y}_n - \mathcal{X}_n) + 2|1-a|(1+\omega_0(\mathcal{X}_n)) \\ \text{or} \\ \omega_n^2 = \omega_0(\mathcal{X}_n) + \omega_0(\mathcal{Y}_n) + 2|1-a|(1+\omega_0(\mathcal{X}_n)) \end{cases}, \\ \mathcal{X}_{n+1} &= \mathcal{Y}_n + \frac{\bar{\omega}_n(\mathcal{Y}_n - \mathcal{X}_n)}{2|a|(1-s_n)}, \\ \delta_{n+1} &= \left(1 + \int_0^1 \omega_0(\mathcal{X}_n + \theta(\mathcal{X}_{n+1} - \mathcal{X}_n))d\theta\right)(\mathcal{X}_{n+1} - \mathcal{X}_n) \\ &\quad + \frac{1}{|a|}(1+\omega_0(\mathcal{X}_n))(\mathcal{Y}_n - \mathcal{X}_n), \\ \mathcal{Y}_{n+1} &= \mathcal{X}_{n+1} + \frac{|a|\delta_{n+1}}{1-\omega_0(\mathcal{X}_{n+1})}. \end{aligned} \tag{3.41}$$

The forthcoming Theorem 3.3 will show that the above sequences serve as majorizers for the method (1.2). However, we offer a general convergence result for the methods.

LEMMA 3.1. *Assume that for each* $n = 0, 1, 2, \ldots \exists\, \kappa \geq 0$ *with*

$$\omega_0(\mathcal{X}_n) < 1, s_n < 1 \text{ and } \mathcal{X}_n \leq \kappa. \tag{3.42}$$

Then, the sequences $\{\mathcal{X}_n\}, \{\mathcal{Y}_n\}$ *as in (3.41) converge to some* $\lambda \in [\mathcal{Y}_0, \kappa]$ *and* $0 \leq \mathcal{X}_n \leq \mathcal{Y}_n \leq \mathcal{X}_{n+1} \leq \lambda$.

Proof. By the properties of ω_0 and ω, (3.41) and (3.42), one can deduce that $\{\mathcal{X}_n\}, \{\mathcal{Y}_n\}$ are non-decreasing sequences bounded above by κ. Thus, they converge to λ. □

REMARK 3.2. *(i) λ is the unique, common least upper bound of $\{\mathcal{X}_n\}$ and $\{\mathcal{Y}_n\}$.*
(ii) Suppose ω_0 is strictly increasing, then one can choose $\kappa = \omega_0^{-1}(1)$.
(iii) If the function $\omega_0(t) - 1$ has a minimal zero $\rho \in (0, +\infty)$, then ω can be constrained to the interval $(0, \rho)$ and $\kappa \geq \rho$.

We now link ω_0, ω, the scalar sequences and the parameter λ to the operator $\wp'$ as given below:

- (C1) $\exists$ an initial point $x_0 \in \mathcal{D}$ and a parameter $\mathcal{Y}_0 \geq 0$ with $\wp'(x_0)^{-1} \in \mathcal{L}(Y, X)$ and $|a|\|\wp'(x_0)^{-1}\wp(x_0)\| \leq \mathcal{Y}_0$.
- (C2) $\|\wp'(x_0)^{-1}(\wp'(v) - \wp'(x_0))\| \leq \omega_0(\|v - x_0\|)$ for all $v \in \mathcal{D}$.
  Set $\mathcal{D}_0 = \mathcal{D} \cap \mathcal{S}(x_0, \rho)$.
- (C3) $\|\wp'(x_0)^{-1}(\wp'(v_2) - \wp'(v_1))\| \leq \omega(\|v_2 - v_1\|)$ for all $v_1, v_2 \in \mathcal{D}_0$.
- (C4) The inequalities in (3.42) hold for $\kappa = \rho$.
- (C5) $\bar{\mathcal{S}}(x_0, \lambda) \subset \mathcal{D}$.

THEOREM 3.3. *Assume (C1)-(C5). Then, the sequence $\{x_n\}$ developed by method (1.2) is convergent to some $\varrho^* \in \mathcal{S}(x_0, \lambda)$ solving (1.1).*

Proof. First, we verify that

$$\|y_n - x_n\| \leq \mathcal{Y}_n - \mathcal{X}_n \tag{3.43}$$

and

$$\|x_{n+1} - y_n\| \leq \mathcal{X}_{n+1} - \mathcal{Y}_n. \tag{3.44}$$

The proof is through induction. By (C1) and (3.41), we get

$$\|y_0 - x_0\| = |a|\|\wp'(x_0)^{-1}\wp(x_0)\| \leq \mathcal{Y}_0 = \mathcal{Y}_0 - \mathcal{X}_0 < \lambda.$$

Thus, (3.43) is true whenever $n = 0$ and $y_0 \in \mathcal{S}(x_0, \lambda)$.

Let $v \in \mathcal{S}(x_0, \lambda)$. By the definition of λ and (C2),

$$\|\wp'(x_0)^{-1}(\wp'(v) - \wp'(x_0))\| \leq \omega_0(\|v - x_0\|) \leq \omega_0(\lambda) < 1 \tag{3.45}$$

showing $\wp'(v)^{-1} \in \mathcal{L}(Y, X)$ and

$$\|\wp'(v)^{-1}\wp'(x_0)\| \leq \frac{1}{1 - \omega_0(\|v - x_0\|)}. \tag{3.46}$$

Further, by (3.45)

$$\|\wp'(x_0)^{-1}\wp'(v)\| \leq 1 + \omega_0(\|v - x_0\|). \tag{3.47}$$

We also need the estimate

$$\begin{aligned}
& \|(2a\wp'(x_0))^{-1}(\wp'(y_k) + (2a-1)\wp'(x_k) - 2a\wp'(x_0))\| \\
\leq\; & \frac{1}{2|a|}(\|\wp'(x_0))^{-1}(\wp'(y_k) - \wp'(x_0)\| + |2a-1|\|\wp'(x_0)^{-1}(\wp'(x_k) - \wp'(x_0))\| \\
\leq\; & \frac{1}{2|a|}(\omega_0(\|y_k - x_0\|) + |2a-1|\omega_0(\|x_k - x_0\|)) \\
\leq\; & \frac{1}{2|a|}(\omega_0(\mathcal{Y}_k) + |2a-1|\omega_0(\mathcal{X}_k)) \\
\leq\; & s_k < 1.
\end{aligned}$$

Thus, by taking $\mathcal{A}_k = \wp'(y_k) + (2a-1)\wp'(x_k)$, we get

$$\|A_k^{-1}\wp'(x_0)\| \leq \frac{1}{2|a|(1 - s_k)}. \tag{3.48}$$

Further, by (1.2)

$$\begin{aligned} x_{k+1}-y_k &= (a\wp'(x_k)^{-1}\wp(x_k)-2a\mathcal{A}_k^{-1}\wp(x_k)) \\ &= (\wp'(x_k)^{-1}-2\mathcal{A}_k^{-1})a\wp(x_k) \\ &= \mathcal{A}_k^{-1}(\mathcal{A}_k-2\wp'(x_k))a\wp'(x_k)^{-1}\wp(x_k) \\ &= A_k^{-1}(2\wp'(x_k)-\wp'(y_k)-(2a-1)\wp'(x_k))(y_k-x_k) \\ &= A_k^{-1}((\wp'(x_k)-\wp'(y_k))+2(1-a)\wp'(x_k))(y_k-x_k). \end{aligned}$$

Consider

$$\begin{aligned} &\|\wp'(x_0)^{-1}[(\wp'(x_k)-\wp'(y_k))+2(1-a)\wp'(x_k)]\| \\ \leq\ &\|\wp'(x_0)^{-1}(\wp'(x_k)-\wp'(y_k))\|+2|1-a|\|\wp'(x_0)^{-1}\wp'(x_k)\| \\ \leq\ &\omega(\|y_k-x_k\|)+2|1-a|(1+\omega_0(\|x_k-x_0\|)) \\ \leq\ &\omega(\mathcal{Y}_k-\mathcal{X}_k)+2|1-a|(1+\omega_0(\mathcal{X}_k)) \\ \leq\ &\omega_k^1\leq\bar{\omega}_k \end{aligned} \tag{3.49}$$

or,

$$\begin{aligned} &\|\wp'(x_0)^{-1}[(\wp'(x_k)-\wp'(y_k))+2(1-a)\wp'(x_k)]\| \\ \leq\ &\|\wp'(x_0)^{-1}[(\wp'(x_k)-\wp'(x_0)+\wp'(x_0)-\wp'(y_k))+2(1-a)\wp'(x_k)]\| \\ \leq\ &\omega_0(\|x_k-x_0\|)+\omega_0(\|y_k-x_0\|)+2|1-a|(1+\omega_0(\|x_k-x_0\|)) \\ \leq\ &\omega_0(\mathcal{X}_k)+\omega_0(\mathcal{Y}_k)+2|1-a|(1+\omega_0(\mathcal{X}_k)) \\ \leq\ &\omega_k^2\leq\bar{\omega}_k. \end{aligned} \tag{3.50}$$

By (3.48)-(3.50), we obtain

$$\|x_{k+1}-y_k\|\leq\frac{\bar{\omega}_k(\mathcal{Y}_k-\mathcal{X}_k)}{2|a|(1-s_k)}=\mathcal{X}_{k+1}-\mathcal{Y}_k \tag{3.51}$$

and

$$\|x_{k+1}-x_0\|\leq\|x_{k+1}-y_k\|+\|y_k-x_0\|\leq\mathcal{X}_{k+1}-\mathcal{Y}_k+\mathcal{Y}_k-\mathcal{X}_0=\mathcal{X}_{k+1}<\lambda.$$

These estimates indicate that the iterate $x_{k+1}\in\mathcal{S}(x_0,\lambda)$ thereby validating (3.44). Now, by the sub-step 1 of the method (1.2),

$$\begin{aligned} \wp(x_{k+1}) &= \wp(x_{k+1})-\wp(x_k)-\wp(x_k) \\ &= \wp(x_{k+1})-\wp(x_k)-\frac{1}{a}\wp'(x_k)(y_k-x_k). \end{aligned} \tag{3.52}$$

It follows by the M.V.T., (3.41), (C2), (C3) and (3.52) that

$$\begin{aligned}
&\|\wp'(x_0)^{-1}\wp(x_{k+1})\| \\
\leq\;& \int_0^1 \|\wp'(x_0)^{-1}(\wp'(x_k+\theta(x_{k+1}-x_k))\|d\theta\|x_{k+1}-x_k\| \\
&+\frac{1}{|a|}\|\wp'(x_0)^{-1}\wp'(x_k)\|\|y_k-x_k\| \\
\leq\;& \left(1+\int_0^1 \omega_0(\|x_k-x_0\|+\theta\|x_{k+1}-x_k\|)d\theta\right)\|x_{k+1}-x_k\| \\
&+\frac{1}{|a|}(1+\omega_0(\|x_k-x_0\|))\|y_k-x_k\| \\
\leq\;& \left(1+\int_0^1 \omega_0(\mathcal{X}_k+\theta(\mathcal{X}_{k+1}-\mathcal{X}_k))d\theta\right)(\mathcal{X}_{k+1}-\mathcal{X}_k)+\frac{1}{|a|}(1+\omega_0(\mathcal{X}_k))(\mathcal{Y}_k-\mathcal{X}_k) \\
=\;& \delta_{k+1}.
\end{aligned} \tag{3.53}$$

Hence, sub-step 1 of the method (1.2), (3.46) (for $v = x_{k+1}$) and (3.53) give

$$\begin{aligned}
\|y_{k+1}-x_{k+1}\| &\leq |a|\|\wp'(x_{k+1})^{-1}\wp'(x_0)\|\|\wp'(x_0)^{-1}\wp(x_{k+1})\| \\
&\leq |a|\frac{\delta_{k+1}}{1-\omega_0(\|x_{k+1}-x_0\|)} \leq |a|\frac{\delta_{k+1}}{1-\omega_0(\mathcal{X}_{k+1})} = \mathcal{Y}_{k+1}-\mathcal{X}_{k+1}
\end{aligned}$$

and

$$\begin{aligned}
\|y_{k+1}-x_0\| &\leq \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\| \\
&\leq \mathcal{Y}_{k+1}-\mathcal{X}_{k+1}+\mathcal{X}_{k+1}-\mathcal{X}_0 = \mathcal{Y}_{k+1} < \lambda.
\end{aligned}$$

Therefore the induction for (3.43), (3.44) is complete and $x_k, y_k \in \mathcal{S}(x_0,\lambda)\ \forall\ k = 0,1,2,\ldots$ By (C4) and Lemma 3.1, $\{\mathcal{X}_k\}$, $\{\mathcal{Y}_k\}$ are Cauchy sequences. Consequently, by (3.43) and (3.44), $\{x_k\}$, $\{y_k\}$ are also Cauchy sequences and converge to some $\varrho^* \in \bar{\mathcal{S}}(x_0,\lambda)$. Finally, (3.53) and the continuity of $\wp$ imply $\wp(\varrho^*) = 0$ (if $k \to +\infty$). □

Next, uniqueness of the solution is established.

PROPOSITION 3.4. *Suppose :*

(i) *$\exists$ a solution (simple) $d \in \mathcal{S}(x_0,r_0)$ of the equation $\wp(x) = 0$ for some $r_0 > 0$.*
(ii) *The condition (C2) holds in $\mathcal{S}(x_0,r_0)$.*
(iii) *$\exists\ r \geq r_0$ such that*

$$\int_0^1 \omega_0((1-\theta)r_0+\theta r)d\theta < 1. \tag{3.54}$$

Set $\mathcal{D}_1 = \mathcal{D}\cap\bar{\mathcal{S}}(x_0,r)$.
Then $\exists$ a unique $d \in \mathcal{D}_1$ such that $\wp(d) = 0$.

Proof. Let $d_1 \in \mathcal{D}_1$ with $\wp(d_1) = 0$. A linear operator M is defined as

$$M = \int_0^1 \wp'(d+\theta(d_1-d))d\theta.$$

Applying (C2) and (3.54) yields

$$\begin{aligned}
\|\wp'(x_0)^{-1}(M-\wp'(x_0))\| &\leq \int_0^1 \omega_0((1-\theta)\|d-x_0\|+\theta\|d_1-x_0\|)d\theta \\
&\leq \int_0^1 \omega_0((1-\theta)r_0+\theta r)d\theta < 1,
\end{aligned}$$

so M is invertible. Thus $d_1 = d$, since $0 = \wp(d_1) - \wp(d) = M(d_1 - d)$. □

REMARK 3.5. *(a) We can replace (C5) by $\bar{S}(x_0, \rho) \subset \mathcal{D}$.*
(b) One can choose $d = \varrho^*$ and $z_0 = \lambda$, assuming the conditions (C1)-(C5).
Similarly, we analyse the semi-local convergence of the method (1.3).

Majorizing sequences $\{\mathcal{X}_n\}$, $\{\mathcal{Y}_n\}$, $\{\mathcal{Z}_n\}$ for the method (1.3) are given by

$$\begin{aligned}
\mathcal{Z}_n &= \mathcal{Y}_n + \frac{\bar{\omega}_n(\mathcal{Y}_n - \mathcal{X}_n)}{2|a|(1-s_n)},\\
p_n &= (1 + \int_0^1 \omega_0(\mathcal{X}_n + \theta(\mathcal{Z}_n - \mathcal{X}_n))d\theta)(\mathcal{Z}_n - \mathcal{X}_n)\\
&\quad + \frac{1}{|a|}(1 + \omega_0(\mathcal{X}_n))(\mathcal{Y}_n - \mathcal{X}_n),\\
\mathcal{X}_{n+1} &= \mathcal{Z}_n + \left(\frac{2}{1-s_n} + \frac{1}{1-\omega_0(\mathcal{X}_n)}\right)p_n,\\
\mathcal{Y}_{n+1} &= \mathcal{X}_{n+1} + \frac{|a|\delta_{n+1}}{1-\omega_0(\mathcal{X}_{n+1})}.
\end{aligned} \tag{3.55}$$

The convergence conditions corresponding to (3.42) are

$$\omega_0(\mathcal{X}_n) < 1, \omega_0(\mathcal{Y}_n) < 1, s_n < 1, \mathcal{X}_n \le \kappa \tag{3.56}$$

and replace (C4). The limit point need not be the same for both the methods. Yet for simplicity, we consider the same notation λ.

THEOREM 3.6. *Assume (C1)-(C5). Then, $\exists\ \varrho^* \in \bar{S}(x_0, \lambda)$ satisfying $\wp(\varrho^*) = 0$ under the method (1.3).*

Proof. The proof is similar to that of Theorem 3.3 with some necessary modifications. Observe that

$$\|z_k - y_k\| \le \frac{\bar{\omega}_k(\mathcal{Y}_k - \mathcal{X}_k)}{2|a|(1-s_k)} = \mathcal{Z}_k - \mathcal{Y}_k.$$

Notice that

$$\begin{aligned}
\wp(z_k) &= \wp(z_k) - \wp(x_k) + \wp(x_k)\\
&= \int_0^1 \wp'(x_k + \theta(z_k - x_k))d\theta(z_k - x_k) - \frac{1}{a}\wp'(x_k)(y_k - x_k).
\end{aligned}$$

Then, by (3.47)

$$\begin{aligned}
\|\wp'(x_0)^{-1}\wp(z_k)\| &\le \left(1 + \int_0^1 \omega_0(\mathcal{X}_k + \theta(\mathcal{Z}_k - \mathcal{X}_k))d\theta\right)(\mathcal{Z}_k - \mathcal{X}_k)\\
&\quad + \frac{1}{|a|}(1 + \omega_0(\mathcal{X}_k))(\mathcal{Y}_k - \mathcal{X}_k) = p_k.
\end{aligned}$$

So, by the third substep of the method (1.3)

$$\begin{aligned}
\|x_{k+1} - z_k\| &\le 4|a|\|\mathcal{A}_k^{-1}\wp(z_k)\| + \|\wp'(x_k)^{-1}\wp(z_k)\|\\
&\le (4|a|\|\mathcal{A}_k^{-1}\wp'(x_0)\| + \|\wp'(x_k)^{-1}\wp'(x_0)\|)\|\wp'(x_0)^{-1}\wp(z_k)\|\\
&\le \left(\frac{2}{1-s_k} + \frac{1}{1-\omega_0(\mathcal{X}_k)}\right)p_k = \mathcal{X}_{k+1} - \mathcal{Z}_k,\\
\|z_k - x_0\| &\le \|z_k - y_k\| + \|y_k - x_0\| \le \mathcal{Z}_k - \mathcal{Y}_k + \mathcal{Y}_k - \mathcal{X}_0 = \mathcal{Z}_k < \lambda
\end{aligned}$$

and

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - z_k\| + \|z_k - x_0\| \leq \mathcal{X}_{k+1} - \mathcal{Z}_k + \mathcal{Z}_k - \mathcal{X}_0 = \mathcal{X}_{k+1} < \lambda.$$

The rest is similar to Theorem 3.3. □

The uniqueness of the solution follows as in Proposition 3.4.

4. Numerical Examples

In this section, we analyse several examples to validate the parameters used in proving the theorems. Further, we observe the convergence of the discussed methods with Newton- Cotes method studied in [17] and a method by Xiao and Yin in [23] through an example.

EXAMPLE 4.1. *Let* $X = Y = C[0,1] = \{g : [0,1] \longrightarrow \mathbb{C} : \text{g is continuous}\}$, *with max norm. Let* $\mathcal{D} = \overline{\mathcal{S}}(0,1)$. *Let* $\wp : \mathcal{D} \subset X \longrightarrow Y$ *be given by*

$$\wp(\varphi)(x) = \varphi(x) - 5\int_0^1 x\tau\varphi(\tau)^3 d\tau. \tag{4.57}$$

Then,

$$\wp'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in \mathcal{D}.$$

Observe $\varrho^* = 0$ *and the conditions (A1)-(A4) hold, provided* $L_1 = 15$, $L_3 = 8.5$ *and* $L_4 = 31$. *The parameter values obtained are shown in Table 1.*

TABLE 1. Parameter values for Example 4.1

Method	r_1	ρ	ρ_1	r
$a = -2$	0.022876	0.015203	0.014416	0.014416
$a = -\frac{3}{2}$	0.021605	0.014819	0.013976	0.013976
(1.4) $a = -1$	0.019453	0.014077	0.013160	0.013160
(1.5) $a = \frac{1}{2}$	0.035168	0.018430	0.018004	0.018004
(1.6) $a = \frac{2}{3}$	0.040370	0.018813	0.018402	0.018402
(1.7) $a = 1$	0.050929	0.019213	0.018824	0.018824
$a = \frac{3}{2}$	0.039566	0.018249	0.017801	0.017801
$a = 2$	0.035727	0.017776	0.017305	0.017305

EXAMPLE 4.2. *Let* $X = Y = \mathbb{R}^3, \mathcal{D} = \bar{\mathcal{S}}(0,1)$ *equipped with max norm and* $\wp : \mathcal{D} \subset X \longrightarrow Y$ *defined for* $w = (w_1, w_2, w_3)$ *by*

$$\wp(w) = (sinw_1, \frac{w_2^2}{5} + w_2, w_3).$$

Then, $\varrho^* = (0,0,0)$. *Observe,*

$$\wp'(w) = \begin{bmatrix} cosw_1 & 0 & 0 \\ 0 & \frac{2w_2}{5} + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\wp''(w) = \left[\begin{array}{ccc|ccc|ccc} -sinw_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The condtions (A1)–(A4) are true if $L_1 = L_3 = 1$ *and* $L_4 = 0.84147$. *The parameter values obtained are shown in Table 2.*

TABLE 2. Parameter values for Example 4.2

Method	r_1	ρ	ρ_1	r
$a=-2$	0.343146	0.247367	0.234681	0.234681
$a=-\frac{3}{2}$	0.324081	0.243619	0.230275	0.230275
(1.4) $a=-1$	0.291796	0.232677	0.218821	0.218821
(1.5) $a=\frac{1}{2}$	0.527525	0.321750	0.323505	0.321750
(1.6) $a=\frac{2}{3}$	0.605551	0.327926	0.333544	0.327926
(1.7) $a=1$	0.763932	0.331341	0.341537	0.331341
$a=\frac{3}{2}$	0.593485	0.306562	0.312249	0.306562
$a=2$	0.535898	0.292498	0.295987	0.292498

EXAMPLE 4.3. *Let $X = Y = \mathbb{R}^3$, $\mathcal{D} = \bar{\mathcal{S}}(0,1)$. Define $\wp$ on $\mathcal{D}$ for $w = (w_1, w_2, w_3)$ by*

$$\wp(w) = \left(e^{w_1} - 1, \frac{e-1}{2}w_2^2 + w_2, w_3\right).$$

Then,

$$\wp'(w) = \begin{bmatrix} e^{w_1} & 0 & 0 \\ 0 & (e-1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\wp''(w) = \left[\begin{array}{ccc|ccc|ccc} e^{w_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (e-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Observe, $\varrho^* = (0,0,0)$. The conditions (A1)–(A4) are true if $L_1 = L_3 = L_4 = e$. The parameter values obtained are shown in Table 3.

TABLE 3. Parameter values for Example 4.3

Method	r_1	ρ	ρ_1	r
$a=-2$	0.126236	0.094984	0.090884	0.090884
$a=-\frac{3}{2}$	0.119223	0.092575	0.088283	0.088283
(1.4) $a=-1$	0.107346	0.087389	0.082923	0.082923
(1.5) $a=\frac{1}{2}$	0.194066	0.121753	0.120938	0.120938
(1.6) $a=\frac{2}{3}$	0.222770	0.125084	0.124993	0.124993
(1.7) $a=1$	0.281035	0.128116	0.128996	0.128116
$a=\frac{3}{2}$	0.218331	0.119267	0.119149	0.119149
$a=2$	0.197146	0.114599	0.114057	0.114057

REMARK 4.4. *Figure 1 displays the behaviour of the parameters r_1, ρ, ρ_1 and r with $a \neq 0$. In the plot, the parameter values for 60 different values of $a \in [-2,0) \cup (0,2]$ have been computed. It is observed that the parameter values are directly proportional to the absolute values of a, whenever $a \in [-1,0) \cup (0,1]$. A slight decrease in the radii is observed beyond $a = 1$, while no much variation in radii is seen beyond $a = -1$ towards the negative axis.*

EXAMPLE 4.5. *Let $X = Y = \mathbb{R}^2$. Consider the system of equations*

$$\begin{aligned} 3t_1^2 t_2 + t_2^2 &= 1 \\ t_1^4 + t_1 t_2^3 &= 1. \end{aligned} \tag{4.58}$$

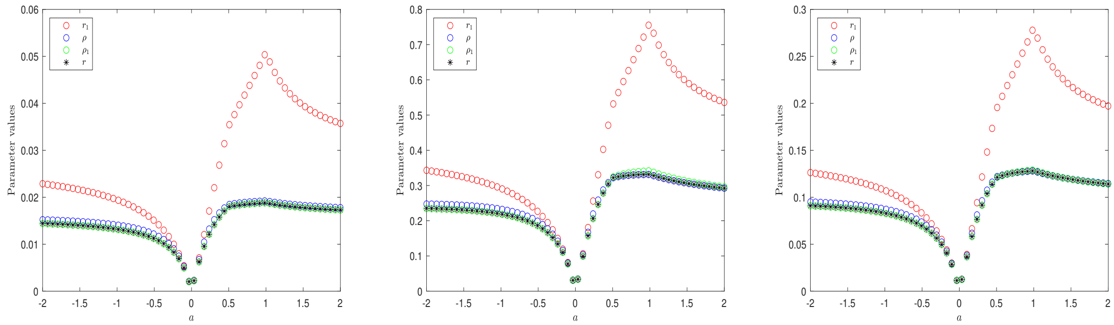

FIGURE 1. Variation of the parameters with $a \neq 0$ for Example 4.1-Example 4.3, respectively.

Observe, $x_1 = (0.9, 0.3)$ *is one of the solutions of (4.58). The approximation to this solution* x_1 *using two particular cases of the class of methods mentioned,* $a = -1, \frac{1}{2}, \frac{2}{3}, 1$ *(method (1.4, method (1.5), method (1.6), method (1.7) are studied (similarly follows for the other values a≠0), starting with* $x_0 = (2, -1)$. *The outcomes are shown in Table 4, 5 and 6.*

TABLE 4. Iterates using the Method (1.4), and the Method (1.7)

k	Method (1.4) $x_k = (t_1^k, t_2^k)$	Ratio $\frac{e_{x_{k+1}}}{(e_{x_k})^5}$	Method (1.7) $x_k = (t_1^k, t_2^k)$	Ratio $\frac{e_{x_{k+1}}}{(e_{x_k})^5}$
0	$(2.000000, -1.000000)$		$(2.000000, -1.000000)$	
1	$(1.113917, 0.121073)$	0.004043	$(1.154301, -0.007924)$	0.004735
2	$(0.993030, 0.306162)$	0.733683	$(0.994370, 0.304228)$	0.334158
3	$(0.992780, 0.306440)$	3.904849	$(0.992780, 0.306440)$	3.837488
4			$(0.992780, 0.306440)$	3.916553

TABLE 5. Iterates using the Method (1.5), and the method from [23]

k	Method (1.5) $x_k = (t_1^k, t_2^k)$	Ratio $\frac{e_{x_{k+1}}}{(e_{x_k})^5}$	Xiao and Yin's method [23] $x_k = (t_1^k, t_2^k)$	Ratio $\frac{e_{x_{k+1}}}{(e_{x_k})^5}$
0	$(2.000000, -1.000000)$		$(2.000000, -1.000000)$	
1	$(1.142149, 0.026558)$	0.004541	$(1.182085, -0.044631)$	0.004999
2	$(0.994022, 0.304907)$	0.411510	$(0.918554, 0.388269)$	0.225379
3	$(0.992780, 0.306440)$	3.856951	$(1.015845, 0.281451)$	7.490528
4	$(0.992780, 0.306440)$	3.916553	$(0.991633, 0.307778)$	2.949116
5			$(0.992778, 0.306443)$	3.971186
6			$(0.992780, 0.306440)$	3.916671
7			$(0.992780, 0.306440)$	3.916553

TABLE 6. Iterates using the Method (1.6) and the method from [17]

k	Method (1.6) $x_k = (t_1^k, t_2^k)$	Ratio $\frac{e_{x_{k+1}}}{(e_{x_k})^5}$	Newton Cotes method [17] $x_k = (t_1^k, t_2^k)$	Ratio $\frac{e_{x_{k+1}}}{(e_{x_k})^5}$
0	$(2.000000, -1.000000)$		$(2.000000, -1.000000)$	
1	$(1.145979, 0.014904)$	0.004605	$(1.263927, -0.166887)$	0.052792
2	$(0.994146, 0.304698)$	0.383700	(1.019452,0.265424)	0.259156
3	$(0.992780, 0.306440)$	3.850384	(0.992853,0.306348)	1.580144
4	$(0.992780, 0.306440)$	3.916553	(0.992780,0.306440)	1.977957
5			(0.992780,0.306440)	1.979028

REMARK 4.6. *Note that the columns corresponding to the Ratios in Table 4, Table 5 and Table 6, show that the method (1.4), method (1.5), method (1.6) and method (1.7) are of order* 5 *(by neglecting few initial iterates). Also, observe that method (1.4), method (1.5), method (1.6) and method (1.7) converge faster than the other two methods under consideration, even though of the same order, hence showing method (1.4) – method (1.7) to be efficient.*

Next, we define and compute the Approximate computational order of convergence of methods (1.4)– (1.7) for the Example 4.3 and Example 4.5.

DEFINITION 4.7. *The 'Approximate computational order of convergence' (ACOC)* [21] *is defined by*

$$ACOC = \frac{ln\left(\left\|\frac{x_{n+1}-x_n}{x_n-x_{n-1}}\right\|\right)}{ln\left(\left\|\frac{x_n-x_{n-1}}{x_{n-1}-x_{n-2}}\right\|\right)},$$

where x_{n-2}, x_{n-1}, x_n *and* x_{n+1} *are the consecutive iterates near the root.*

TABLE 7. ACOC of method (1.4)–method (1.7) for Example 4.3 and Example 4.5.

Example	Root	x_0	ACOC for method (1.4)	ACOC for method (1.5)	ACOC for method (1.6)	ACOC for method (1.7)
4.3	$(0, 0, 0)$	$(1.9, 0.5, 0.5)$	4.5995	4.1051	4.8288	4.8625
		$(1, 0.03, 0.03)$	5.8235	4.6622	4.3906	4.5331
4.5	$(0.9, 0.3)$	$(2, -0.05)$	4.2016	4.8295	4.8047	4.7514
		$(0.3, -0.05)$	5.7595	4.9737	4.8962	4.6328

REMARK 4.8. *The calculated value ACOC, does not estimate the theoretical order of convergence accurately when 'pathological behavior' of the iterative method (for instance, slow convergence at the beginning of the implemented iterative method, oscillating behavior of approximations, etc.) exists. Therefore, it is not regarded as an ideal tool for estimating the order in most of the cases.*

5. CONCLUSION

The local convergence of a class of fifth order iterative methods is discussed. Earlier works rely on Taylor's expansion and require higher order derivatives to exist, which is not the case in several real life problems. But, our work requires the operator to be just twice differentiable and the order of convergence is proved without Taylor expansion. Further, weaker assumptions are taken and the theory is in the more general setting of a Banach space. The semi-local convergence analysis has also been discussed. The theory developed is verified through examples.

ACKNOWLEDGMENTS

The work of S. George is supported by the Science and Engineering Research Board, Govt. of India under the Project Grant No.CRG/2021/004776. Manjusree Gopal would like to thank National Institute of Technology Karnataka, India, for the support.

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