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# On simple normal structure and best proximity points in reflexive Banach space

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ABSTRACT. We introduce the concept of simple normal structure (see Definition 2.3) for a pair of subsets in a normed space that is not proximal. Using this concept, we show that if $\mathcal{E}$ is a reflexive Banach space, $\mathcal{A}$ and $\mathcal{B}$ are two nonempty, convex, bounded and closed subsets of $\mathcal{E}$ having a simple normal structure, and $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ is a cyclic relatively nonexpansive map, then $\mathcal{T}^2$ admits a fixed point in $\mathcal{A}$. Moreover, if $\mathcal{T}$ satisfies a $\min$-$\max$ condition, then this fixed point of $\mathcal{T}^2$ is also a best proximity point for $\mathcal{T}$.
Using this concept, we obtain the same result for the best proximity point of a cyclic contraction map. We also provide an example of a reflexive Banach space that is strictly convex but not uniformly convex.

## 1. INTRODUCTION AND PRELIMINARY

Let $(\mathcal{E}, \|.\|)$ be a normed linear space. Consider two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{E}$. Recall that a mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ is said to be *relatively nonexpansive* if, $\|\mathcal{T}(x) - \mathcal{T}(y)\| \leqslant \|x - y\|$ for every pair $(x, y)$ of $\mathcal{A} \times \mathcal{B}$. Notice that a relatively nonexpansive mapping need not be continuous in general.

A self-mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called *cyclic* if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$. For such a mapping, a *best proximity point* is an element $p$ of $\mathcal{A}$ that solves the minimization problem :

$$\|p - \mathcal{T}(p)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}),$$

where $\operatorname{dist}(\mathcal{A}, \mathcal{B}) := \inf\{\|x - y\| : (x, y) \in \mathcal{A} \times \mathcal{B}\}$. Note that in this case, if $\mathcal{T}$ is relatively nonexpansive, then $q = \mathcal{T}(p) \in \mathcal{B}$ is also a proximity point of $\mathcal{T}$, since

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) \leqslant \|q - \mathcal{T}(q)\| \leqslant \|p - \mathcal{T}(p)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

The importance of the best proximity points stems in their ability to offer optimal solutions to the problem of the best approximation between two sets. Relevant references on best proximity points can be found in [9, 10, 16, 2, 3, 4, 19]. In [10], Eldred and Veeramani proved a theorem establishing the existence of a best proximity point for cyclic contractions within the setting of uniformly convex Banach spaces. In [19], Suzuki, Kikkawa, and Vetro introduced the concept of the property UC and extended Eldred and Veeramani's result to metric spaces with the UC property. Subsequent contributions by Espínola and Fernández-León [11], as well as Sintunavarat and Kumam [18], have continued to explore the existence of best proximity points for cyclic maps under various domain-related conditions. These works reflect the growing interest in this area in recent years.

For the reader's convenience, we provide some vocabulary and results. The pair $(\mathcal{A}, \mathcal{B})$ is said to be *proximal*, if for each $(x, y) \in \mathcal{A} \times \mathcal{B}$ there exists $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $\|x - b\| = \|y - a\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. It is said to have *proximal normal structure* (see [9]), if for any nonempty closed bounded convex proximal pair $(\mathcal{H}, \mathcal{K})$ in $(\mathcal{A}, \mathcal{B})$ with $\delta(\mathcal{H}, \mathcal{K}) > \operatorname{dist}(\mathcal{H}, \mathcal{K})$ and $\operatorname{dist}(\mathcal{H}, \mathcal{K}) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$, the inequality $\max\{\delta_p(\mathcal{K}), \delta_q(\mathcal{H})\} < \delta(\mathcal{H}, \mathcal{K})$

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holds for some $(p, q) \in (\mathcal{H}, \mathcal{K})$. Here $\delta(\mathcal{H}, \mathcal{K}) = \sup\{\|x - y\| : x \in \mathcal{H}, y \in \mathcal{K}\}$ and $\delta_x(\mathcal{K}) = \sup\{\|x - y\| : y \in \mathcal{K}\}$, for any $x \in \mathcal{H}$.

Recall that a Banach space $(\mathcal{E}, \|.\|)$ is said to be uniformly convex [6], if there exists a strictly increasing function $\zeta : (0, 2] \to (0, 1]$ (the modulus of uniform convexity) such that the following implication holds for all $x, y, p \in \mathcal{E}$, $R > 0$ and $r \in [0, 2R]$ :

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ \|x - y\| \geqslant r \end{cases} \Rightarrow \|\frac{x + y}{2} - p\| \leqslant (1 - \zeta(\frac{r}{R}))R.$$

It is said to be strictly convex if the following implication holds for all $x, y, p \in \mathcal{E}$ and $R > 0$ :

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \|\frac{x + y}{2} - p\| < R.$$

In the entire paper, we will say that $(\mathcal{A}, \mathcal{B})$ satisfies a specific property if both the sets $\mathcal{A}$ and $\mathcal{B}$ satisfy that property.

The study of the existence of a best proximity pair was initially introduced and explored in [9]. Specifically, the following key result was established :

**Theorem 1.1** (Corollary 2.1 in [9]). *If $(\mathcal{A}, \mathcal{B})$ is a nonempty closed bounded convex pair of a uniformly convex Banach space $\mathcal{E}$, then every cyclic relatively nonexpansive mapping defined on $\mathcal{A} \cup \mathcal{B}$ has a best proximity pair.*

The concepts of reflexivity and strict convexity are closely linked to the existence of projection onto a closed convex set.

**Definition 1.1.** *(see Definition 5.1.17 in* [15]*) Let $(\mathcal{E}, \|.\|)$ be a normed linear space and $\mathcal{A}$ be a nonempty subset of $\mathcal{E}$.*

- *$\mathcal{A}$ is a set of uniqueness if, for every $x$ of $\mathcal{E}$, there is no more than one element $y$ of $\mathcal{A}$ such that $\|x - y\| = \operatorname{dist}(x, \mathcal{A})$, where $\operatorname{dist}(x, \mathcal{A}) = \inf\{\|x - a\| : a \in \mathcal{A}\}$.*
- *$\mathcal{A}$ is a set of existence or proximinal if, for every $x$ of $\mathcal{E}$, there is at least one element $y$ of $\mathcal{A}$ such that $\|x - y\| = \operatorname{dist}(x, \mathcal{A})$.*
- *$\mathcal{A}$ is a Chebyshev set if, for every $x$ of $\mathcal{E}$, there is exactly one element $y$ of $\mathcal{A}$ such that $\|x - y\| = \operatorname{dist}(x, \mathcal{A})$; that is, if $\mathcal{A}$ is both a set of uniqueness and a set of existence.*

**Theorem 1.2** (Corollary 2.12 in [12]). *Let $(\mathcal{E}, \|.\|)$ be a Banach space. The following are equivalent:*

- *$\mathcal{E}$ is reflexive;*
- *every nonempty weakly closed subset of $\mathcal{E}$ is proximinal;*
- *every nonempty closed convex subset of $\mathcal{E}$ is proximinal;*
- *every closed subspace of $\mathcal{E}$ is proximinal.*

**Remark 1.1.** *It is shown (see Theorem 5.1.18, Corollary 5.1.19, and the accompanying comments in* [15]*) that a normed space $\mathcal{E}$ is reflexive and strictly convex if and only if every nonempty convex and closed subset of $\mathcal{E}$ is a Chebyshev set. Therefore, in this setting, the projection mapping $\mathcal{P}_{\mathcal{A}}$ on a nonempty closed convex subset $\mathcal{A}$ is well-defined and single-valued.*

In the following proposition, we establish the relationship between best proximity point and fixed point.

**Proposition 1.1.** *Let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ be a cyclic relatively nonexpansive, where $(\mathcal{A}, \mathcal{B})$ is a pair of nonempty subsets of a normed space $\mathcal{E}$. Suppose that $\mathcal{A}$ is a set of uniqueness and that $\mathcal{T}$ admits a best proximity point $x \in \mathcal{A}$. Then $x$ is also a fixed point for $\mathcal{T}^2$.*

*Proof.* Since $\mathcal{T}$ admits a best proximity point $x \in \mathcal{A}$, i.e. $\|x - \mathcal{T}(x)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$, we have $\|x - \mathcal{T}(x)\| = \operatorname{dist}(\mathcal{T}(x), \mathcal{A})$. As $\mathcal{A}$ is a set of uniqueness, $x$ is the projection of $\mathcal{T}(x)$ onto $\mathcal{A}$, we also have

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) \leqslant \|\mathcal{T}(x) - \mathcal{T}(\mathcal{T}(x))\| \leqslant \|x - \mathcal{T}(x)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}),$$

so $\|\mathcal{T}(\mathcal{T}(x)) - \mathcal{T}(x)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$ and $\mathcal{T}(\mathcal{T}(x))$ is also the projection of $\mathcal{T}(x)$ onto $\mathcal{A}$. Thus $\mathcal{T}^2(x) = x$. □

**Remark 1.2.** *The result of Proposition 1.1 holds true when the set $\mathcal{A}$ is nonempty, closed, and convex, and when $\mathcal{E}$ is reflexive and strictly convex. In particular, under the hypothesis of Theorem 1.1, $\mathcal{T}$ admits a best proximity point which is also a fixed point of $\mathcal{T}^2$.*

The notion of a *cyclic contraction map* was introduced in [10].

**Definition 1.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of a normed space. A cyclic mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is termed a cyclic contraction if there is $\kappa \in (0, 1)$ such that, for all $(x, y) \in \mathcal{A} \times \mathcal{B}$,*

$$\|\mathcal{T}(x) - \mathcal{T}(y)\| \leqslant \kappa\|x - y\| + (1 - \kappa)\operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

**Theorem 1.3** (Theorem 3.10. in [10]). *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ is a cyclic contraction. Then, there exists a unique best proximity point $x \in \mathcal{A}$. Further, for every $x_0 \in \mathcal{A}$ the sequence $(\mathcal{T}^{2n}(x_0))_{n \in \mathbb{N}}$ converges to $x$.*

**Theorem 1.4** (Theorem 11. in [1]). *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty closed and convex subsets of a reflexive and strictly convex Banach space. Suppose $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclic contraction and assume that $(\mathcal{A} - \mathcal{A}) \cap (\mathcal{B} - \mathcal{B}) = \{0\}$. Then, there exists a unique $x \in \mathcal{A}$ such that $\mathcal{T}^2(x) = x$ and $\|x - \mathcal{T}(x)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$.*

The rest of the paper is organized as follows :

The following section contains the main results of the paper. In Subsection 2.1, we introduce a new concept called “simple normal structure,” which is a natural generalization of the notion of “normal structure” introduced in [5] and studied by W. A. Kirk in [14], and distinct from the concept of “proximal normal structure” introduced in [10]. Remark 2.3 justifies that every pair of nonempty convex subsets of $\mathcal{E}$ having simple normal structure also has proximal normal structure. Theorem 2.5 shows that if $(\mathcal{A}, \mathcal{B})$ is a nonempty bounded convex closed pair having simple normal structure in a reflexive Banach space, then for all cyclic, relatively nonexpansive mappings $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$, the map $\mathcal{T}^2$ admits a fixed point in $\mathcal{A}$. Moreover, if $\mathcal{T}$ satisfies the $\min$-$\max$ condition, then this fixed point of $\mathcal{T}^2$ is also a best proximity point for $\mathcal{T}$.

In Subsection 2.2, we deal with the case of cyclic contraction mappings. Theorem 2.6 justifies that if $(\mathcal{A}, \mathcal{B})$ is a nonempty, bounded, and closed convex pair having simple normal structure in a reflexive and strictly convex Banach space, then any cyclic contraction $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ admits a unique best proximity point in $\mathcal{A}$. We thus obtain the existence of a best proximity point for a cyclic contraction in a reflexive and strictly convex Banach space, which provides a positive answer to Eldred and Veeramani’s question (see [10]). We end the article with an example of a cyclic contraction in a reflexive Banach space that is strictly convex but not uniformly convex.

## 2. MAIN RESULT

2.1. **Simple normal structure and best proximity points of a cyclic relatively nonexpansive map.** Let $(\mathcal{E}, \|.\|)$ be a reflexive Banach space. For any nonempty closed convex and bounded subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{E}$, we set

$$\delta_x(\mathcal{B}) = \sup\{\|x-y\| : y \in \mathcal{B}\},$$

$$\delta_{\mathcal{A}}(\mathcal{B}) = \inf\{\delta_x(\mathcal{B}) : x \in \mathcal{A}\} \text{ and}$$

$$\mathcal{A}_{\mathcal{B}} = \{x \in \mathcal{A} : \delta_x(\mathcal{B}) = \delta_{\mathcal{A}}(\mathcal{B})\}.$$

We define in the same manner $\delta_y(\mathcal{A})$, for every $y \in \mathcal{B}$, $\delta_{\mathcal{B}}(\mathcal{A})$ and $\mathcal{B}_{\mathcal{A}}$. We denote also

$$\delta(\mathcal{A}, \mathcal{B}) := \sup\{\|x-y\| : (x, y) \in \mathcal{A} \times \mathcal{B}\}.$$

**Definition 2.3.** *Let $\mathcal{A}, \mathcal{B}$ be nonempty subsets of $\mathcal{E}$. We say that $(\mathcal{A}, \mathcal{B})$ has a* **simple normal structure***, if for every nonempty closed bounded and convex pair $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{E}$ such that $\mathcal{X} \subset \mathcal{A}$ and $\mathcal{Y} \subset \mathcal{B}$, the following implications hold true :*

(i) *if $\mathcal{X}$ contains more than one point then there is some point $x \in \mathcal{X}$ such that : $\delta_x(\mathcal{Y}) < \delta(\mathcal{X}, \mathcal{Y})$;*

(ii) *if $\mathcal{Y}$ contains more than one point then there is some point $y \in \mathcal{Y}$ such that : $\delta_y(\mathcal{X}) < \delta(\mathcal{X}, \mathcal{Y})$.*

**Remark 2.3.** *In Definition 2.3, the pair $(\mathcal{X}, \mathcal{Y})$ is not necessarily proximal; therefore, a pair $(\mathcal{A}, \mathcal{B})$ can have proximal normal structure but not a simple normal structure. But the reciprocal is always true : if $(\mathcal{A}, \mathcal{B})$ is a convex pair, in a Banach space $\mathcal{E}$, having a simple normal structure, then $(\mathcal{A}, \mathcal{B})$ has a proximal normal structure.*

Indeed, suppose that $(\mathcal{A}, \mathcal{B})$ has a simple normal structure and let $(\mathcal{H}, \mathcal{K})$ be a nonempty closed bounded convex proximal pair in $(\mathcal{A}, \mathcal{B})$ with $\delta(\mathcal{H}, \mathcal{K}) > \operatorname{dist}(\mathcal{H}, \mathcal{K})$. It suffices to show that $\mathcal{H}$ and $\mathcal{K}$ are not singletons. Suppose $\mathcal{H} = \{h_0\}$ for some $h_0 \in \mathcal{E}$. Since $(\mathcal{H}, \mathcal{K})$ is proximal then, for all $k \in \mathcal{K}$, $\|k - h_0\| = \operatorname{dist}(\mathcal{H}, \mathcal{K})$, whence $\delta(\mathcal{H}, \mathcal{K}) = \delta_{h_0}(\mathcal{K}) = \operatorname{dist}(\mathcal{H}, \mathcal{K})$, which contradicts the hypothesis. Thus $\mathcal{H}$ is not a singleton. Similarly we show that $\mathcal{K}$ is not a singleton.

**Example 2.1.** *Let $\mathcal{E} = \mathbb{R}^2$ be equipped with the norm $\|.\|_\infty$, defined for every $(x, y) \in \mathbb{R}^2$ by $\|(x, y)\|_\infty = \max(|x|, |y|)$. Consider $a = (0, 0)$, $a' = (1, 0)$, $b = (2, 0)$, $b' = (3, 0)$, and let $\mathcal{A}$ be the line segment $[a, a']$ and $\mathcal{B} = [b, b']$. It is not difficult to show that $(\mathcal{A}, \mathcal{B})$ has a simple normal structure. Note that in this case, if $\mathcal{T}$ is a relatively nonexpansive cyclic mapping on $\mathcal{A} \cup \mathcal{B}$, then necessarily $\mathcal{T}(a') = b$ and $\mathcal{T}(b) = a'$. Hence, $\mathcal{T}^2(a') = a'$ and $\mathcal{T}^2(b) = b$. Therefore, $\mathcal{T}^2$ admits fixed points in $\mathcal{A}$ and $\mathcal{B}$, which are also proximity points.*

**Example 2.2.** *Let $\ell_n^n = (\ell_n, \|.\|_n)$, $n \in \mathbb{N} \setminus \{0\}$, where the Banach space $\ell_n$ is of the zero real sequences starting from the index $n+1$, endowed with the norm $\|.\|_n$ defined by*

$$\forall \;\; x(n) = (x(n)(i))_{k\geqslant 1} \in \ell_n, \;\; \|x(n)\|_n = \left(\sum_{i=1}^{+\infty} |x(n)(i)|^n\right)^{\frac{1}{n}}.$$

*Let $\mathcal{L}^2 = \{x = (x(n))_{n\geqslant 1} : (\forall n \in \mathbb{N} \setminus \{0\},\; x(n) \in \ell_n^n)$ and $\sum_{n=1}^{+\infty} \|x(n)\|_n^2 < +\infty\}$. The set $\mathcal{L}^2$ endowed with the norms $\|.\|$ defined by $\|x\| = (\sum_{i=1}^{+\infty} \|x(n)\|_n^2)^{\frac{1}{2}}$, is a Banach space. Since the $\ell_n^n$, $n \in \mathbb{N} \setminus \{0\}$, are finite-dimensional vector spaces endowed with the norms $\|.\|_n$, they are reflexive and strictly convex. Thus, by* [17] *and Theorem 2 of* [7]*, the space $(\mathcal{L}^2, \|.\|)$ is a strictly convex* *reflexive Banach space which is not uniformly convex.*

• *Let* $n \in \mathbb{N} \setminus \{0\}$. *We consider the four elements* $U_n = (x(k))_{k\geqslant 1}, V_{2n} = (v(k))_{k\geqslant 1}$, *and* $V_{2n+1} = (w(k))_{k\geqslant 1}$ *of* $\mathcal{L}^2$ *defined by:*

$$x(n)(i) = \begin{cases} \dfrac{1}{n^{\frac{1}{n}}} \text{ if } 1 \leqslant i \leqslant n, \\ 0 \text{ if } i > n \end{cases}, \text{ and } u(k) = 0 \text{ if } k \neq n,$$

$$v(2n)(i) = \begin{cases} \dfrac{1}{(2n)^{\frac{1}{2n}}} \text{ if } 1 \leqslant i \leqslant n, \\ \dfrac{-1}{(2n)^{\frac{1}{2n}}} \text{ if } n+1 \leqslant i \leqslant 2n, \\ 0 \text{ if } i > n \end{cases}, \text{ and } v(k) = 0 \text{ if } k \neq 2n,$$

$$w(2n+1)(i) = \begin{cases} \dfrac{1}{(2n+1)^{\frac{1}{2n+1}}} \text{ if } 1 \leqslant i \leqslant n+1, \\ \dfrac{-1}{(2n+1)^{\frac{1}{2n+2}}} \text{ if } n+2 \leqslant i \leqslant 2n+1, \\ 0 \text{ if } i > 2n+1 \end{cases} \text{ and } w(k) = 0 \text{ if } k \neq 2n+1.$$

*It is clear that* $\|U_n\| = \|U_{n+1}\| = \|V_{2n}\| = \|V_{2n+1}\| = 1$, *for all* $n \in \mathbb{N} \setminus \{0\}$. *Therefore the subsets* $\mathcal{A} = co(U_n, V_{2n})$ *and* $\mathcal{B} = co(U_{n+1}, V_{2n+1})$ *are closed convex of* $B_{\mathcal{L}^2}([0,1])$. *We have* $\mathcal{A} = \{\lambda.U_n + (1-\lambda).V_{2n} : \lambda \in [0,1]\}$ *and* $\mathcal{B} = \{\mu.U_{n+1} + (1-\mu).V_{2n+1} : \mu \in [0,1]\}$.

• $(\mathcal{A}, \mathcal{B})$ *have a simple normal structure. Indeed, let* $(\mathcal{X}, \mathcal{Y})$ *be a nonempty closed convex pair contained in* $(\mathcal{A}, \mathcal{B})$. *Then, there exist* $a, b, a', b' \in [0,1]$ *such that*

$$\mathcal{X} = \{\lambda.U_n + (1-\lambda).V_{2n} \in \mathcal{A} : \lambda \in [a,b]\}$$

$$\mathcal{Y} = \{\mu.U_{n+1} + (1-\mu).V_{2n+1} \in \mathcal{B} : \mu \in [a',b']\},$$

*with* $a \leqslant b$ *and* $a' \leqslant b'$.

*Suppose the set* $\mathcal{Y}$ *is not a singleton, so* $a' < b'$. *We put* $\rho(t) = t^2 + (1-t)^2$, *for all* $t \in [0,1]$.

**Case 1:** $a', b' \in [0, \frac{1}{2}]$, *so* $\rho(b') < \rho(a')$. *We have*

$$\begin{aligned} \delta_{b'.U_{n+1}+(1-b').V_{2n+1}}(\mathcal{X}) &= \sup_{\lambda\in[a,b]} \|(\lambda.U_n + (1-\lambda).V_{2n}) - (b'.U_{n+1} + (1-b').V_{2n+1})\| \\ &= \sup_{\lambda\in[a,b]} \sqrt{\lambda^2 + (1-\lambda)^2 + b'^2 + (1-b')^2} \\ &\leqslant \sqrt{\max\{\rho(a), \rho(b)\} + b'^2 + (1-b')^2} \\ &< \sqrt{\max\{\rho(a), \rho(b)\} + \max\{\rho(a'), \rho(b')\}}, \\ &\leqslant \sup_{(\lambda,\mu)\in[a,b]\times[a',b']} \sqrt{\lambda^2 + +(1-\lambda)^2 + \mu^2 + (1-\mu)^2} \\ &= \delta(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

**Case 2:** $a', b' \in ]\frac{1}{2}, 1]$*, so* $\rho(a') < \rho(b')$*. We have*

$$\begin{aligned}\delta_{a'.U_{n+1}+(1-a').V_{2n+1}}(\mathcal{X}) &= \sup_{\lambda\in[a,b]} \|(\lambda.U_n + (1-\lambda).V_{2n}) - (a'.U_{n+1} + (1-a').V_{2n+1})\| \\ &= \sup_{\lambda\in[a,b]} \sqrt{\lambda^2 + (1-\lambda)^2 + a'^2 + (1-a')^2} \\ &< \sqrt{\max\{\rho(a), \rho(b)\} + \max\{\rho(a'), \rho(b')\}}, \\ &\leqslant \delta(\mathcal{X}, \mathcal{Y}).\end{aligned}$$

**Case 3:** $0 \leqslant a' \leqslant \frac{1}{2} \leqslant b' \leqslant 1$*. We have*

$$\begin{aligned}\delta_{\frac{1}{2}.(U_{n+1}+V_{2n+1})}(\mathcal{X}) &= \sup_{\lambda\in[a,b]} \|(\lambda.U_n + (1-\lambda).V_{2n}) - \frac{1}{2}.(U_{n+1} + V_{2n+1})\| \\ &= \sup_{\lambda\in[a,b]} \sqrt{\lambda^2 + (1-\lambda)^2 + \frac{1}{2}} \\ &< \sqrt{\max\{\rho(a), \rho(b)\} + \max\{\rho(a'), \rho(b')\}}, \quad \textit{because } a' < b', \\ &\leqslant \delta(\mathcal{X}, \mathcal{Y}).\end{aligned}$$

*Similarly, since* $a < b$*, if the set* $\mathcal{X}$ *is not a singleton, then there exists* $\lambda_0 \in [0, 1]$ *such that* $\delta_{\lambda_0.U_n+(1-\lambda_0).V_{2n}}(\mathcal{Y}) < \delta(\mathcal{X}, \mathcal{Y})$*.*

In the following example, we give a compact convex pair $(\mathcal{A}, \mathcal{B})$ which has proximal normal structure but does not have a simple normal structure.

**Example 2.3.** *As in Example 2.1, we consider* $\mathcal{E} = \mathbb{R}^2$ *equipped with the norm* $\|.\|_\infty$*. Let* $a = (0, 0)$*,* $a' = (0, 1)$*,* $b = (2, 0)$*,* $b' = (2, 1)$*,* $\mathcal{A} = [a, a']$ *and* $\mathcal{B} = [b, b']$*. On the one hand, since for every* $(x, y) \in \mathcal{A} \times \mathcal{B}$*,* $\|x - y\|_\infty = 2$*, then* $(\mathcal{A}, \mathcal{B})$ *does not have a simple normal structure. On the other hand, as* $(\mathcal{A}, \mathcal{B})$ *is compact convex pair of a Banach space, according to Proposition 2.2. of* [9]*,* $(\mathcal{A}, \mathcal{B})$ *has proximal normal structure.*

Following almost the same steps of the proof of Proposition 2.1 in [9], without using the notion of proximal pair, we obtain the following result.

**Proposition 2.2.** *If* $(\mathcal{A}, \mathcal{B})$ *is a nonempty convex pair in a uniformly convex Banach space* $\mathcal{E}$*, then* $(\mathcal{A}, \mathcal{B})$ *has a simple normal structure.*

*Proof.* Let $(\mathcal{X}, \mathcal{Y})$ be a nonempty closed bounded and convex pair in $\mathcal{E}$ such that $\mathcal{X} \subset \mathcal{A}$ (resp. $\mathcal{Y} \subset \mathcal{B}$). Suppose that $\mathcal{X}$ is not a singleton. Then there exists $(x_1, x_2) \in \mathcal{X}^2$ such that $x_1 \neq x_2$. For $y \in \mathcal{Y}$, by the definition of $\delta(\mathcal{X}, \mathcal{Y})$, we have

$$\|x_1 - y\| \leqslant \delta(\mathcal{X}, \mathcal{Y}) \text{ and } \|x_2 - y\| \leqslant \delta(\mathcal{X}, \mathcal{Y}).$$

Put $\eta = 1 - \zeta \frac{\|x_1 - x_2\|}{\delta(\mathcal{X}, \mathcal{Y})}$ (for the function $\zeta$, see the definition of a uniformly convex space). By the convexity of $\mathcal{X}$, $x^* = \frac{x_1 + x_2}{2} \in \mathcal{X}$ and by the uniform convexity of $\mathcal{E}$, one has

$$\|x^* - y\| \leqslant \eta\, \delta(\mathcal{X}, \mathcal{Y}).$$

Hence, $\delta_{x^*}(\mathcal{Y}) \leqslant \eta\, \delta(\mathcal{X}, \mathcal{Y}) < \delta(\mathcal{X}, \mathcal{Y})$. Similarly, if $\mathcal{Y}$ is not a singleton there exists $y^* \in \mathcal{Y}$ such that $\delta_{y^*}(\mathcal{X}) < \delta(\mathcal{X}, \mathcal{Y})$. That is $(\mathcal{A}, \mathcal{B})$ has a simple normal structure. □

Example 2.3 shows that a compact convex pair in a Banach space does not necessarily have a simple normal structure. But it is the case in a strictly convex Banach space.

**Proposition 2.3.** *Let $(\mathcal{A}, \mathcal{B})$ be a nonempty compact convex pair in a Banach space $\mathcal{E}$. If $\mathcal{E}$ is strictly convex, then $(\mathcal{A}, \mathcal{B})$ has a simple normal structure.*

*Proof.* Let $(\mathcal{X}, \mathcal{Y})$ be any nonempty bounded closed and convex pair contained in $(\mathcal{A}, \mathcal{B})$. Suppose that $\mathcal{X}$ is not a singleton and let $x_1 \neq x_2$ be elements of $\mathcal{X}$. By the convexity of $\mathcal{X}$, $x_0 = \dfrac{x_1 + x_2}{2} \in \mathcal{X}$. On the one hand, since $\mathcal{Y}$ is compact, there is $y_0 \in \mathcal{Y}$ such that $\|x_0 - y_0\| = \delta_{x_0}(\mathcal{Y})$. On the other hand, $\|x_i - y_0\| \leq \delta(\mathcal{X}, \mathcal{Y})$ for $i = 1, 2$. Therefore, by the strict convexity of $\mathcal{E}$, $\delta_{x_0}(\mathcal{Y}) = \|x_0 - y_0\| < \delta(\mathcal{X}, \mathcal{Y})$. □

**Proposition 2.4.** *Let $(\mathcal{A}, \mathcal{B})$ be a nonempty weakly compact convex pair in a Banach space $\mathcal{E}$. Suppose $\delta(\mathcal{X}, \mathcal{Y}) > \max\{\delta(\mathcal{X}), \delta(\mathcal{Y})\} + \operatorname{dist}(\mathcal{X}, \mathcal{Y})$ for every nonempty bounded closed and convex pair $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X}$ or $\mathcal{Y}$ is not a singleton, and $\mathcal{X} \subset \mathcal{A}$ and $\mathcal{Y} \subset \mathcal{B}$. Then, $(\mathcal{A}, \mathcal{B})$ has a simple normal structure.*

*Proof.* Let $(\mathcal{X}, \mathcal{Y})$ be any nonempty bounded closed and convex pair contained in $(\mathcal{A}, \mathcal{B})$. Suppose $\mathcal{Y}$ is not a singleton. If we assume that $\delta_y(\mathcal{X}) = \delta(\mathcal{X}, \mathcal{Y})$, for each $y \in \mathcal{Y}$, then there exists a sequence $((x_n, y_n))_{n\geqslant 0}$ of $\mathcal{X} \times \mathcal{Y}$ such that $\lim\limits_{n\to+\infty} \|x_n - y_n\| = \operatorname{dist}(\mathcal{X}, \mathcal{Y})$. Since $(\mathcal{X}, \mathcal{Y})$ is a closed convex pair, it is therefore weakly closed, and by hypothesis $(\mathcal{A}, \mathcal{B})$ is weakly compact. Thus there exists a subsequence $((x_{\phi(n)}, y_{\phi(n)}))_{n\geqslant 0}$ of $((x_n, y_n))_{n\geqslant 0}$ which weakly converges to $(h, k) \in \mathcal{X} \times \mathcal{Y}$, and we have

$$\|h - k\| \leqslant \lim_{n\to+\infty} \|x_{\phi(n)} - y_{\phi(n)}\| = \operatorname{dist}(\mathcal{X}, \mathcal{Y}).$$

Hence, $\|h - k\| = \operatorname{dist}(\mathcal{X}, \mathcal{Y})$.
Let $m \in \mathbb{N}$. There exists a sequence $(z_n)_{n\geqslant 0}$ of $\mathcal{X}$ such that

$$\lim_{n\to+\infty} \|z_n - y_{\phi(m)}\| = \delta_{y_{\phi(m)}}(\mathcal{X}).$$

For each $n \in \mathbb{N}$, we have

$$\|z_n - y_{\phi(m)}\| \leqslant \|z_n - x_{\phi(m)}\| + \|x_{\phi(m)} - y_{\phi(m)}\| \leqslant \delta(\mathcal{X}) + \|x_{\phi(m)} - y_{\phi(m)}\|.$$

Taking the limit as $n \to +\infty$ in the previous inequality, we obtain

$$\delta(\mathcal{X}, \mathcal{Y}) = \delta_{y_{\phi(m)}}(\mathcal{X}) \leqslant \delta(\mathcal{X}) + \|x_{\phi(m)} - y_{\phi(m)}\|, \text{ for all } m \in \mathbb{N}. \tag{2.1}$$

Letting $m \to +\infty$ in the inequality (2.1), we get

$$\delta(\mathcal{X}, \mathcal{Y}) \leqslant \delta(\mathcal{X}) + \operatorname{dist}(\mathcal{X}, \mathcal{Y}) \leqslant \max\{\delta(\mathcal{X}), \delta(\mathcal{Y})\} + \operatorname{dist}(\mathcal{X}, \mathcal{Y}),$$

which contradicts hypothesis of the Proposition 2.4. Thus there exists $k_0 \in \mathcal{Y}$ such that $\delta_{k_0}(\mathcal{X}) < \delta(\mathcal{X}, \mathcal{Y})$.
Similarly, if $\mathcal{X}$ is a singleton, there exists $h_0 \in \mathcal{X}$ such that $\delta_{h_0}(\mathcal{Y}) < \delta(\mathcal{X}, \mathcal{Y})$. □

In order to establish our first result (Theorem 2.5 below), we need to prove two technical lemmas.

**Lemma 2.1.** *If $\mathcal{A}$ and $\mathcal{B}$ are nonempty closed bounded and convex subsets of a reflexive Banach space $\mathcal{E}$, then $\mathcal{A}_{\mathcal{B}}$ and $\mathcal{B}_{\mathcal{A}}$ are nonempty closed bounded and convex.*

*Proof.* For every $\varepsilon > 0$ and for every $y \in \mathcal{B}$, we set

$$\mathcal{C}(y, \varepsilon) = \left\{x \in \mathcal{A} : \|x - y\| \leqslant \delta_{\mathcal{A}}(\mathcal{B}) + \frac{1}{\varepsilon}\right\} \text{ and; } \mathcal{C}_\varepsilon = \bigcap_{y\in\mathcal{B}} \mathcal{C}(y, \varepsilon). \tag{2.2}$$

Let $\varepsilon > 0$. According to the characterization of the lower bound, for this $\varepsilon$, there exists $x \in \mathcal{A}$ such that $\delta_x(\mathcal{B}) \leqslant \delta_{\mathcal{A}}(\mathcal{B}) + \varepsilon$, whence, for every $y \in \mathcal{B}$,

$$\|x - y\| \leq \delta_x(\mathcal{B}) \leqslant \delta_{\mathcal{A}}(\mathcal{B}) + \varepsilon,$$

which shows that $x \in \mathcal{C}_\varepsilon$. Thus $\mathcal{C}_\varepsilon \neq \emptyset$.

$\mathcal{A}_\mathcal{B} = \bigcap_{\varepsilon>0} \mathcal{C}_\varepsilon$. Indeed, if $x \in \mathcal{A}_\mathcal{B}$, then for all $y \in \mathcal{B}$,

$$\|x - y\| \leqslant \delta_x(\mathcal{B}) = \delta_\mathcal{A}(\mathcal{B}) < \delta_\mathcal{A}(\mathcal{B}) + \frac{1}{\varepsilon}, \text{ for all } \varepsilon > 0.$$

So $x \in \bigcap_{y\in\mathcal{B}} \mathcal{C}(y, \varepsilon)$, for all $\varepsilon > 0$, i.e. $x \in \bigcap_{\varepsilon>0} \mathcal{C}_\varepsilon$.

Conversely, if $x \in \bigcap_{\varepsilon>0} \mathcal{C}_\varepsilon$, then for all $\varepsilon > 0$ and $y \in \mathcal{B}$, $\|x - y\| \leqslant \delta_\mathcal{A}(\mathcal{B}) + \frac{1}{\varepsilon}$. Therefore,

$$\delta_\mathcal{A}(\mathcal{B}) \leqslant \delta_x(\mathcal{B}) \leqslant \delta_\mathcal{A}(\mathcal{B}) + \frac{1}{\varepsilon}, \text{ for all } \varepsilon > 0.$$

By letting $\varepsilon$ tend to $+\infty$ in the inequality, we obtain $\delta_\mathcal{A}(\mathcal{B}) = \delta_x(\mathcal{B})$, which shows that $x \in \mathcal{A}_\mathcal{B}$.

For each $\varepsilon > 0$ and $y \in \mathcal{B}$, the set $\mathcal{C}(y, \varepsilon)$ is closed and convex, so $\mathcal{C}_\varepsilon$ and $\mathcal{A}_\mathcal{B}$ are too. As $\mathcal{A}$ is bounded, so is $\mathcal{A}_\mathcal{B}$.

The family $\{\mathcal{C}_\varepsilon : \varepsilon > 0\}$ is made up of decreasing (in the sense of inclusion), nonempty bounded closed and convex parts and as $\mathcal{E}$ is a reflexive Banach space, then

$$\mathcal{A}_\mathcal{B} = \bigcap_{\varepsilon>0} \mathcal{C}_\varepsilon \neq \emptyset.$$

which was to be shown. □

**Lemma 2.2.** *If $(\mathcal{A}, \mathcal{B})$ is nonempty closed bounded and convex subset of a reflexive Banach space $\mathcal{E}$ having a simple normal structure. Then, for every pair $(\mathcal{X}, \mathcal{Y})$ of nonempty closed bounded and convex pair in $(\mathcal{A}, \mathcal{B})$ we have the following implication :*

$$(\mathcal{X} \text{ and } \mathcal{Y} \text{ contains more than on point}) \Rightarrow \max\{\delta(\mathcal{Y}_\mathcal{X}, \mathcal{X}), \delta(\mathcal{X}_\mathcal{Y}, \mathcal{Y})\} < \delta(\mathcal{X}, \mathcal{Y}).$$

*Proof.* By the assumed simple normal structure, there is some $x \in \mathcal{X}$ such that $\delta_x(\mathcal{Y}) < \delta(\mathcal{X}, \mathcal{Y})$. For every $(a, y) \in \mathcal{X}_\mathcal{Y} \times \mathcal{Y}$,

$$\|a - y\| \leq \delta_a(\mathcal{Y}) = \delta_\mathcal{X}(\mathcal{Y}).$$

Therefore,

$$\delta(\mathcal{X}_\mathcal{Y}, \mathcal{Y}) \leqslant \delta_\mathcal{X}(\mathcal{Y}) \leqslant \delta_x(\mathcal{Y}) < \delta(\mathcal{X}, \mathcal{Y}).$$

Similarly we show that $\delta(\mathcal{Y}_\mathcal{X}, \mathcal{X}) < \delta(\mathcal{X}, \mathcal{Y})$. □

**Definition 2.4.** *Given a map $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$, we say that $\mathcal{T}$ satisfies the $min$-$max$ condition if, for all $(x, y) \in \mathcal{A} \times \mathcal{B}$, we have*

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) < d(x, y) \ \Rightarrow \max(\mathcal{T}x, \mathcal{T}y) \neq \min(\mathcal{T}x, \mathcal{T}y),$$

*where* $\min(\mathcal{T}x, \mathcal{T}y)$ *and* $\max(\mathcal{T}x, \mathcal{T}y)$ *are defined as*

$$\min(\mathcal{T}x, \mathcal{T}y) = \min\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}^2y), d(y, \mathcal{T}^2x), d(\mathcal{T}x, \mathcal{T}^2x), d(\mathcal{T}y, \mathcal{T}^2y), d(\mathcal{T}^2x, \mathcal{T}^2y)\}$$

$$\max(\mathcal{T}x, \mathcal{T}y) = \min\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}^2y), d(y, \mathcal{T}^2x), d(\mathcal{T}x, \mathcal{T}^2x), d(\mathcal{T}y, \mathcal{T}^2y), d(\mathcal{T}^2x, \mathcal{T}^2y)\}.$$

*Here, for all $u, v \in \mathcal{E}$, $d(u, v)$ denotes the real $\|u - v\|$.*

**Theorem 2.5.** *Let $\mathcal{E}$ be a reflexive Banach space, and let $\mathcal{A}$, $\mathcal{B}$ be two nonempty, convex, bounded, and closed subsets of $\mathcal{E}$ having a simple normal structure. Let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ a cyclic relatively nonexpansive. Then, there exists $(x^*, y^*) \in \mathcal{A} \times \mathcal{B}$ such that*

$$\mathcal{T}x^* = y^* \text{ and } \mathcal{T}y^* = x^* \tag{2.3}$$

$$\text{and so } \mathcal{T}^2x^* = x^* \text{ and } \mathcal{T}^2y^* = y^*.$$

*If furthermore, $\mathcal{T}$ satisfies the min-max condition, then*

$$\operatorname{dist}(x^*, \mathcal{T}x^*) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) = \operatorname{dist}(y^*, \mathcal{T}y^*). \tag{2.4}$$

*Proof.* Let $\mathcal{F}$ denote the collection of all the pairs $(X, Y)$ such that $X$ and $Y$ are nonempty bounded closed and convex subsets of $\mathcal{E}$ verifying $X \subset \mathcal{A}, Y \subset \mathcal{B}, \mathcal{T}(X) \subset Y$ and $\mathcal{T}(Y) \subset X$. Obviously, $\mathcal{F}$ is nonempty since $(\mathcal{A}, \mathcal{B}) \in \mathcal{F}$. The set $\mathcal{F}$ is equipped with the order relation '$\preceq$' defined by for all $(X, Y), (X', Y') \in \mathcal{F}$,

$$(X, Y) \preceq (X', Y') \iff (X \subseteq X', Y \subseteq Y').$$

Let $(X_\alpha, Y_\alpha)_{\alpha\in\Gamma}$ be a decreasing chain in $\mathcal{F}$. Define $(\mathcal{L}, \mathcal{K})$ by

$$\mathcal{L} = \bigcap_{\alpha\in\Gamma} X_\alpha \text{ and } \mathcal{K} = \bigcap_{\alpha\in\Gamma} Y_\alpha.$$

It is clear that $(\mathcal{L}, \mathcal{K}) \neq \emptyset$, since for all $\alpha \in \Gamma$, $X_\alpha$ and $Y_\alpha$ are nonempty bounded closed and convex subsets of reflexive space $\mathcal{E}$. Moreover, $\mathcal{T}(\mathcal{L}) \subset \mathcal{K}$, $\mathcal{T}(\mathcal{K}) \subset \mathcal{L}$ and $(\mathcal{L}, \mathcal{K}) \subseteq (\mathcal{A}, \mathcal{B})$. Thus $(\mathcal{L}, \mathcal{K}) \in \mathcal{F}$. Hence, Zorn's lemma implies that $\mathcal{F}$ has a minimal element, which we denote by $(U, V)$.

Let $x \in U_V$. For all $y \in V$, $\|\mathcal{T}x - \mathcal{T}y\| \leqslant \|x - y\| \leqslant \delta_U(V)$. Then

$$\mathcal{T}(V) \subset U \cap \mathcal{B}(\mathcal{T}x, \delta_U(V)),$$

Similarly, let $y' \in V_U$, for all $x' \in U$, $\|\mathcal{T}x' - \mathcal{T}y'\| \leqslant \|x' - y'\| \leq \delta_V(U)$. Then

$$\mathcal{T}(U) \subset V \cap \mathcal{B}(\mathcal{T}y', \delta_V(U)),$$

where $\mathcal{B}(\mathcal{T}x, \delta_U(V))$ (resp. $\mathcal{B}(\mathcal{T}y', \delta_V(U))$) is the closed ball with radius $\delta_U(V)$ centered at $\mathcal{T}x$ (resp. radius $\delta_V(U)$ centered at $\mathcal{T}y'$). We have $U \cap \mathcal{B}(\mathcal{T}x, \delta_U(V))$ and $V \cap \mathcal{B}(\mathcal{T}y', \delta_V(U))$ are nonempty bounded closed and convex subsets of $\mathcal{A}$ and $\mathcal{B}$ resp., and

$$\mathcal{T}(V \cap \mathcal{B}(\mathcal{T}y', \delta_V(U))) \subset U \cap \mathcal{B}(\mathcal{T}x, \delta_U(V))$$

$$\mathcal{T}(U \cap \mathcal{B}(\mathcal{T}x, \delta_U(V))) \subset V \cap \mathcal{B}(\mathcal{T}y', \delta_V(U)).$$

By minimality of $(U, V)$, we have

$$U = U \cap \mathcal{B}(\mathcal{T}x, \delta_U(V)) \text{ and } V = V \cap \mathcal{B}(\mathcal{T}y', \delta_V(U))).$$

So

$$U \subseteq \mathcal{B}(\mathcal{T}x, \delta_U(V)) \text{ and } V \subseteq \mathcal{B}(\mathcal{T}y', \delta_V(U))).$$

Hence, $\delta_{\mathcal{T}x}(U) \leqslant \delta_U(V)$ and $\delta_{\mathcal{T}y'}(V) \leqslant \delta_V(U)$. Therefore $\delta_V(U) \leqslant \delta_U(V)$ and $\delta_U(V) \leqslant \delta_V(U)$. This give $\delta_U(V) = \delta_V(U)$, which implies that, for all $z \in U$, $\|\mathcal{T}x - z\| \leqslant \delta_U(V) = \delta_V(U)$, whence $\delta_{\mathcal{T}x}(V) = \delta_V(U)$ and $\mathcal{T}x \in V_U$. Therefore $\mathcal{T}U_V \subset V_U$. Similarly one has $\mathcal{T}V_U \subset U_V$.

By Lemma 2.1, $(U_V, V_U) \in \mathcal{F}$ and since $U_V \subseteq U$, $V_U \subseteq V$ and $\mathcal{F}$ has a minimal element $(U, V)$, then $(U_V, V_U) = (U, V)$. Consequently $\delta(U_V, V) = \delta(U, V)$, so

$$\max\{\delta(U_V, V), \delta(V_U, U)\} = \delta(U, V).$$

Hence by Lemma 2.2, $U$ or $V$ is a singleton. If for example $U$ is a singleton, there exists $x^* \in \mathcal{A}$ such that $U = \{x^*\}$. The following inclusions $\mathcal{T}(V) \subseteq U$ give us $\mathcal{T}^2x^* = x^*$. Setting $y^* = \mathcal{T}x^*$, we obtain $\mathcal{T}y^* = x^*$ and $\mathcal{T}^2y^* = y^*$. Similarly, if we consider $V$ a singleton.

As a result, $\max(\mathcal{T}x^*, \mathcal{T}y^*) = d(x^*, y^*) = \min(\mathcal{T}x^*, \mathcal{T}y^*)$.

Since, $\mathcal{T}$ satisfies the *min-max* condition, we obtain that $d(x^*, y^*) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. It follows that

$$d(x^*, \mathcal{T}x^*) = \operatorname{dist}(\mathcal{A}, \mathcal{B}) = d(y^*, \mathcal{T}y^*),$$

which completes the proof. □

**Example 2.4.** *Consider the real space $\mathbb{R}^m$, $m \geq 2$, endowed with the norm $\|.\|_1$ defined by*

$$\forall x = (x_1, \cdots, x_m) \in \mathbb{R}^m, \ \|x\|_1 = \sum_{k=1}^{m} |x_k|.$$

*Let*

$$\mathcal{A} = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : 1 \leqslant x_1 \leqslant 2 \text{ and } x_2 = \cdots = x_m = 1\};$$
$$\mathcal{B} = \{y = (y_1, \cdots, y_m) \in \mathbb{R}^m : -2 \leqslant y_1 \leqslant -1 \text{ and } y_2 = \cdots = y_m = -1\}.$$

*The space $\mathbb{R}^m$ equipped with the norm $\|.\|_1$, is reflexive and not strictly convex. Then the sets $\mathcal{A}$ and $\mathcal{B}$ are convex, nonempty, closed and the pair $(\mathcal{A}, \mathcal{B})$ has a simple normal structure. Moreover,* $\operatorname{dist}(\mathcal{A}, \mathcal{B}) = 2m$. *Consider the mapping $\mathcal{T} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B}$ defined by*

$$\text{for } x = (x_1, 1, \cdots, 1) \in \mathcal{A}, \ \mathcal{T}(x) = (-\sqrt{x_1}, -1, \cdots, -1)$$
$$\text{for } y = (y_1, -1, \cdots, -1) \in \mathcal{B}, \ \mathcal{T}(y) = (\sqrt{-y_1}, 1, \cdots, 1).$$

*We have $\mathcal{T}^2(x) = (x_1^{\frac{1}{4}}, 1, \cdots, 1)$ and $\mathcal{T}^2(y) = (-(-y_1)^{\frac{1}{4}}, -1, \cdots, -1)$. Moreover, for all $(x, y) \in \mathcal{A} \times \mathcal{B}$,*

$$\begin{aligned}\|\mathcal{T}(x) - \mathcal{T}(y)|_1 &= \|(-\sqrt{x_1}, -1, \cdots, -1) - (\sqrt{-y_1}, 1, \cdots, 1)\|_1 = |\sqrt{x_1} + \sqrt{-y_1}| + 2(m-1)\\ &\leqslant (x_1 - y_1) + 2(m-1)\\ &= \|x - y\|_1.\end{aligned}$$

*Furthermore, the mapping $\mathcal{T}$ satisfies the min-max condition. By Theorem 2.5, there exists $(x^*, y^*) \in \mathcal{A} \times \mathcal{B}$ such that $x^*$ and $y^*$ are best proximity points, and*

$$\mathcal{T}^2 x^* = x^* \text{ and } \mathcal{T}^2 y^* = y^*,$$

*with $x^* = (1, 1, \cdots, 1)$, $y^* = (-1, -1, \cdots, -1)$.*

The notion of "simple normal structure" is a natural extension of the concept of "normal structure" introduced by Milman and Brodskii [5] and studied by W. A. Kirk in [14]. Using Theorem 2.5, we can prove (Corollary 2.1 below) the well-known Kirk's fixed point theorem [14].

**Corollary 2.1.** *Let $\mathcal{E}$ be a reflexive Banach space, and let $\mathcal{A}$ be a nonempty, convex, bounded, and closed subset of $\mathcal{E}$ such that $(A, A)$ has a simple normal structure. If a mapping $\mathcal{R} : A \longrightarrow A$ is such that*

(2.5) $$\forall (x, y) \in \mathcal{A} \times \mathcal{A} : \|\mathcal{R}x - \mathcal{R}y\| \leqslant \|x - y\|,$$

*then there exists $a \in \mathcal{A}$ such that $\mathcal{R}a = a$.*

*Proof.* Taking $\mathcal{A} = \mathcal{B}$ and $\mathcal{T} = \frac{1}{2}(\mathcal{R} + \mathrm{Id}_{\mathcal{A}})$ in Theorem 2.5, one can deduce that there is $a \in A$ such that $\mathcal{T}^2 a = a$. To prove the result, note that

$$\mathcal{T}a - a = \mathcal{R}a - \mathcal{T}a = \frac{1}{2}(\mathcal{R}a - a).$$

Since $a = \mathcal{T}^2 a$, we hence get

$$\mathcal{R}a - a = \mathcal{R}a - \mathcal{T}^2 a = \mathcal{R}a - \frac{1}{2}(\mathcal{R}\mathcal{T}a + \mathcal{T}a) = \frac{1}{2}(\mathcal{R}a - \mathcal{R}\mathcal{T}a) + \frac{1}{2}(\mathcal{R}a - \mathcal{T}a)$$

and, consequently,

$$\|\mathcal{R}a - a\| \leqslant \frac{1}{2}\|a - \mathcal{T}a\| + \frac{1}{4}\|\mathcal{R}a - a\| = \frac{1}{2}\|\mathcal{R}a - a\|,$$

which implies $\mathcal{R}a = a$. □

2.2. **Simple normal structure and best proximity points of a cyclic contraction map.** In the following we present some results on cyclic contractions.

**Lemma 2.3.** *Let $(\mathcal{A}, \mathcal{B})$ be a pair of nonempty subset of a normed space $\mathcal{E}$. If $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ is a cyclic contraction and $x \in \mathcal{A}$ is a fixed point of $\mathcal{T}^2$, then for every $x_0 \in \mathcal{A}$, the sequence $\left(\|\mathcal{T}^{2n}(x_0) - \mathcal{T}(x)\|\right)_{n\in\mathbb{N}}$ converges to* $\operatorname{dist}(\mathcal{A}, \mathcal{B})$.

*Proof.* Let $\kappa \in (0, 1)$. Suppose

$$\forall (x, y) \in \mathcal{A} \times \mathcal{B},\ \|\mathcal{T}(x) - \mathcal{T}(y)\| \leqslant \kappa\|x - y\| + (1 - \kappa) \operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

By induction we can show that, for every $n \in \mathbb{N}$,

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) \leqslant \|\mathcal{T}^{2n}(x_0) - \mathcal{T}(x)\| \leqslant \kappa^{2n}\|x_0 - \mathcal{T}(x)\| + (1 - \kappa^{2n}) \operatorname{dist}(\mathcal{A}, \mathcal{B}).$$

The result follows by tending $n$ to infinity. □

**Theorem 2.6.** *Let $\mathcal{E}$ be a reflexive Banach space, and let $(\mathcal{A}, \mathcal{B})$ be a pair of nonempty, convex, bounded, and closed subsets of $\mathcal{E}$ with a simple normal structure. Let $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$ be a cyclic contraction mapping. Then $\mathcal{T}^2$ admits at least one fixed point $x^* \in \mathcal{A}$, which is also a best proximity point for $\mathcal{T}$ in $\mathcal{A}$.*
*If moreover $\mathcal{E}$ is strictly convex space, then*

- *(i) $x^*$ is the unique best proximity point of $\mathcal{T}$;*
- *(ii) for every $x_0 \in \mathcal{A}$, the sequence $(\mathcal{T}^{2n}(x_0))_{n\in\mathbb{N}}$ converges weakly to $x^*$.*

*Proof.* As $\mathcal{T}$ is a cyclic contraction, $\mathcal{T}$ is cyclic relatively nonexpansive. $\mathcal{T}$ satisfies the condition min-max. Indeed, let $(x, y) \in \mathcal{A} \times \mathcal{B}$ such that $\operatorname{dist}(\mathcal{A}, \mathcal{B}) < \|x - y\|$. Since $\mathcal{T}$ is a cyclic contraction with $\kappa \in (0, 1)$, then

$$\|\mathcal{T}x - \mathcal{T}y\| \leqslant \kappa\|x - y\| + (1 - \kappa)\operatorname{dist}(\mathcal{A}, \mathcal{B}) < \|x - y\|,$$

therefore $\|\mathcal{T}x - \mathcal{T}y\| < \|x - y\|$, thus $\min(\mathcal{T}x, \mathcal{T}y) \neq \max(\mathcal{T}x, \mathcal{T}y)$.
According to Theorem 2.5, $\mathcal{T}$ admits at least one best proximity point $x^* \in \mathcal{A}$, which is also a fixed point of $\mathcal{T}^2$ in $\mathcal{A}$.
We assume that $\mathcal{E}$ is strictly convex.

(i) Suppose there exist $x_1, x_2 \in \mathcal{A}$, two best proximity points of $\mathcal{T}$ such that, $\mathcal{T}^2(x_1) = x_1$ and $\mathcal{T}^2(x_2) = x_2$. Taking $x_0 = x_2$ in Lemma 2.3, we get

$$\lim_{n\to+\infty} \|\mathcal{T}^{2n}(x_0) - \mathcal{T}(x_1)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}),$$

whence

$$\|x_2 - \mathcal{T}(x_1)\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}) = \|x_1 - \mathcal{T}(x_1)\|.$$

Moreover, $\operatorname{dist}(\mathcal{A}, \mathcal{B}) \leqslant \operatorname{dist}(\mathcal{T}(x_1), \mathcal{A}) \leqslant \|\mathcal{T}(x_1) - x_1\| = \operatorname{dist}(\mathcal{A}, \mathcal{B})$. So $\operatorname{dist}(\mathcal{T}(x), \mathcal{A}) = \operatorname{dist}(\mathcal{A}, \mathcal{B})$, and $x_1, x_2 \in \mathcal{P}_{\mathcal{A}}(\mathcal{T}(x_1))$.

Since $\mathcal{A}$ is nonempty closed and convex subset of reflexive strictly convex Banach space $\mathcal{E}$, by Remark 1.1, $\mathcal{A}$ is a Chebyshev set. Hence, $x_1 = x_2$.

(ii) Let $x_0 \in \mathcal{A}$ and define the sequence $(x_n)_{n\in\mathbb{N}}$ by, $x_{n+1} = \mathcal{T}(x_n)$. According to the Lemma 2.3, the sequence $(\|x_{2n} - \mathcal{T}(x)\|)_{n\geqslant 0}$ converges to $\operatorname{dist}(\mathcal{A}, \mathcal{B})$. Since the sequence $(x_{2n})_{n\in\mathbb{N}}$ is in the closed, bounded and convex subset $\mathcal{A}$ of the reflexive Banach space

$\mathcal{E}$, so, by the Banach-Alaoglu theorem, it admits a subsequence $(x_{2\varphi(n)})_{n\in\mathbb{N}}$ converging weakly to an element of $\mathcal{A}$.
Now let us show that $(x_{2n})_{n\geqslant 0}$ converges weakly to $x^*$. Indeed, let $(x_{2\psi(n)})_{n\geqslant 0}$ be a subsequence of $(x_{2n})_{n\geqslant 0}$. Then, there exists a subsequence $(x_{2(\psi\circ\sigma)(n)})_{n\geqslant 0}$ of $(x_{2\psi(n)})_{n\geqslant 0}$ which converges weakly to an element $a \in \mathcal{A}$. We have

$$\begin{aligned}\operatorname{dist}(\mathcal{A},\mathcal{B}) &\leqslant \|a-\mathcal{T}(x^*)\|\\ &\leqslant \liminf_{n\to+\infty}\|x_{2(\psi\circ\sigma)(n)}-\mathcal{T}(x^*)\|\\ &= \operatorname{dist}(\mathcal{A},\mathcal{B}),\end{aligned}$$

which implies that $a = x^*$, and this is true for any subsequence $(x_{2\psi(n)})_{n\geqslant 0}$ of $(x_{2n})_{n\geqslant 0}$. Thus the sequence $(x_{2n})_{n\geqslant 0}$ converges weakly to $x^*$. □

**Example 2.5.** *In the previous Example 2.2, $\mathcal{A}$ and $\mathcal{B}$ are nonempty convex bounded closed subsets of $\mathfrak{L}^2$, and $(\mathcal{A},\mathcal{B})$ have a simple normal structure. Since $\|U_n\| = \|V_{2n}\| = \|U_{n+1}\| = \|V_{2n+1}\| = 1$, for all $n \in \mathbb{N}\setminus\{0\}$,*

$$\begin{aligned}\operatorname{dist}(\mathcal{A},\mathcal{B}) &= \inf_{(u,v)\in\mathcal{A}\times\mathcal{B}}\|u-v\|\\ &= \inf_{(\lambda,\mu)\in[0,1]^2}\|(\lambda.U_n+(1-\lambda).V_{2n})-(\mu.U_{n+1}+(1-\mu).V_{2n+1})\|\\ &= \inf_{(\lambda,\mu)\in[0,1]^2}\sqrt{\lambda^2+(1-\lambda)^2+\mu^2+(1-\mu)^2}=1,\end{aligned}$$

*Define $\mathcal{T}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ by :*

$$\begin{cases}\text{if } x=\lambda.U_n+(1-\lambda).V_{2n}\in\mathcal{A},\ \text{with } \lambda\in[0,1],\ \mathcal{T}(x)=\dfrac{1}{2}(U_n+V_{2n}),\\ \text{if } y=\mu.U_{n+1}+(1-\mu).V_{2n+1}\in\mathcal{B},\ \text{with } \mu\in[0,1],\ \mathcal{T}(y)=\dfrac{1}{2}.(U_{n+1}+V_{2n+1}).\end{cases}$$

*We have*

$$\|x-y\|=\sqrt{\lambda^2+(1-\lambda)^2+\mu^2+(1-\mu)^2}\geqslant 1=\operatorname{dist}(\mathcal{A},\mathcal{B}).$$

*So*

$$\begin{aligned}\|\mathcal{T}(x)-\mathcal{T}(y)\| &= \|\frac{1}{2}(U_n+V_{2n})-\frac{1}{2}.(U_{n+1}+V_{2n+1})\|=1\\ &= \operatorname{dist}(\mathcal{A},\mathcal{B})=\lambda\operatorname{dist}(\mathcal{A},\mathcal{B})+(1-\lambda)\operatorname{dist}(\mathcal{A},\mathcal{B})\\ &\leqslant \kappa\,\|x-y\|+(1-\kappa)\operatorname{dist}(\mathcal{A},\mathcal{B}),\ \text{where } \kappa \text{ an element of } [0,1].\end{aligned}$$

*Hence, $\mathcal{T}$ is a cyclic contraction. By Theorem 2.6, $\mathcal{T}$ admits a unique best proximity point $x^* = \frac{1}{2}(U_n+V_{2n}) \in \mathcal{A}$.*

**Remark 2.4.** *To summarize, consider a nonempty closed and convex pair $(\mathcal{A},\mathcal{B})$ in a reflexive and strictly convex normed space. Then, each of the hypotheses below implies the existence of a unique $x\in\mathcal{A}\cup\mathcal{B}$ such that $\mathcal{T}^2x=x$ and $\|x-\mathcal{T}x\|=\operatorname{dist}(\mathcal{A},\mathcal{B})$ :*

1. *$\mathcal{T}$ is weakly continuous on $\mathcal{A}$ or $\mathcal{B}$* [1, Theorem 12];
2. *For every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{A}\cup\mathcal{B}$ that converges weakly to an $x\in\mathcal{A}\cup\mathcal{B}$ and verifying $\lim_{n\to\infty}\|x_n-Tx_n\|=\operatorname{dist}(\mathcal{A},\mathcal{B})$, one has $\|x-\mathcal{T}x\|=\operatorname{dist}(\mathcal{A},\mathcal{B})$* [1, Theorem 12];
3. $(\mathcal{A}-\mathcal{A})\cap(\mathcal{B}-\mathcal{B})=\{0\}$ [1, Theorem 11];
4. *$(\mathcal{A},\mathcal{B})$ is bounded and has a simple normal structure (Theorem 2.6).*

*The condition "$(\mathcal{A},\mathcal{B})$ has a simple normal structure" does not generally imply the condition "$(\mathcal{A}-\mathcal{A})\cap(\mathcal{B}-\mathcal{B})=\{0\}$", as example 2.4 shows.*
*It is remarkable to note that in Theorem 2.6, the strict convexity of the reflexive Banach space $\mathcal{E}$ is only used to study the uniqueness of a best proximity point for the map $\mathcal{T}$.*

*Finally, we note that in the two main theorems 2.5 and 2.6, a delimitation of the domain sets $\mathcal{A}$ and $\mathcal{B}$ is assumed. This assumption is closely related to observations made by Zhelinski and Zlatanov* [20]*, where a bounded UC property for the ordered pair* $(\mathcal{A}, \mathcal{B})$ *is introduced.*

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