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Global-fixed-point property of gyrogroup actions

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ABSTRACT. The notion of a gyrogroup action generalizes that of a group action. This work, inspired by the Extended Cartan Fixed Point Theorem, is devoted to a fixed-point property of gyrogroup actions. In the case when a gyrogroup G acts on a non-empty set X, X is said to have the global-fixed-point property if there exists an element x in X such that $a \cdot x = x$ for all $a \in G$. In this paper, several conditions for X to have the global-fixed-point property are determined. A few examples regarding the results are also discussed.

1. INTRODUCTION

A fixed-point property may be appropriately used in the investigation of the structure of a mathematical object such as groups, vector spaces, topological spaces, graphs, functions, and so on. For instance, the proof of Sylow's Theorems for finite groups in a modern approach makes use of a fixed-point property. More precisely, the proof that two Sylow subgroups P and Q of a finite group G are conjugate in G (known as the Second Sylow Theorem; see, for example, Section 4.5 of [2]) can be done by considering the action of Q on $G/P = \{gP : g \in P\}$ by left multiplication and then proving that the set of fixed points of this action, $\{X \in G/P : g \cdot X = X \text{ for all } g \in Q\}$, is non-empty; that is, this action has a fixed-point property. Furthermore, the concept of fixed-points appears frequently in several important theorems, as discussed below, and becomes one of the most important branches in mathematics with applications in real-world problems. In this paper, we aim to study a fixed-point property of a gyrogroup action, which generalizes the notion of group actions.

2. PRELIMINARIES

Basic definitions and notations used in the paper can be found in [1–3,5,6,10,12,13]. In this section, we recall relevant definitions and notations for easy reference.

A pair $(G, \oplus)$, where G is a non-empty set and $\oplus$ is a binary operation on G, is called a *gyrogroup* if (i) there is an element e in G such that $e \oplus a = a$ for all $a \in G$; (ii) for each $b \in G$, there is an element $a \in G$ such that $a \oplus b = e$; (iii) for all elements $a, b \in G$, there is a (unique) automorphism $\operatorname{gyr}[a, b]$ of $(G, \oplus)$ such that $a \oplus (b \oplus c) = (a \oplus b) \oplus \operatorname{gyr}[a, b](c)$ for all $c \in G$ (called the *left gyroassociative law*); and (iv) $\operatorname{gyr}[a \oplus b, b] = \operatorname{gyr}[a, b]$ for all $a, b \in G$ (called the *left loop property*). In fact, the element e in (i) is the unique two-sided identity of G, and the element a in (ii) is the unique two-sided inverse of b in G denoted by $\ominus b$. The automorphism $\operatorname{gyr}[a, b]$ is called the *gyroautomorphism* generated by a and b.

Let G be a gyrogroup. A subset H of G is called a *subgyrogroup* of G if H forms a gyrogroup under the operation inherited from G and $\operatorname{gyr}[a, b](H) = H$ for all $a, b \in H$. A subgyrogroup H of G is called an *L-subgyrogroup* if $\operatorname{gyr}[a, h](H) = H$ for all $a \in G, h \in H$. Furthermore, for each $a \in G$, the set $a \oplus H$ is defined as $a \oplus H = \{a \oplus h : h \in H\}$. The

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index of H in G is defined as the size of $G/H = \{a \oplus H : a \in G\}$ and is denoted by $[G : H]$. For each $a \in G$, the *cyclic subgyrogroup* generated by a is given by $\langle a \rangle = \{ma : m \in \mathbb{Z}\}$.

Recall that an action of a group Γ on a non-empty set X is a map $\Gamma \times X \to X$, written $(g, x) \mapsto g \cdot x$, such that $1 \cdot x = x$ for all $x \in X$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in \Gamma, x \in X$. It turns out that this notion can be generalized to the case of gyrogroups, which are non-associative algebraic structures having common properties with groups, as mentioned in [6]. In fact, a *gyrogroup action* of a gyrogroup G on a non-empty set X is a map $G \times X \to X$, written $(a, x) \mapsto a \cdot x$, such that $e \cdot x = x$ for all $x \in X$ and $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G, x \in X$, where e is the identity of G. In this case, X is called a *G-set*. The gyrogroup action induces the equivalence relation on X given by $x \sim y$ if and only if $y = a \cdot x$ for some element a in G. The equivalence class containing a point $x \in X$ is called the *orbit* of x, denoted by $G \cdot x$, given by the formula

$$G \cdot x = \{y \in X : y \sim x\} = \{a \cdot x : a \in G\}. \tag{2.1}$$

Moreover, the action of G on X induces the permutation σ_a, $a \in G$, of X defined by $\sigma_a(x) = a \cdot x$ for all $x \in X$ so that we can consider the set of fixed points of σ_a, denoted by X_a (instead of $\mathrm{Fix}\,(\sigma_a)$ for simplicity), is given by the formula

$$X_a = \{x \in X : a \cdot x = x\}. \tag{2.2}$$

The set of common fixed-points of σ_a, where $a \in G$, denoted by X_G, is defined as $X_G = \bigcap_{a \in G} X_a$. Therefore,

$$X_G = \{x \in X : a \cdot x = x \text{ for all } a \in G\}. \tag{2.3}$$

An element of X_G (if any) is called a *global fixed-point* of the action of G on X. A duality of a fixed-point set is a stabilizer subgyrogroup of G, which is defined as

$$G_x = \{a \in G : a \cdot x = x\} \tag{2.4}$$

for all $x \in X$. Also, a duality of X_G is the kernel of the action given by $G_X = \bigcap_{x \in X} G_x$. Therefore,

$$G_X = \{a \in G : a \cdot x = x \text{ for all } x \in X\}. \tag{2.5}$$

It is clear by definition that if $x \in X$, then the orbit of x is a singleton set if and only if x is a global fixed-point; that is, $G \cdot x = \{x\}$ if and only if $x \in X_G$. Recall that a gyrogroup G *acts isometrically* or *acts by isometry* on a metric space X if the induced permutation σ_a is a surjective isometry of X for all $a \in G$.

Suppose that a gyrogroup G acts on a non-empty set X. Then G_x is a subgyrogroup invariant under all the gyroautomorphisms of G. In particular, if $c \in G$ and $c \cdot x = x$, then $\mathrm{gyr}[a, b](c) \cdot x = x$. This implies that G_x forms an L-subgyrogroup of G so that the index formula holds: $|G| = [G : G_x]|G_x|$ whenever G is finite. Hence, if G is finite, then the order of G_x divides the order of G for all $x \in X$. The following proposition lists basic properties of fixed-point sets and stabilizer subgyrogroups.

Proposition 2.1 (Proposition 3.26, [9]). *Suppose that a gyrogroup G acts on a non-empty set X, let $x \in X$, and let $a \in G$.*

(1) Then $a \in G_X$ if and only if $X_a = X$.
(2) Then $X_a = X_{\ominus a}$.
(3) If $b \in \langle a \rangle$, then $X_a \subseteq X_b$.
(4) Then $a \in G_x$ if and only if $x \in X_a$.

The following theorem, called the Orbit-Stabilizer Theorem for Gyrogroup Actions, states that the product of $|G \cdot x|$ and $|G_x|$ is constant no matter what x is.

Theorem 2.1 (Theorem 3.9, [6]). *Let G be a gyrogroup acting on a non-empty set X. For each $x \in X$, there exists a bijection from the orbit of x to the set G/G_x of left cosets of the stabilizer of x. In particular, if G is finite, then*

$$|G| = |G \cdot x||G_x|. \tag{2.6}$$

The previous theorem is applied to establish the following theorem, which yields a numerical formula relating the size of X, the number of global fixed-points, and the sum of indices of stabilizer subgyrogroups, referred to as the Orbit Decomposition Theorem for Gyrogroup Actions.

Theorem 2.2 (Theorem 3.10, [6]). *Let G be a gyrogroup acting on a finite non-empty set X. Let $x_1, x_2, \ldots, x_n$ be representatives for the distinct non-singleton orbits in X (if any). Then*

$$|X| = |X_G| + \sum_{i=1}^{n} [G : G_{x_i}]. \tag{2.7}$$

Finally, we quote a theorem that emphasizes the importance of fixed-point sets, referred to as the Orbit Counting Theorem for Gyrogroup Actions. This theorem shows that the number of distinct orbits of G is related to the number of fixed-points in X in a fascinating way.

Theorem 2.3 (Theorem 3.11, [6]). *Let G be a finite gyrogroup acting on a finite non-empty set X. Then the number of distinct orbits in X is equal to $\dfrac{1}{|G|}\displaystyle\sum_{a \in G} |X_a|$.*

3. Main results

As noted in the introduction, the concept of fixed points is crucial, and the set of global fixed-points can be used to examine the structure of a finite group whenever it is not empty. Therefore, in this section, we focus on the problem of determining whether the set of global fixed-points in X is empty for a given G-set X, where G is a gyrogroup. In some cases, the set of global fixed-points is non-empty, and in some cases, it is empty. This motivates the following definition.

Definition 3.1. *Let G be a gyrogroup, and let X be a G-set. We say that X has the global-fixed-point property (with respect to G) if the set of global fixed-points in X is not empty, that is, if $X_G \neq \emptyset$.*

Example 3.1. *Here is a concrete example showing that a G-set may have no the global-fixed-point property. Let $(\mathbb{B}, \oplus_E)$ be the n-dimensional Einstein gyrogroup (see [11]). Let d_e be the gyronorm metric induced by the Euclidean norm on $\mathbb{B}$ (see [8, p. 534]), given by the formula $d_e(\mathbf{u}, \mathbf{v}) = \| -\mathbf{u} \oplus_E \mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. Define $\widehat{\mathbb{B}} = \{L_{\mathbf{v}} : \mathbf{v} \in \mathbb{B}\}$, where $L_{\mathbf{v}}$ is the left gyrotranslation by $\mathbf{v}$ defined by $L_{\mathbf{v}}(\mathbf{w}) = \mathbf{v} \oplus_E \mathbf{w}$, let $\mathrm{Sym}\,(\mathbb{B})$ be the set of permutations of $\mathbb{B}$, and let $\mathrm{Iso}\,(\mathbb{B})$ be the set of isometries of $(\mathbb{B}, d_e)$. Also, define $\mathrm{Sym}_{\mathbf{0}}(\mathbb{B}) = \{\rho \in \mathrm{Sym}\,(\mathbb{B}) : \rho(\mathbf{0}) = \mathbf{0}\}$. Then $\mathrm{Sym}\,(\mathbb{B})$ forms a gyrogroup under the operation defined by the formula*

$$(L_{\mathbf{a}} \circ \alpha) \oplus (L_{\mathbf{b}} \circ \beta) = L_{\mathbf{a} \oplus_E \mathbf{b}} \circ (\alpha \circ \beta) \tag{3.8}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{B}, \alpha, \beta \in \mathrm{Sym}_{\mathbf{0}}(\mathbb{B})$. Furthermore, $\mathrm{Iso}\,(\mathbb{B})$ is a subgyrogroup of $\mathrm{Sym}\,(\mathbb{B})$. As a consequence of the results in Section 5.2 of [6], $\mathrm{Iso}\,(\mathbb{B})$ acts on $\mathrm{Sym}\,(\mathbb{B})/\widehat{\mathbb{B}}$ by the formula

$$\sigma \cdot (\tau \oplus \widehat{\mathbb{B}}) = (\sigma \oplus \tau) \oplus \widehat{\mathbb{B}} \tag{3.9}$$

for all $\sigma \in \mathrm{Iso}\,(\mathbb{B}), \tau \in \mathrm{Sym}\,(\mathbb{B})$. *As shown in [4], the set of global fixed-points of* $\mathrm{Sym}\,(\mathbb{B})/\widehat{\mathbb{B}}$ *is empty. Hence,* $\mathrm{Sym}\,(\mathbb{B})/\widehat{\mathbb{B}}$ *has no the global-fixed-point property.*

Example 3.2. *This example gives a family of gyrogroup actions whose corresponding G-sets have the global-fixed-point property. Let G be a gyrogroup, and let X be a non-empty set. Define*

$$\mathcal{L}(G, X) = \{f : f \text{ is a function from } G \text{ to } X\}.$$

A function f in $\mathcal{L}(G, X)$ is said to be gyro-invariant if $f(a \oplus \mathrm{gyr}[x, y](z)) = f(a \oplus z)$ for all $a, x, y, z \in G$. Let $\mathcal{L}^{\mathrm{gyr}}(G, X)$ be the set of gyro-invariant functions in $\mathcal{L}(G, X)$, that is,

$$\mathcal{L}^{\mathrm{gyr}}(G, X) = \{f \in \mathcal{L}(G, X) : f \text{ is gyro-invariant}\}. \tag{3.10}$$

Then, as in Section 4 of [7], G acts on $\mathcal{L}^{\mathrm{gyr}}(G, X)$ by the formula

$$a \cdot f = f \circ L_{\ominus a} \tag{3.11}$$

for all $a \in G, f \in \mathcal{L}^{\mathrm{gyr}}(G, X)$, called the left standard action of G on $\mathcal{L}^{\mathrm{gyr}}(G, X)$. For each $\alpha \in X$, define f_α by the formula $f_\alpha(x) = \alpha$ for all $x \in G$, called a constant function in $\mathcal{L}(G, X)$. As in the proof of Theorem 4.6 of [7], one can show that the set of global fixed-points in $\mathcal{L}^{\mathrm{gyr}}(G, X)$ is indeed

$$\mathcal{L}^{\mathrm{gyr}}(G, X)_G = \{f_\alpha : \alpha \in X\}. \tag{3.12}$$

Therefore, $\mathcal{L}^{\mathrm{gyr}}(G, X)_G \neq \emptyset$, and so $\mathcal{L}^{\mathrm{gyr}}(G, X)$ has the global-fixed-point property.

In light of Example 3.2, we obtain an interesting consequence of the fact that $\mathcal{L}^{\mathrm{gyr}}(G, X)$ has the global-fixed-point property: the action of G on $\mathcal{L}^{\mathrm{gyr}}(G, X)$ is neither transitive nor free (and hence is not regular) whenever X has at least two distinct elements. In contrast to Example 3.1, we obtain a characterization of the global-fixed-point property in the case when a gyrogroup acts isometrically on a complete $\mathrm{CAT}(0)$ space as a consequence of the Extended Cartan Fixed Point Theorem (see Theorem 3.4 of [4]).

Theorem 3.4 (Proposition 3.7, [4]). *Let G be a gyrogroup acting on a complete* $\mathrm{CAT}(0)$ *space X by isometry. Then the following conditions are equivalent:*

(i) X has the global-fixed-point property.
(ii) Each orbit of G is bounded.
(iii) G has a bounded orbit.

In Section 4 of [6], the author gives a sufficient and necessary condition for an arbitrary gyrogroup G to act on its left coset space $G/H = \{a \oplus H : a \in G\}$ by left gyroaddition: $a \cdot (x \oplus H) = (a \oplus x) \oplus H$. Here, we show that this action induces a G-set that has no the global-fixed-point property, in general. In fact, we obtain the following result.

Proposition 3.2. *Let G be a gyrogroup, and let H be a subgyrogroup of G satisfying the property that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b, x \in G$. Then G acts on G/H by left gyroaddition, and G/H has the global-fixed-point property if and only if $H = G$.*

Proof. That G acts on G/H by left gyroaddition was proved in Theorem 4.3 of [6]. Suppose that $H = G$. Then $G/H = \{e \oplus H\}$. Let $a \in G$. Then

$$a \cdot (e \oplus H) = (a \oplus e) \oplus H = a \oplus H = e \oplus H$$

since $a \in G = H$ implies $a \oplus H = e \oplus H$. This shows that $G/H_G = \{e \oplus H\} \neq \emptyset$, and so G/H has the global-fixed-point property. To prove the converse, suppose that $H \neq G$. Hence, there is an element in $G \setminus H$, say $y \in G \setminus H$. Thus, $e \oplus H \neq y \oplus H$. Let $X \in G/H$. Then $X = x \oplus H$ for some $x \in G$. In the case when $x \in H$, we obtain that $X = e \oplus H$. Hence, $y \cdot X = y \cdot (e \oplus H) = (y \oplus e) \oplus H = y \oplus H \neq X$. In the case when $x \notin H$, we obtain that $X \neq e \oplus H$. Note that $\ominus x \cdot X = \ominus x \cdot (x \oplus H) = (\ominus x \oplus x) \oplus H = e \oplus H \neq X$. This shows that $G/H_G = \emptyset$, and so G/H has no the global-fixed-point property. □

The next proposition shows that a free action induces a G-set that has no the global-fixed-point property. Recall that a gyrogroup action of G on X is *free* if $G_x = \{e\}$ for all $x \in X$.

Proposition 3.3. *Suppose that a non-trivial gyrogroup G acts on a non-empty set X. If the action of G on X is free, then X has no the global-fixed-point property.*

Proof. We prove the contrapositive: if X has the global-fixed-point property, then the action of G on X is not free. Suppose that $X_G \neq \emptyset$, say $x \in X_G$. Let a be a non-identity element of G. Then $a \neq e$ and $a \cdot x = x$. Thus, $a \in G_x$, and so $G_x \neq \{e\}$. This shows that the action of G on X is not free. □

Fortunately, any action of a non-degenerate gyrogroup (which is a gyrogroup having a non-identity gyroautomorphism) is never free, as proved in Theorem 3.1 of [9]. Moreover, the converse of Proposition 3.3 is not generally true. In fact, let Γ be a non-trivial group, and suppose that Ξ is a proper non-trivial subgroup of Γ (for example, let Γ be the symmetric group S_3, and let Ξ be the subgroup of S_3 generated by $(1\ 2\ 3)$). Then, by Proposition 3.2, Γ acts on Γ/Ξ by the formula $g \cdot (x\Xi) = (gx)\Xi$ such that Γ/Ξ has no the global-fixed-point property. This action is not free because $\Gamma_{x\Xi} = x\Xi x^{-1} \neq \{1\}$ for all $x \in \Gamma$.

Next, we give a characterization of the global-fixed-point property via the notion of Schreier graphs and Schreier digraphs (see [9] for relevant definitions). This provides visualization of the global-fixed-point property. Recall that any gyrogroup action gives rise to a digraph as well as a graph. In fact, if G is a gyrogroup and if X is a finite G-set, then for each subset A of G, the *Schreier digraph* $\overrightarrow{\Gamma}(X, A)$ is defined to be a digraph consisting of the vertex set X and the arc set $E = \{(x, a \cdot x) : x \in X, a \in A\}$. The Schreier graph $\Gamma(X, A)$ may be defined as the underlying graph of $\overrightarrow{\Gamma}(X, A)$. We are now in a position to state the aforementioned characterization in the following proposition.

Proposition 3.4. *Suppose that a gyrogroup G acts on a finite non-empty set X, and let A be a left generating set for G. Then X has the global-fixed-point property if and only if the Schreier digraph $\overrightarrow{\Gamma}(X, A)$ (or the Schreier graph $\Gamma(X, A)$) contains a bouquet (which is a connected component with one vertex whose arc is a self-loop).*

Proof. Suppose that X has the global-fixed-point property. Hence, $X_G \neq \emptyset$, say $x \in X_G$. Hence, $a \cdot x = x$ for all $a \in A$. This implies that the connected component of $\overrightarrow{\Gamma}(X, A)$ containing the vertex x is a bouquet. Conversely, suppose that the Schreier digraph $\overrightarrow{\Gamma}(X, A)$ contains a bouquet whose vertex is x. Therefore, $a \cdot x = x$ for all $a \in A$. Let $g \in G$. By assumption, there exist elements $a_1, a_2, \ldots, a_n$ in A such that

$$g = a_n \oplus (a_{n-1} \oplus (\cdots \oplus (a_2 \oplus a_1) \cdots)).$$

It follows that

$$g \cdot x = a_n \oplus (a_{n-1} \oplus (\cdots \oplus (a_2 \oplus a_1) \cdots)) \cdot x = a_n \cdot (a_{n-1} \cdot (\cdots (a_1 \cdot x))) = x.$$

This shows that $x \in X_G$, and so $X_G \neq \emptyset$. Thus, X has the global-fixed-point property. □

Example 3.3. *In Example 1 of [5], the gyrogroup $G_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is given. Its gyroaddition and gyration tables are given in Tables 3 and 4 of [5], respectively. We have by inspection that $A = \{1, 6\}$ is a left generating set for G_8, similar to Example 3.10 of [9]. Let $\mathbb{F}_2 = \{0, 1\}$ be the field of two elements. Identifying a function from G_8 to $\mathbb{F}_2$ with an 8-tuple, we obtain that*

$$\mathcal{L}^{\mathrm{gyr}}(G_8, \mathbb{F}_2) = \{(\alpha, \beta, \beta, \alpha, \gamma, \delta, \gamma, \delta) : \alpha, \beta, \gamma, \delta \in \mathbb{F}_2\}. \tag{3.13}$$

As in Example 3.2, G_8 acts on $\mathcal{L}^{\rm gyr}(G_8,\mathbb{F}_2)$ by formula (3.11). *To depict the Schreier digraph $\overrightarrow{\Gamma}(\mathcal{L}^{\rm gyr}(G_8,\mathbb{F}_2),\{1,6\})$, set*

$$\begin{aligned} x_1 &= (0,0,0,0,0,0,0,0), & x_2 &= (0,0,0,0,0,1,0,1),\\ x_3 &= (0,0,0,0,1,0,1,0), & x_4 &= (0,0,0,0,1,1,1,1),\\ x_5 &= (0,1,1,0,0,0,0,0), & x_6 &= (0,1,1,0,0,1,0,1),\\ x_7 &= (0,1,1,0,1,0,1,0), & x_8 &= (0,1,1,0,1,1,1,1),\\ x_9 &= (1,0,0,1,0,0,0,0), & x_{10} &= (1,0,0,1,0,1,0,1),\\ x_{11} &= (1,0,0,1,1,0,1,0), & x_{12} &= (1,0,0,1,1,1,1,1),\\ x_{13} &= (1,1,1,1,0,0,0,0), & x_{14} &= (1,1,1,1,0,1,0,1),\\ x_{15} &= (1,1,1,1,1,0,1,0), & x_{16} &= (1,1,1,1,1,1,1,1). \end{aligned}$$

The Schreier digraph $\overrightarrow{\Gamma}(\mathcal{L}^{\rm gyr}(G_8,\mathbb{F}_2),\{1,6\})$ is represented pictorially in Figure 1. According to Proposition 3.4, $\mathcal{L}^{\rm gyr}(G_8,\mathbb{F}_2)$ has the global-fixed-point property for the corresponding Schreier digraph has a bouquet. In fact, we also know that the set of global fixed-points in $\mathcal{L}^{\rm gyr}(G_8,\mathbb{F}_2)$ is $\{x_1,x_{16}\}$.

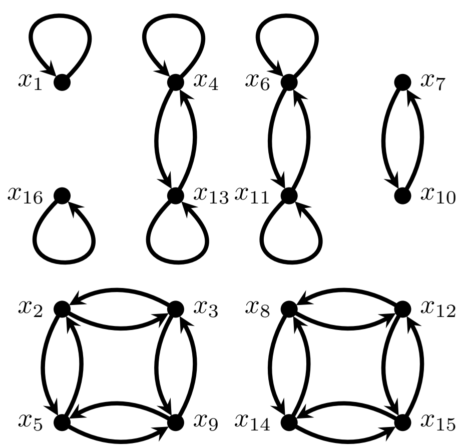

FIGURE 1. The Schreier digraph $\overrightarrow{\Gamma}(\mathcal{L}^{\rm gyr}(G_8,\mathbb{F}_2),\{1,6\})$.

We close this section with the following proposition, which is a nice application of the Orbit-Stabilizer Theorem, together with the Orbit Decomposition Theorem. This result is motivated by the proof of the Third Sylow Theorem for finite groups. Let p be a prime. We say that a finite gyrogroup is a *p-gyrogroup* if its order is a power of p (that is, its order is of the form p^n for some non-negative integer n).

Proposition 3.5. *Suppose that a finite non-trivial gyrogroup G acts on a finite non-empty set X. If G is a p-gyrogroup, then*

$$|X| \equiv |X_G| \pmod{p}. \tag{3.14}$$

Proof. Let $x\in X$. By Theorem 2.1, $|G| = |G\cdot x||G_x|$, and so $[G:G_x] = |G\cdot x|$ is a divisior of $|G|$. Hence, if x is a representative for a non-singleton orbit in X, that is, if $|G\cdot x|>1$, then $[G:G_x]$ is of the form p^k for some $k\in\mathbb{N}$. In the case when there is no non-singleton orbit,

$|X| = |X_G|$. Otherwise, we obtain from Theorem 2.2 that $|X| - |X_G| = \sum_{i=1}^{n}[G : G_{x_i}]$, where $x_1, x_2, \ldots, x_n$ are representatives for the distinct non-singleton orbits in X. As above, p divides $\sum_{i=1}^{n}[G : G_{x_i}]$. Thus, $|X| \equiv |X_G| \pmod{p}$, which completes the proof. □

We immediately obtain a sufficient condition for a finite G-set, where G is a finite p-gyrogroup, to have the global-fixed-point property.

Corollary 3.1. *Suppose that a non-trivial p-gyrogroup G acts on a finite non-empty set X. If p is not a divisor of $|X|$, then X has the global-fixed-point property.*

Proof. We prove the contrapositive. Suppose that $X_G = \emptyset$. Then, by Proposition 3.5, $|X| \equiv 0 \pmod{p}$, and so p divides $|X|$. □

We remark that the converse of Corollary 3.1 is not generally true. In fact, as in Example 3.3, G_8 is a 2-gyrogroup that acts on $\mathcal{L}^{\mathrm{gyr}}(G_8, \mathbb{F}_2)$ by formula (3.11), and $\mathcal{L}^{\mathrm{gyr}}(G_8, \mathbb{F}_2)$ has the global-fixed-point property (see Example 3.2). However, 2 divides $|\mathcal{L}^{\mathrm{gyr}}(G_8, \mathbb{F}_2)|$ because $|\mathcal{L}^{\mathrm{gyr}}(G_8, \mathbb{F}_2)| = 16$.

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