CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 525-541

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.17

Two generalized cyclic projection algorithms for solving a class of the split feasibility problem in real Hilbert spaces

PHAN THI VAN HUYEN[1], NGUYEN SONG HA[2], AND TRUONG MINH TUYEN[3]

ABSTRACT. In the present paper, we propose two new cyclic projection algorithms for solving a class of the split feasibility problem and analyse their strong convergence. Our algorithms are based on the hybrid or shrinking projections methods and use the general index control mapping. Our algorithm can be implemented without any need for information about the norm of the transfer mappings or the cost operators' inverse strong monotone coefficient.

1. INTRODUCTION

In the present paper, we concern the split common solution problem for monotone operator equations (SCSP-MOE, for short), which is to find an element $p \in \Xi$ under the following assumptions:

(O1) $i \in J = \{0, 1, 2, \ldots, N\}$, $\mathbb{H}_i$ is a real Hilbert space and $\mathfrak{f}_i \in \mathbb{H}_i$ is a given element.
(O2) For each $i \in J$, $\mathbb{F}_i : \mathbb{H}_i \to \mathbb{H}_i$ is a γ_i-cocoercive operator on $\mathbb{H}_i$.
(O3) $\mathbb{L}_0 = I_0$ is the identity mapping on $\mathbb{H}_0$, $\mathbb{L}_j : \mathbb{H}_0 \to \mathbb{H}_j$ $(j = 1, 2, 3, \ldots, N)$ are bounded linear operators on $\mathbb{H}_j$ with the adjoint operator $\mathbb{L}_j^* : \mathbb{H}_j \to \mathbb{H}_0$, respectively.
(O4) $\Xi := \cap_{i \in J} \Xi_{A_i} \neq \emptyset$, where $\Xi_{A_i} = \{p \in \mathbb{H}_0 : \mathbb{F}_i(\mathbb{L}_i(p)) = \mathfrak{f}_i\}$ for each $i \in J$.

The operators $\mathbb{L}_i$ and $\mathbb{F}_i$ are called the transfer mapping and the cost operator, respectively.

It is easy to see that the SCSP-MOE is a generalized split problem (see, for example, [10]) and covers various important known split problems. We mention, for instance, the split feasibility problem with multiple output sets (see, [11, 14, 15, 17, 19]), the split common fixed point problem with multiple output sets (see, [7, 8, 18, 20]) and the split common minimum point problem with multiple output sets (see, [10, 16]).

To find a solution to the SCSP-MOE, Ha et al. [10] proposed three algorithms using the combination of the inertial proximal point algorithm with the hybrid and shrinking projection methods. They have first proved the weak convergence of the sequence $\{x_n\}$ generated by

$$\mathbb{F}(x_{n+1}) + \sum_{i=1}^{N} \mathbb{L}_i^* \mathbb{F}_i(\mathbb{L}_i(x_{n+1})) - f - \sum_{i=1}^{N} \mathbb{L}_i^* f_i + x_{n+1} = x_n + \theta(x_n - x_{n-1}), \tag{1.1}$$

for all $n \geq 1$ and with arbitrary initial points x_0 and x_1 belong to $\mathbb{H}_0$. To arrive at strong convergence, they have constructed the second and third algorithms by combining the iterative method (1.1) with the hybrid and shrinking projection methods, respectively. However, these proposed projection algorithms need to know the cost operators' inverse strong monotone coefficient information. Besides, defining x_{n+1} in equation (1.1), we

Received: 15.06.2024. In revised form: 05.11.2024. Accepted: 10.11.2024
2020 *Mathematics Subject Classification.* 47H05, 47H09, 49J53, 90C25.
Key words and phrases. *Hilbert space, cyclic algorithms, metric projection, cocoercive operator.*
Corresponding author: Truong Minh Tuyen (tuyentm@tnus.edu.vn) and Nguyen Song Ha (hans@tnus.edu.vn)

need to solve an equation concerning all operators $\mathbb{F}_i$ and $\mathbb{L}_i$ per each iteration. This may be cause to consume more time and cost to compute in the implementation process. We also note that the equation (1.1) implicit includes a one-step inertial component on the right side. This has improved and accelerated part of the convergence rate more than algorithms without this factor. Additionally, in some recent related literature, replacing one-step inertial extrapolation with more-step inertial extrapolation has also shown that this could be more beneficial numerically and provide acceleration over the one-step inertial extrapolation (see, for instance, [9, 12] and references therein). Thus, there are two natural questions which are posed as follows:

(Q1) Can we construct a new algorithm only concerning operators $\mathbb{F}_i$ and $\mathbb{L}_i$ at each iteration step?

(Q2) Can we include multiple inertial components to accelerate the algorithm's convergence rate?

This paper aims to answer affirmative about two questions above. To this end, we propose a new equation to replace equation (1.1) by using the cyclic iterative method with the generalized index control mapping. Note that the cyclic iterative method is an efficient and powerful method for solving some important problems, see, for instance, the common fixed point problem (see, [3, 4]), the split multiple-set split feasibility problem (see, [21]). Moreover, we also use multiple inertial components in the proposed algorithm to speed up the convergence rate.

The paper is organized as follows. In the next section, we provide concepts related to the SCSP-MOS and lemmas used in the proofs of the main theorems. In Section 3, we propose two cyclic projection algorithms using the proximal point algorithm with multiple inertial components and the hybrid or shrinking projection method. We respectively prove the strong convergence of the first and the second algorithms in Theorem 3.1 and Theorem 3.2. Our algorithms do not depend on the norm of the transfer mappings and the cost operators' inverse strong monotone coefficient. Next, we introduce new algorithms for solving some related problems, such as the split common fixed point problem, the split common null point problem and the split common minimum point problem with multiple output sets. Section 4 presents two numerical experiments and compares them with several known algorithms in [8, 10, 15, 20] to illustrate the effectiveness of the proposed algorithms.

2. Preliminaries

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the induced norm, respectively, on a real Hilbert space $\mathbb{H}$. The symbols $\to$ and $\rightharpoonup$ are for strong and weak convergence, respectively.

Let C be a nonempty closed and convex subset of $\mathbb{H}$. As we know, for each $\mathfrak{a} \in \mathbb{H}$, there is a unique point $P_C(\mathfrak{a}) \in C$ which has the following property

$$\|\mathfrak{a} - P_C(\mathfrak{a})\| = \inf_{w\in C} \|\mathfrak{a} - w\|. \tag{2.2}$$

Thus, we can define the mapping $P_C : \mathbb{H} \to C$ by (2.2). This mapping is called the metric projection of $\mathbb{H}$ onto C. It is well known that a mapping $P_C : \mathbb{H} \to C$ is the metric projection of $\mathbb{H}$ onto C if and only if the following inequality holds true [6, Theorem 3.4]:

$$\langle \mathfrak{a} - P_C(\mathfrak{a}), \mathfrak{b} - P_C(\mathfrak{a})\rangle \le 0, \quad \forall \mathfrak{a} \in \mathbb{H},\ \mathfrak{b} \in C. \tag{2.3}$$

Definition 2.1. An operator $\mathbb{F} : \mathbb{H} \to \mathbb{H}$ is called

(i) monotone if

$$\langle \mathbb{F}(\mathfrak{a}) - \mathbb{F}(\mathfrak{b}), \mathfrak{a} - \mathfrak{b}\rangle \ge 0, \quad \forall \mathfrak{a}, \mathfrak{b} \in \mathbb{H};$$

(ii) β-strongly monotone if there exists $\beta \in (0, \infty)$ such that

$$\langle \mathbb{F}(\mathfrak{a}) - \mathbb{F}(\mathfrak{b}), \mathfrak{a} - \mathfrak{b} \rangle \geq \beta \|\mathfrak{a} - \mathfrak{b}\|^2, \quad \forall \mathfrak{a}, \mathfrak{b} \in \mathbb{H};$$

(iii) γ-cocoercive (or γ-inverse strongly monotone) if there exists $\gamma \in (0, \infty)$ such that

$$\langle \mathbb{F}(\mathfrak{a}) - \mathbb{F}(\mathfrak{b}), \mathfrak{a} - \mathfrak{b} \rangle \geq \gamma \|\mathbb{F}(\mathfrak{a}) - \mathbb{F}(\mathfrak{b})\|^2, \quad \forall \mathfrak{a}, \mathfrak{b} \in \mathbb{H}.$$

Definition 2.2. An operator $\mathbb{F} : \mathbb{H} \to \mathbb{H}$ is said to be L-Lipschitz continuous if there exists a real number $L > 0$ such that

$$\|\mathbb{F}(\mathfrak{a}) - \mathbb{F}(\mathfrak{b})\| \leq L \|\mathfrak{a} - \mathfrak{b}\|, \quad \forall \mathfrak{a}, \mathfrak{b} \in \mathbb{H}.$$

In the case that $L = 1$, $\mathbb{F}$ is called a nonexpansive mapping, and if $0 \leq L < 1$ then $\mathbb{F}$ is called a strictly contraction mapping.

Remark 2.1. It is easy to see that a γ-cocoercive operator is a γ^{-1}-Lipschitz continuous operator.

Definition 2.3. An operator $\mathbb{F} : \mathbb{H} \to \mathbb{H}$ is called hemicontinuous if for every $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{H}$, we have

$$\lim_{\alpha \downarrow 0} \langle \mathfrak{c}, \mathbb{F}(\mathfrak{a} + \alpha \mathfrak{b}) \rangle = \langle \mathfrak{c}, \mathbb{F}(\mathfrak{a}) \rangle.$$

Definition 2.4. A set-valued operator $\mathfrak{G} : \mathbb{H} \to 2^{\mathbb{H}}$ is called

(i) monotone if

$$\langle u - v, \mathfrak{a} - \mathfrak{b} \rangle \geq 0, \quad \forall (\mathfrak{a}, u) \in \text{gra}(\mathfrak{G}), \quad \forall (\mathfrak{b}, v) \in \text{gra}(\mathfrak{G}),$$

where $\text{gra}(\mathfrak{G}) = \{(\mathfrak{a}, u) \in \mathbb{H} \times \mathbb{H} : u \in \mathfrak{G}(\mathfrak{a})\}$, the graph of $\mathfrak{G}$.

(ii) maximal monotone if it is monotone and there exists no monotone operator $\widehat{\mathfrak{G}} : \mathbb{H} \to \mathbb{H}$ such that graph of $\widehat{\mathfrak{G}}$ properly contains graph of $\mathfrak{G}$, i.e., for all $(\mathfrak{a}, u) \in \mathbb{H} \times \mathbb{H}$,

$$(\mathfrak{a}, u) \in \text{gra}(\mathfrak{G}) \Leftrightarrow \langle \mathfrak{a} - \mathfrak{b}, u - v \rangle \geq 0, \quad \forall (\mathfrak{b}, v) \in \text{gra}(\mathfrak{G}).$$

The following lemmas are used in the sequel in the proofs of the main results.

Lemma 2.1. [6, Corollary 2.42 and Lemma 2.35] *Let $\{\mathfrak{a}_n\}$ be a sequence in $\mathbb{H}$. Then the following statements hold true:*

(i) *If $\mathfrak{a}_n \rightharpoonup \mathfrak{a}$ and $\|\mathfrak{a}_n\| \to \|\mathfrak{a}\|$ as $n \to \infty$, then $\mathfrak{a}_n \to \mathfrak{a}$ as $n \to \infty$.*
(ii) *If $\mathfrak{a}_n \rightharpoonup \mathfrak{a}$ as $n \to \infty$, then $\|\mathfrak{a}\| \leq \liminf_{n\to\infty} \|\mathfrak{a}_n\|$.*

Lemma 2.2. (see [13, Lemma 2.1]) *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Then for all $\mathfrak{a} \in H$ and $\mathfrak{b} \in C$, we have*

$$\|\mathfrak{a} - P_C(\mathfrak{a})\|^2 + \|\mathfrak{b} - P_C(\mathfrak{a})\|^2 \leq \|\mathfrak{a} - \mathfrak{b}\|^2.$$

Lemma 2.3. (see, [2, Theorem 1.7.5] and [6, Proposition 20.24]) *If $\mathbb{F} : \mathbb{H} \to \mathbb{H}$ is a monotone, hemicontinuous and coercive operator then $R(\mathbb{F}) = \mathbb{H}$, where $R(\mathbb{F})$ is the range of $\mathbb{F}$.*

Lemma 2.4. (see, [10, Lemma 2.8] *If $\mathbb{F} : \mathbb{H} \to \mathbb{H}$ is γ-cocoercive operator in $\mathbb{H}$, then the set $\mathcal{S} = \{\mathfrak{a} \in \mathbb{H} : \mathbb{F}(\mathfrak{a}) = f\}$ is convex and closed for every $f \in R(\mathbb{F})$.*

3. Main results

We first recall the definition of index control mapping. Let $J = \{0, 1, 2, \dots, N\}$. A mapping $c: \mathbb{N} \to J$ is called an index control mapping if for each $i \in J$, there is a natural number M_i such that

$$i \in \{c(n), c(n+1), \dots, c(n+M_i-1)\}, \quad \forall n \in \mathbb{N}.$$

Example 3.1. The mapping $c: \mathbb{N} \to J$ defined by

$$c(n) = n \mod (N+1), \quad \forall n \in \mathbb{N}$$

is an index control mapping (see, for example, [5]).

Remark 3.2. Ξ is a closed and convex subset of $\mathbb{H}_0$ thanks to Lemma 2.4.

3.1. **Hybrid projection algorithm.** To find a solution to the SCSP-MOE, we first propose the following algorithm.

Algorithm 1.

Step 1. Choose bounded real sequences $\{\varrho_{t,n}\}$ for all $t \in \{1, 2, 3, \dots, k\}$. Select arbitrary points $x_{-k}, x_{-(k-1)}, x_{-(k-2)}, \dots, x_0 \in \mathbb{H}_0$ and set $n := 0$.
Step 2. Compute

$$y_n = x_n + \sum_{t=1}^{k} \varrho_{t,n}(x_{n+1-t} - x_{n-t}).$$

Step 3. Define z_n from the following equation:

$$\mathbb{L}^*_{c(n)}\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n)) - \mathbb{L}^*_{c(n)}\mathfrak{f}_{c(n)} + z_n = y_n. \tag{3.4}$$

Step 4. Define subsets $\mathcal{A}_n$ and $\mathcal{B}_n$ as follows:

$$\mathcal{A}_n = \{a \in \mathbb{H}_0 : \|z_n - a\| \le \|y_n - a\|\}, \tag{3.5}$$

$$\mathcal{B}_n = \{b \in \mathbb{H}_0 : \langle x_0 - x_n, b - x_n\rangle \le 0\}. \tag{3.6}$$

Step 5. Compute x_{n+1} as follows:

$$x_{n+1} = P_{\mathcal{A}_n \cap \mathcal{B}_n}(x_0). \tag{3.7}$$

Step 6. Set $n \leftarrow n+1$ and go to Step 2.

We first prove the following proposition to confirm the unique existence of z_n.

Proposition 3.1. *For each $n \in \mathbb{N}$, the equation* (3.4) *has a unique solution z_n.*

Proof. For each $x \in \mathbb{H}_0$, we define the operator $\mathfrak{G} : \mathbb{H}_0 \to \mathbb{H}_0$ as follows:

$$\mathbb{G}(x) = \mathbb{L}^*_{c(n)}\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x)) + I_0(x),$$

where I_0 is the identity mapping on $\mathbb{H}_0$.
It follows from **(O2)** that

$$\begin{aligned}\langle \mathbb{G}(x) - \mathbb{G}(y), x - y\rangle &= \langle \mathbb{L}^*_{c(n)}\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x)) - \mathbb{L}^*_{c(n)}\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(y)), x - y\rangle \\ &\quad + \|x-y\|^2 \\ &= \langle \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x)) - \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(y)), \mathbb{L}_{c(n)}(x) - \mathbb{L}_{c(n)}(y)\rangle \\ &\quad + \|x-y\|^2 \\ &\ge \gamma_{c(n)}\|\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x)) - \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(y))\|^2 + \|x-y\|^2 \ge 0,\end{aligned}$$

which implies that $\mathbb{G}$ is a monotone operator on $\mathbb{H}_0$.

In view of Remark 2.1, we find that

$$\begin{aligned}\|\mathbb{G}(x)-\mathbb{G}(y)\| &= \|\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x))-\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(y))+x-y\| \\ &\le \|\mathbb{L}_{c(n)}^*\|\|\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x))-\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(y))\|+\|x-y\| \\ &\le \frac{1}{\gamma_{c(n)}}\|\mathbb{L}_{c(n)}\|\|\mathbb{L}_{c(n)}(x))-\mathbb{L}_{c(n)}(y)\|+\|x-y\| \\ &\le \left(\frac{1}{\gamma_{c(n)}}\|\mathbb{L}_{c(n)}\|^2+1\right)\|x-y\|,\end{aligned}$$

which yields to $\mathbb{G}$ is a L-Lipschitz operator on $\mathbb{H}_0$ with $L := \dfrac{1}{\gamma_{c(n)}}\|\mathbb{L}_{c(n)}\|^2+1$.

On the other hand, we observe that

$$\begin{aligned}\langle\mathbb{G}(x),x\rangle &= \langle\mathbb{G}(x)-\mathbb{G}(0),x-0\rangle+\langle\mathbb{G}(0),x\rangle \\ &= \langle\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x))-\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(0)),x-0\rangle+\|x\|^2+\langle\mathbb{G}(0),x\rangle \\ &= \langle\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(x))-\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(0)),\mathbb{L}_{c(n)}(x)-\mathbb{L}_{c(n)}(0)\rangle \\ &\quad +\|x\|^2+\langle\mathbb{G}(0),x\rangle \\ &\ge \|x\|^2-\|\mathbb{G}(0)\|\|x\|.\end{aligned}$$

This leads to

$$\lim_{\|x\|\to\infty}\frac{\langle\mathbb{G}(x),x\rangle}{\|x\|}\ge\lim_{\|x\|\to\infty}(\|x\|-\|\mathbb{G}(0)\|)=\infty,$$

which shows that $\mathbb{G}$ is a coercive operator on $\mathbb{H}_0$.

From the above facts, we can conclude that for each $n\in\mathbb{N}$, the equation (3.4) always has solution z_n thanks to Lemma 2.3.

We now prove the uniqueness of z_n. Indeed, suppose that w_n is also a solution to (3.4), that is,

$$\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(w_n))-\mathbb{L}_{c(n)}^*\mathfrak{f}_{c(n)}+w_n=y_n.$$

Combining this with (3.4), we obtain

$$\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n))-\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(w_n))+(z_n-w_n)=0.$$

This implies that

$$\begin{aligned}0 &= \langle z_n-w_n,\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n))-\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(w_n))+(z_n-w_n)\rangle \\ &= \|z_n-w_n\|^2+\langle z_n-w_n,\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n))-\mathbb{L}_{c(n)}^*\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(w_n))\rangle \\ &= \|z_n-w_n\|^2+\langle\mathbb{L}_{c(n)}z_n-\mathbb{L}_{c(n)}w_n,\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n))-\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(w_n))\rangle \\ &\ge \|z_n-w_n\|^2+\gamma_{c(n)}\|\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n))-\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(w_n))\|^2.\end{aligned}$$

This shows that $z_n=w_n$. □

We now prove the following proposition to confirm the existence of the sequence $\{x_n\}$.

Proposition 3.2. *The sequence $\{x_n\}$ generated by Algorithm 1 is well defined.*

Proof. We divide the proof into several steps.
Claim 1. $\mathcal{A}_n$ and $\mathcal{B}_n$ are two closed half-spaces of $\mathbb{H}_0$. It follows that they are closed and convex subset of $\mathbb{H}_0$.

It is easy to see that (3.5) and (3.6) can be rewritten in the following forms

$$\mathcal{A}_n = \left\{ a \in \mathbb{H}_0 : \langle y_n - z_n, a \rangle \le \frac{\|y_n\|^2 - \|z_n\|^2}{2} \right\},$$
$$\mathcal{B}_n = \{ b \in \mathbb{H}_0 : \langle x_0 - x_n, b \rangle \le \langle x_0 - x_n, x_n \rangle \}.$$

Thus, both of them are two closed half-spaces of $\mathbb{H}_0$ for each $n \in \mathbb{N}$.

Claim 2. $\Xi \subset \mathcal{A}_n \cap \mathcal{B}_n$ for all $n \in \mathbb{N}$.

We first take any $p \in \Xi$, that is

$$\mathbb{F}_i(\mathbb{L}_i(p)) = \mathfrak{f}_i, \quad \forall i \in J. \tag{3.8}$$

We now observe that

$$\langle z_n - p, y_n - p \rangle = \frac{1}{2} \left(\|z_n - p\|^2 + \|y_n - p\|^2 - \|z_n - y_n\|^2 \right). \tag{3.9}$$

On the other hand, it follows from (3.4), (3.8) and **(O2)** that

$$\begin{aligned} \langle z_n - p, y_n - p \rangle &= \langle z_n - p, \mathbb{L}^*_{c(n)} \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n)) - \mathbb{L}^*_{c(n)} \mathfrak{f}_{c(n)} + z_n - p \rangle \\ &= \langle z_n - p, \mathbb{L}^*_{c(n)} \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n)) - \mathbb{L}^*_{c(n)} \mathfrak{f}_{c(n)} \rangle + \|z_n - p\|^2 \\ &= \langle \mathbb{L}_{c(n)}(z_n) - \mathbb{L}_{c(n)}(p), \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n)) - \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(p)) \rangle \\ &\quad + \|z_n - p\|^2 \\ &\ge \gamma_{c(n)} \|\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n)) - \mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(p))\|^2 + \|z_n - p\|^2. \end{aligned} \tag{3.10}$$

Using (3.9) and (3.10), we infer that

$$\|z_n - p\|^2 \le \|y_n - p\|^2 - \|z_n - y_n\|^2 - 2\gamma_{c(n)} \|\mathbb{F}_{c(n)}(\mathbb{L}_{c(n)}(z_n)) - \mathfrak{f}_{c(n)}\|^2 \tag{3.11}$$
$$\le \|y_n - p\|^2, \tag{3.12}$$

which implies that $p \in \mathcal{A}_n$ for all $n \in \mathbb{N}$.

Next, with $n = 0$, it is clear that $\Xi \subset \mathcal{B}_0 = \mathbb{H}_0$. Suppose that $\Xi \subset \mathcal{B}_n$ for some $n \ge 0$. Using (3.7) and (2.3), we have

$$\langle x_0 - x_{n+1}, w - x_{n+1} \rangle \le 0, \quad \forall w \in \mathcal{A}_n \cap \mathcal{B}_n.$$

Thus, it follows from $p \in \mathcal{A}_n \cap \mathcal{B}_n$ that

$$\langle x_0 - x_{n+1}, p - x_{n+1} \rangle \le 0,$$

which implies that $p \in \mathcal{B}_{n+1}$. By employing mathematical induction, we now conclude that $p \in \mathcal{B}_n$ for all $n \ge 0$, that is, $\Xi \subset \mathcal{B}_n$ for all $n \ge 0$. Therefore, we infer that $\Xi \subset \mathcal{A}_n \cap \mathcal{B}_n$ for all $n \in \mathbb{N}$, as claimed.

Furthermore, these also show that the sequence $\{x_n\}$ is well defined. □

The strong convergence of the sequence $\{x_n\}$ generated by Algorithm 1 is established in the following theorem.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 1, converges strongly to $P_\Xi(x_0)$ as n tends to infinite.*

Proof. We divide the proof into several steps. We first take any $p \in \Xi$.

Claim 1. The sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.

It follows from (3.6) that $x_n = P_{\mathcal{B}_n}(x_0)$. Thus, we infer that

$$\|x_n - x_0\| \le \|x_0 - p\|, \quad \forall n \ge 0 \tag{3.13}$$

thanks to (2.2) and $p \in \mathcal{B}_n$ (Claim 2 in Proposition 3.2). This ensures that the sequence $\{x_n\}$ is bounded. Hence, the sequences $\{y_n\}$ and $\{z_n\}$ are also bounded thanks to (3.4) and (3.12).

Claim 2. The sequence $\{x_n\}$ is asymptotically regular, that is,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in \mathcal{B}_n$ and $x_n = P_{\mathcal{B}_n}(x_0)$, using Lemma 2.2, we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \quad \forall n \ge 0, \tag{3.14}$$

which implies that the sequence $\{\|x_n - x_0\|\}$ is increasing for all $n \ge 0$. Because of the boundedness of $\{x_n\}$, we can infer that there exists the finite limit

$$\lim_{n\to\infty} \|x_n - x_0\| = l.$$

Combining this fact with (3.14), we obtain the desired.

Claim 3. The following limits exist:

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{3.15}$$

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{3.16}$$

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.17}$$

Using the definition of y_n, we find that

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|x_{n+1} - x_n + \sum_{t=1}^{k} \varrho_{t,n}(x_{n+1-t} - x_{n-t})\| \\ &\le \|x_{n+1} - x_n\| + \sum_{t=1}^{k} |\varrho_{t,n}|\|x_{n+1-t} - x_{n-t}\|.\end{aligned}$$

Using Claim 2 and the boundedness of the real number sequences $\{\varrho_{t,n}\}$, we obtain the existence of the limit (3.15), as asserted.

On the other hand, since $x_{n+1} \in \mathcal{A}_n$, we have

$$\|z_n - x_{n+1}\| \le \|y_n - x_{n+1}\|,$$

which implies that

$$\|z_n - x_{n+1}\| \to 0. \tag{3.18}$$

Using Claim 2, (3.15) and (3.18) and the following estimates

$$\|x_n - z_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - z_n\|,$$
$$\|y_n - z_n\| \le \|x_{n+1} - y_n\| + \|x_{n+1} - z_n\|,$$

we also get the limits (3.16) and (3.17), as claimed.

Claim 4. All weak cluster points of the sequence $\{x_n\}$ belong to Ξ.

Assume that $\widehat{p}$ is an arbitrary cluster point of the sequence $\{x_n\}$. Then there exists a subsequence $\{x_{m_n}\}$ of $\{x_n\}$ such that

$$x_{m_n} \rightharpoonup \widehat{p}.$$

For any $i \in J$, there exists a natural number M_i such that

$$i \in \{c(m_n), c(m_n+1), c(m_n+2), \ldots, c(m_n+M_i-1)\}, \quad \forall n \in \mathbb{N}.$$

We can remove some elements of the subsequence $\{x_{m_n}\}$, if necessary, to obtain a new subsequence, which is also denoted by $\{x_{m_n}\}$ such that

$$m_{n+1} \ge m_n + M_i.$$

Then there is another subsequence $\{x_{p_n}\}$ of $\{x_n\}$ for which

$$m_n \le p_n \le m_n + M_i - 1 < m_{n+1} \le p_{n+1}, \quad i = c(p_n).$$

Furthermore, we can see that

$$\|x_{p_n} - x_{m_n}\| \le \sum_{j=m_n}^{m_n+M_i-2} \|x_{j+1} - x_j\| \le (M_i - 1) \max_{m_n \le j \le m_n+M_i-2} \|x_{j+1} - x_j\|.$$

Using Claim 2, we find that

$$\|x_{p_n} - x_{m_n}\| \to 0,$$

which yields to $x_{p_n} \rightharpoonup \widehat{p}$, and, thus, we have $z_{p_n} \rightharpoonup \widehat{p}$ thanks to (3.16). Since $\mathbb{L}_i$ is bounded linear operator, we also have

$$\mathbb{L}_i(z_{p_n}) \rightharpoonup \mathbb{L}_i(\widehat{p}). \tag{3.19}$$

Next, we note that $\mathbb{F}_{c(p_n)} = \mathbb{F}_i$, $\mathbb{L}_{c(p_n)} = \mathbb{L}_i$ for all $n \ge 1$ and

$$\begin{aligned}
0 &\le \gamma_i \|\mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathbb{F}_i(\mathbb{L}_i(\widehat{p}))\|^2 \\
&\le \langle \mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathbb{F}_i(\mathbb{L}_i(\widehat{p})), \mathbb{L}_i(z_{p_n}) - \mathbb{L}_i(\widehat{p})\rangle \\
&= \langle \mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathfrak{f}_i, \mathbb{L}_i(z_{p_n}) - \mathbb{L}_i(\widehat{p})\rangle + \langle \mathfrak{f}_i - \mathbb{F}_i(\mathbb{L}_i(\widehat{p})), \mathbb{L}_i(z_{p_n}) - \mathbb{L}_i(\widehat{p})\rangle \\
&\le \|\mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathfrak{f}_i\| \|\mathbb{L}_i(z_{p_n}) - \mathbb{L}_i(\widehat{p})\| + \langle \mathfrak{f}_i - \mathbb{F}_i(\mathbb{L}_i(\widehat{p})), \mathbb{L}_i(z_{p_n}) - \mathbb{L}_i(\widehat{p})\rangle \\
&\le K_1 \|\mathbb{L}_i\| \|\mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathfrak{f}_i\| + \langle \mathfrak{f}_i - \mathbb{F}_i(\mathbb{L}_i(\widehat{p})), \mathbb{L}_i(z_{p_n}) - \mathbb{L}_i(\widehat{p})\rangle, \qquad (3.20)
\end{aligned}$$

where $K_1 = \sup_n \|z_{p_n} - \widehat{p}\|$. On the other hand, it follows from (3.11) that

$$\begin{aligned}
\gamma_i \|\mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathfrak{f}_i\|^2 &\le \frac{1}{2}\left(\|y_{p_n} - p\|^2 - \|z_{p_n} - p\|^2\right) \\
&= \frac{1}{2}(y_{p_n} - p\| - \|z_{p_n} - p\|)(y_{p_n} - p\| + \|z_{p_n} - p\|) \\
&\le K_2 \|y_{p_n} - z_{p_n}\|,
\end{aligned}$$

where $K_2 = \dfrac{1}{2}\sup_n\{(y_{p_n} - p\| + \|z_{p_n} - p\|)\}$. Combining this with (3.17), we obtain

$$\|\mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathfrak{f}_i\| \to 0. \tag{3.21}$$

Thus, from (3.19), (3.20) and (3.21), we infer that

$$\|\mathbb{F}_i(\mathbb{L}_i(z_{p_n})) - \mathbb{F}_i(\mathbb{L}_i(\widehat{p}))\| \to 0.$$

Combining this with (3.21), we obtain that $\mathbb{F}_i(\mathbb{L}_i(\widehat{p})) = \mathfrak{f}_i$. Since $i \in J$ is an arbitrary element, we infer that $\widehat{p} \in \Xi$.

Claim 5. The sequence $\{x_n\}$ converges strongly to $x_\star = P_\Xi(x_0)$.

Suppose that $\{x_{q_n}\}$ is a subsequence of $\{x_n\}$ converges weakly to q. Because of Claim 4, we find that $q \in \Xi$. Moreover, since $x_\star = P_\Xi(x_0)$ and $q \in \Xi$, using (3.13) and Lemma 2.1 (ii), we have

$$\begin{aligned}
\|x_\star - x_0\| \le \|q - x_0\| &\le \liminf_{k\to\infty} \|x_{q_n} - x_0\| \\
&\le \limsup_{k\to\infty} \|x_{q_n} - x_0\| \le \|x_\star - x_0\|.
\end{aligned}$$

Using the uniqueness of the nearest point $x_\star$, we infer that $x_\star = q$. Moreover, we also have

$$\|x_{q_n} - x_0\| \to \|x_\star - x_0\|.$$

In view of Lemma 2.1 (i), we obtain $x_{q_n} \to x_\star$. Using again the uniqueness of $x_\star$, we conclude that $x_n \to x_\star$ as $n \to \infty$. □

3.2. **Shrinking projection algorithm.** We now propose a modification of Algorithm 1 using the shrinking projection method to solve the SCSP-MOE.

Algorithm 2.

Step 1. Choose bounded real sequences $\{\varrho_{t,n}\}$ for all $t \in \{1, 2, 3, \dots, k\}$. Select arbitrary points $x_{-k}, x_{-(k-1)}, x_{-(k-2)}, \dots, x_0 \in \mathbb{H}_0$, $\mathcal{C}_{-1} = \mathbb{H}_0$ and set $n := 0$.
Step 2. Compute y_n as in Step 2 of Algorithm 1.
Step 3. Define z_n as in Step 3 of Algorithm 1.
Step 4. Define subset $\mathcal{C}_n$ as follows:

$$\mathcal{C}_n = \{a \in \mathcal{C}_{n-1} : \ \|z_n - a\| \leq \|y_n - a\|\}.$$

Step 5. Compute x_{n+1} as follows:

$$x_{n+1} = P_{\mathcal{C}_n}(x_0).$$

Step 6. Set $n \leftarrow n + 1$ and go to Step 2.

Theorem 3.2. *The sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $P_\Xi(x_0)$ as n tends to infinite.*

Proof. We will break down the proof into multiple steps.
Claim 1. We have $\Xi \subset \mathcal{C}_n$ for all $n \geq 0$.

It is easy to see that $\mathcal{C}_n$ is a closed and convex subset of $\mathbb{H}_0$ for all $n \geq 0$. Employing an argument similar to the one used in the proof of Claim 2 of the proof of Proposition 3.2, and using mathematical induction, we also conclude that $\Xi \subset \mathcal{C}_n$ for all $n \geq 0$. Hence, the sequence $\{x_n\}$ is well defined.
Claim 2. The sequence $\{x_n\}$ converges strongly to $x^\dagger \in \mathbb{H}_0$.

For each $p \in \Xi \subset \mathcal{C}_n$, it follows from $x_{n+1} = P_{\mathcal{C}_n}(x_0)$ that

$$\|x_{n+1} - x_0\| \leq \|p - x_0\|, \tag{3.22}$$

which ensures that the sequence $\{x_n\}$ is bounded.

For any $m \geq n \geq 1$, since $x_m \in \mathcal{C}_{m-1} \subset \mathcal{C}_{n-1}$ and $x_n = P_{\mathcal{C}_{n-1}}(x_0)$, in view of Lemma 2.2, we have

$$\|x_m - x_n\| \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \tag{3.23}$$

when $n, m \to \infty$. This implies that $\{x_n\}$ is a Cauchy sequence. Hence, the sequence $\{x_n\}$ converges strongly to an element $x^\dagger$, as asserted.
Claim 3. The following limits exist:

$$\lim_{n\to\infty} \|x_{n+1} - x_n\|_{\mathbb{H}} = 0, \tag{3.24}$$

$$\lim_{n\to\infty} \|x_{n+1} - y_n\|_{\mathbb{H}} = 0, \tag{3.25}$$

$$\lim_{n\to\infty} \|x_n - y_n\|_{\mathbb{H}} = 0, \tag{3.26}$$

$$\lim_{n\to\infty} \|y_n - z_n\|_{\mathbb{H}} = 0. \tag{3.27}$$

Using (3.23), we get the limit (3.24), as claimed. Furthermore, by employing an argument similar to the one used in the proof of Claim 3 of the proof of Theorem 3.1, we also obtain the limits (3.25), (3.26) and (3.27), as asserted.
Claim 4. The sequence $\{x_n\}$ converges strongly to $x_\star = P_\Xi(x_0)$.

Since $x_n \to x^\dagger$, we have $y_n \to x^\dagger$ and $z_n \to x^\dagger$ thanks to (3.26). Employing an argument similar to the one used in the proof Claim 4 of the proof of Theorem 3.1, we also infer that $x^\dagger \in \Xi$.

Finally, in (3.22), letting $n \to \infty$, we infer that $\|x^\dagger - a_0\| \le \|x_\star - a_0\|$. It now follows from (2.2) that $x^\dagger = x_\star$. □

Remark 3.3. Let $\mathcal{T}_i : \mathbb{H}_i \to \mathbb{H}_i$ be a nonexpansive mapping on $\mathbb{H}_i$ and I_i be the identity mapping on $\mathbb{H}_i$ for each $i \in J = \{0, 1, 2, \dots, N\}$. Observe that if we take $\mathbb{F}_i = I_i - \mathcal{T}_i$ and $\mathfrak{f}_i = 0$ for all $i \in J$, then the SCSP-MOE becomes the following split common fixed point problem with multiple output sets (SCFPP-MOS, for short):

$$\text{Find } p \in \widehat{\Xi} := \mathrm{Fix}(\mathcal{T}_0) \cap (\cap_{i=1}^N \mathbb{L}_i^{-1}(\mathrm{Fix}(\mathcal{T}_i))).$$

It is also easy to see that $I_i - \mathcal{T}_i$ is 0.5-cocoercive operator on $\mathbb{H}_i$ for all $i \in J$ (see, for instance, [10]). Therefore, from Algorithms 1 and 2, we arrive at the following algorithms concerning the SCFPP-MOS.

The following algorithm is established from Algorithm 1.

Algorithm 3.

Step 1. Choose $\{\varrho_{t,n}\}$ $(t = 1, 2, 3, \dots, k)$ as in Step 1 Algorithm 1. Select arbitrary points $x_{-k}, x_{-(k-1)}, x_{-(k-2)}, \dots, x_0 \in \mathbb{H}_0$ and set $n := 0$.
Step 2. Compute y_n as in Step 2 of Algorithm 1.
Step 3. Define z_n from the following equation:

$$\mathbb{L}_{c(n)}^*(I_{c(n)} - \mathcal{T}_{c(n)})(\mathbb{L}_{c(n)}(z_n)) + z_n = y_n.$$

Step 4. Define subsets $\widehat{\mathcal{A}}_n$ and $\widehat{\mathcal{B}}_n$ as follows:

$$\widehat{\mathcal{A}}_n = \{a \in \mathbb{H}_0 : \|z_n - a\| \le \|y_n - a\|\},$$
$$\widehat{\mathcal{B}}_n = \{b \in \mathbb{H}_0 : \langle x_0 - x_n, b - x_n\rangle \le 0\}.$$

Step 5. Compute x_{n+1} as follows:

$$x_{n+1} = P_{\widehat{\mathcal{A}}_n \cap \widehat{\mathcal{B}}_n}(x_0).$$

Step 6. Set $n \leftarrow n + 1$ and go to Step 2.

Corollary 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to $P_{\widehat{\Xi}}(x_0)$ as n tends to infinite.*

The following algorithm is obtained from Algorithm 2.

Algorithm 4.

Step 1. Choose $\{\varrho_{t,n}\}$ $(t = 1, 2, 3, \dots, k)$ as in Step 1 Algorithm 1. Select arbitrary points $x_{-k}, x_{-(k-1)}, x_{-(k-2)}, \dots, x_0 \in \mathbb{H}_0$ and set $n := 0$.
Step 2. Compute y_n as in Step 2 of Algorithm 1.
Step 3. Define z_n as in Step 3 of Algorithm 3.
Step 4. Define subset $\widehat{\mathcal{C}}_n$ as follows:

$$\widehat{\mathcal{C}}_n = \{a \in \mathcal{C}_{n-1} : \|z_n - a\| \le \|y_n - a\|\}.$$

Step 5. Compute x_{n+1} as follows:

$$x_{n+1} = P_{\widehat{\mathcal{C}}_n}(x_0).$$

Step 6. Set $n \leftarrow n+1$ and go to Step 2.

Corollary 3.2. *The sequence $\{x_n\}$ generated by Algorithm 4 converges strongly to $P_{\widehat{\Xi}}(x_0)$ as n tends to infinite.*

Remark 3.4. Let U_i be a nonempty closed and convex subsets of $\mathbb{H}_i$ for each $i \in J = \{0, 1, 2, \ldots, N\}$. Let P_{U_i} be the metric projections from $\mathbb{H}_i$ onto U_i for each $i \in J$. It is well known that P_{U_i} is a nonexpansive mapping and $\text{Fix}(P_{U_i}) = U_i$ for all $i \in J$ (see, for example, [6, Proposition 4.8]). Thus, in Remark 3.3, if we replace $\mathcal{T}_i$ by P_{U_i}, then the SCSP-MOS reduce to the split feasibility problem with multiple output sets (SFP-MOS, for short) which is introduced and studied in [15] and take the following form:

$$\text{Find } p \in \overline{\Xi} := U_0 \cap (\cap_{i=1}^N \mathbb{L}_i^{-1}(U_i)).$$

Accordingly, we can use Algorithms 3 and 4 to solve the SFP-MOS.

Remark 3.5. Let $i \in J = \{0, 1, 2, \ldots, N\}$ and $\Theta_i : \mathbb{H}_i \to 2^{\mathbb{H}_i}$ be a maximal monotone operator on $\mathbb{H}_i$. For each $r_i > 0$, let $J_{r_i}^{\Theta_i}$ be the resolvent of Θ_i, that is, $J_{r_i}^{\Theta_i} := (I_i + r_i\Theta_i)^{-1}$. As we know, $J_{r_i}^{\Theta_i}$ is a nonexpansive mapping and $\text{Zer}(\Theta_i) = \text{Fix}(J_{r_i}^{\Theta_i})$ (see, for instance, [6, Corollary 23.10 and Proposition 23.38]), where $\text{Zer}(\Theta_i) = \{\mathfrak{a} \in \mathbb{H}_i :\ 0 \in \Theta_i(\mathfrak{a})\}$, the zero point set of Θ_i. Thus, in Remark 3.3, if we replace $\mathcal{T}_i$ by $J_{r_i}^{\Theta_i}$, then the SCSP-MOS becomes following split common zero point problem with multiple output sets (SCZP-MOS, for short):

$$\text{Find } p \in \Xi^\circ := \text{Zer}(\Theta_0) \cap (\cap_{i=1}^N \mathbb{L}_i^{-1}(\text{Zer}(\Theta_i))).$$

Hence, we also can apply Algorithms 3 and 4 to find a solution to the SCZP-MOS.

Let $\Phi_i :\ \mathbb{H}_i \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. The subdifferential of Φ_i is the set-valued operator $\partial\Phi_i : \mathbb{H}_i \to 2^{\mathbb{H}_i}$ defined by

$$\partial\Phi_i(x) := \{\mathfrak{a} \in \mathbb{H}_i : \langle \mathfrak{a}, y - x\rangle \le \Phi_i(y) - \Phi_i(x),\ \forall y \in \mathbb{H}_i\},$$

for each $x \in \mathbb{H}_i$. It is known that $\partial\Phi$ is a maximal monotone operator and

$$x_0 \in \text{Arg}\min_{x \in \mathbb{H}_i} \Phi_i(x) \text{ if and only if } \partial\Phi_i(x_0) \ni 0, \text{ that is, } x_0 \in \text{Zer}(\Phi_i),$$

(see, for instance, [6, Theorem 16.2 and Theorem 21.2]). Thus, we can apply Algorithms 3 and 4 for solving the split common minimum point problem with multiple output sets (SCMP-MOS, for short), that is,

$$\text{Find } p \in \Xi^\star := \text{Arg}\min \Phi_0 \cap \left(\cap_{i=1}^N \mathbb{L}_i^{-1}(\text{Arg}\min \Phi_i)\right).$$

4. Numerical experiments

In this section, all methods are implemented in MATLAB 14a running on the DESKTOP-8LDGIN0, Intel(R) Core(TM) i5-4210U CPU @ 1.70GHz with 2.40 GHz and 16GB RAM.

Example 4.2. We now consider the SCMP-MOS for which the following details the hypotheses:

(a) $\Phi_0 : \mathbb{R}^5 \to \mathbb{R}$, $\Phi_1 : \mathbb{R}^4 \to \mathbb{R}, \Phi_2 : \mathbb{R}^3 \to \mathbb{R}$ and $\Phi_3 : \mathbb{R}^2 \to \mathbb{R}$ are convex and Fréchet differentiable functions defined as follows:

$$\Phi_0(a) = (a_1 + 2a_2 + a_3 + 2a_4 + a_5 - 1)^2, \quad \forall a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5;$$
$$\Phi_1(b) = (b_1 + 2b_2 - b_3 + b_4 - 1)^2, \quad \forall b = (b_1, b_2, b_3, b_4) \in \mathbb{R}^4;$$
$$\Phi_2(c) = (2c_1 - c_2 + c_3 - 1)^2, \quad \forall c = (c_1, c_2, c_3) \in \mathbb{R}^3;$$
$$\Phi_3(d) = (d_1 + d_2 - 1)^2, \quad \forall d = (d_1, d_2) \in \mathbb{R}^2;$$

(b) $\mathbb{L}_0 = I_0$ is the identity mapping on $\mathbb{R}^5$ while the linear bounded operators $\mathbb{L}_1 : \mathbb{R}^5 \to \mathbb{R}^4, \mathbb{L}_2 : \mathbb{R}^5 \to \mathbb{R}^3$ and $\mathbb{L}_3 : \mathbb{R}^5 \to \mathbb{R}^2$ are respectively defined by

$$\mathbb{L}_1(a) = (a_1 + a_2 - 4a_5, 2a_3 - a_4 + a_5, a_1 + 2a_2 + a_3 - a_4, 2a_2 + 3a_5);$$
$$\mathbb{L}_2(a) = (a_1 - a_2 + a_4, -2a_3 + 2a_4 - a_5, -2a_1 + 2a_2 - a_3 - a_4);$$
$$\mathbb{L}_3(a) = (a_1 - a_2 - a_3 + a_4 + a_5, -a_1 + a_2 + a_3 - 2a_5);$$

for all $a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$.

It is not difficult to show that

$$\Xi^\star = \{x = (5 - 3t, -4, 2, 1 + t, t) \in \mathbb{R}^5 : t \in \mathbb{R}\}.$$

We now test the convergence of the sequence $\{x_n\}$ generated by Algorithms 3 and 4 with

$$\mathcal{T}_i = J^{\partial\Phi_i} = (I_i + \partial\Phi_i)^{-1}, \ i = 0, 1, 2, 3,$$

where I_1, I_2 and I_3 are identity mappings on $\mathbb{R}^4$, $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. For this experiment, we use Algorithms 1 and 2 and select the initial points as follows:

$$x_{-3} = (3, 0, 3, 0, 3), \quad x_{-2} = (2, 1, 2, 1, 2),$$
$$x_{-1} = (1, 2, 1, 2, 1), \quad x_0 = (-4, 0, 4, 0, -4).$$

It is also easy to see that $x_\star = P_{\Xi^\star}(x_0) = (-1, -4, 2, 3, 2)$.

We also compare our algorithms with some known algorithms (Algorithms 2 and 3 in [10] (ALGO-H1 and ALGO-H2, for short, respectively). The stopping rule for all algorithms was taken as

$$\text{err} = \|x_n - x_\star\| < \text{TOL},$$

where TOL is a given tolerance.

The numerical results are presented in Table 1 and Table 2.

Example 4.3. We now consider the SFP-MOS for which the following details the hypotheses:

(a) $\mathbb{H}_i = \mathcal{L}^2[0, 1]$ $(i = 0, 1, 2, 3)$ with the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ and the induced norm $\|x\| = \left(\int_0^1 x^2(t)dt\right)^{\frac{1}{2}}$ for all $x = x(t) \in \mathcal{L}^2[0, 1]$ and $y = y(t) \in \mathcal{L}^2[0, 1]$.

(b) The set U_i $(i = 0, 1, 2, 3)$ are given by

$$U_i = \{x \in \mathcal{L}^2[0, 1] : \langle \mathfrak{a}_i, x\rangle = \mathfrak{b}_i\},$$

where $\mathfrak{a}_i = \cos((i+1)t)$ and $\mathfrak{b}_i = \dfrac{(i + 1)\sin(i + 1) + \cos(i + 1) - 1}{(i + 1)^3}$, for all $t \in [0, 1]$.

(c) $\mathbb{L}_0 = I_0$ is the identity mapping on $\mathcal{L}^2[0, 1]$. For each $i = 1, 2, 3$, the linear bounded operator $\mathbb{L}_i : \mathcal{L}^2[0, 1] \to \mathcal{L}^2[0, 1]$ is defined by $\mathbb{L}_i(x)(t) = \dfrac{x(t)}{i + 1}$ for all $x = x(t) \in \mathcal{L}^2[0, 1]$.

TOL	θ_n	Algorithms	n	err	CPU-time (s)
10^{-4}	50	ALGO-H1	1607	$9.9735e-05$	5.3989
		ALGO-H2	303	$9.9301e-05$	1.6858
	100	ALGO-H1	3202	$9.9604e-05$	11.3877
		ALGO-H2	325	$8.8377e-05$	2.1084
	$1000/n$	ALGO-H1	263	$9.1738e-05$	1.5724
		ALGO-H2	319	$9.6868e-05$	2.1917
10^{-5}	50	ALGO-H1	1826	$9.9617e-06$	5.9028
		ALGO-H2	362	$9.9748e-06$	2.4305
	100	ALGO-H1	3638	$9.9611e-06$	13.3407
		ALGO-H2	384	$9.8051e-06$	3.0366
	$1000/n$	ALGO-H1	281	$9.1894e-06$	1.6091
		ALGO-H2	373	$9.0063e-06$	2.9550
10^{-6}	50	ALGO-H1	2040	$9.9300e-07$	7.3342
		ALGO-H2	448	$9.6430e-07$	3.9373
	100	ALGO-H1	4065	$9.9030e-07$	14.9199
		ALGO-H2	474	$9.4845e-07$	5.0596
	$1000/n$	ALGO-H1	298	$9.2445e-07$	1.6584
		ALGO-H2	431	$8.3679e-07$	3.7227

TABLE 1. Table of numerical results with different choices of θ_n

It is easy to see that $\overline{\Xi} := U_0 \cap (\cap_{i=1}^3 \mathbb{L}_i^{-1}(U_i))$ is a nonempty set because $t \in \overline{\Xi}$. For this experiment, we use Algorithms 3 and 4 with the initial points

$$x_{-3} = \sin t, \quad x_{-2} = \cos t, \quad x_{-1} = e^t, \quad x_0 = \sqrt{t}, \quad \forall t \in [0,1].$$

We also compare our algorithms with some previous algorithms (Algorithm 5.2 in [8] (ALGO-C, for short), Algorithms 1.5 and 1.6 in [15] (ALGO-R1 and ALGO-R2, for short, respectively), Algorithm 4.10 in [20] (ALGO-W, for short)). The data for each algorithm are chosen as follows:

• ALGO-C: $\alpha_n = \dfrac{1}{n}$, $\tau = \dfrac{0.05}{1+\sum_{i=1}^3 \|\mathbb{L}_i\|^2}$ and $u := u(t) = t+1$ for all $t \in [0,1]$.

• ALGO-R1: $\gamma_n = \dfrac{0.0005}{3\max_{0\le i\le 3}\{\|\mathbb{L}_i\|^2\}}$.

• ALGO-R2: $\gamma_n = \dfrac{0.1}{3\max_{0\le i\le 3}\{\|\mathbb{L}_i\|^2\}}$, $\alpha_n = \dfrac{1}{n^{0.75}}$ and $f(x)(t) = 0.5x(t)$ for all $x = x(t) \in \mathcal{L}^2[0,1]$.

• ALGO-W: $\gamma_n = \dfrac{1}{n^{0.75}}$ and $f(x)(t) = 0.5x(t)$ for all $x = x(t) \in \mathcal{L}^2[0,1]$.

The stopping rule for all algorithms was taken as

$$\text{err} = \frac{\sum_{i=0}^3 \|\mathbb{L}_i(x_n) - P_{U_i}(\mathbb{L}_i(x_n))\|^2}{4} < \text{TOL},$$

where TOL is a given tolerance. The numerical results are showed in Table 3 and Table 4.

Remark 4.6. Tables 1, 2, 3 and 4 show that our proposed algorithms outperform several known algorithms proposed in [8, 10, 15, 20] concerning the number of iterations and the CPU time.

	TOL		Algorithm 1	Algorithm 2
$\varrho_{1,n}=1$, $\varrho_{2,n}=-3$, $\varrho_{3,n}=2$	10^{-4}	n	114	72
		err	$9.9000e-05$	$9.9920e-05$
		CPU-time (s)	0.2854	0.4212
	10^{-5}	n	130	83
		err	$9.9781e-06$	$7.7280e-06$
		CPU-time (s)	0.3011	0.5535
	10^{-6}	n	146	95
		err	$9.3456e-07$	$6.5143e-07$
		CPU-time (s)	0.3232	0.7336
$\varrho_{1,n}=50$, $\varrho_{2,n}=-25$, $\varrho_{3,n}=100$	10^{-4}	n	134	123
		err	$9.0839e-05$	$8.4315e-05$
		CPU-time (s)	0.2949	0.6706
	10^{-5}	n	150	135
		err	$9.1640e-06$	$7.6039e-06$
		CPU-time (s)	0.3569	0.8289
	10^{-6}	n	166	147
		err	$8.8274e-07$	$7.1771e-07$
		CPU-time (s)	0.3900	0.9956
$\varrho_{1,n}=100$, $\varrho_{2,n}=-30$, $\varrho_{3,n}=-10$	10^{-4}	n	164	151
		err	$9.3564e-05$	$4.7631e-05$
		CPU-time (s)	0.3636	1.1650
	10^{-5}	n	188	161
		err	$8.0723e-06$	$9.8930e-06$
		CPU-time (s)	0.4018	1.3968
	10^{-6}	n	206	175
		err	$9.9109e-07$	$6.4223e-07$
		CPU-time (s)	0.4511	1.6950
$\varrho_{1,n}=1000/n$, $\varrho_{2,n}=-50$, $\varrho_{3,n}=25$	10^{-4}	n	119	95
		err	$9.0477e-05$	$6.7640e-05$
		CPU-time (s)	0.2712	0.4592
	10^{-5}	n	135	106
		err	$9.1169e-06$	$8.0157e-06$
		CPU-time (s)	0.3115	0.5611
	10^{-6}	n	151	115
		err	$8.4995e-07$	$8.7889e-07$
		CPU-time (s)	0.3393	0.6583

TABLE 2. Table of numerical results with different choices of $\varrho_{t,n}$

5. CONCLUSIONS

We have studied the split common solution problem for monotone operator equations. To solve this problem, we have introduced two new cyclic projection algorithms, which are based on the hybrid or shrinking projections methods and use the general index control mapping. Our algorithm can be implemented without any need for information about

	TOL		Algorithm 3	Algorithm 4
$\varrho_{1,n} = -1$, $\varrho_{2,n} = -2$, $\varrho_{3,n} = -3$	10^{-4}	n	52	36
		err	$9.5057e-05$	$7.8498e-05$
		CPU-time (s)	0.2188	0.2664
	10^{-5}	n	70	65
		err	$8.7951e-06$	$9.4747e-06$
		CPU-time (s)	0.2639	0.5076
	10^{-6}	n	171	106
		err	$9.8082e-07$	$9.4637e-07$
		CPU-time (s)	0.4298	1.2527
$\varrho_{1,n} = 1$, $\varrho_{2,n} = 2$, $\varrho_{3,n} = 3$	10^{-4}	n	12	11
		err	$7.7042e-05$	$6.4461e-05$
		CPU-time (s)	0.1527	0.1730
	10^{-5}	n	25	25
		err	$9.5718e-06$	$9.4475e-06$
		CPU-time (s)	0.1782	0.2365
	10^{-6}	n	322	41
		err	$9.5877e-07$	$6.1331e-07$
		CPU-time (s)	0.7363	0.3728
$\varrho_{1,n} = -10$, $\varrho_{2,n} = 20$, $\varrho_{3,n} = -3$	10^{-4}	n	42	36
		err	$5.9390e-05$	$7.8754e-05$
		CPU-time (s)	0.2031	0.2928
	10^{-5}	n	61	50
		err	$3.0716e-06$	$9.5896e-06$
		CPU-time (s)	0.2363	0.4035
	10^{-6}	n	160	76
		err	$9.4149e-07$	$9.1679e-07$
		CPU-time (s)	0.4026	0.7797
$\varrho_{1,n} = -10$, $\varrho_{2,n} = 20$, $\varrho_{3,n} = 3$	10^{-4}	n	37	32
		err	$4.9442e-05$	$9.0814e-05$
		CPU-time (s)	0.1873	0.2751
	10^{-5}	n	89	48
		err	$4.3360e-06$	$8.7057e-06$
		CPU-time (s)	0.2877	0.4083
	10^{-6}	n	269	72
		err	$9.7636e-07$	$8.7536e-07$
		CPU-time (s)	0.6195	0.7357

TABLE 3. Table of numerical results with different choices of $\varrho_{t,n}$

the norm of the transfer mappings or the cost operators' inverse strong monotone coefficient. Thus, we can avoid the difficult task of estimating them. Two numerical experiments also show that our proposed algorithms have effectively performed better than several previous existing algorithms proposed in [8, 10, 15, 20].

TOL	Algorithms	n	err	CPU-time (s)
10^{-4}	ALGO-C	18980	$9.9994e-05$	0.4136
	ALGO-R1	35148	$9.9999e-05$	1.2278
	ALGO-R2	16733	$9.9993e-05$	1.1826
	ALGO-W	572	$9.9985e-05$	0.2361
10^{-5}	ALGO-C	60420	$9.9998e-06$	1.3188
	ALGO-R1	304782	$9.9999e-06$	10.4006
	ALGO-R2	81820	$9.9998e-06$	2.9567
	ALGO-W	2686	$9.9959e-06$	1.0471
10^{-6}	ALGO-C	191291	$9.9999e-07$	4.1379
	ALGO-R1	577617	$9.9999e-07$	19.8820
	ALGO-R2	411309	$9.9999e-07$	14.7744
	ALGO-W	13288	$9.9997e-07$	5.0946

TABLE 4. Table of numerical results

ACKNOWLEDGMENTS

The authors sincerely thank the editor and two anonymous referees for their careful reading, constructive comments and useful suggestions which helped us improve our paper. The first author was supported by the Science and Technology Fund of Thai Nguyen University of Technology.

REFERENCES

[1] Agarwal, R. P.; O'Regan, D.; Sahu, D. R. *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, 2009.
[2] Alber, Y.; Ryazantseva, I. *Nonlinear ill posed problems of monotone type*, Springer, Dordrecht, 2006.
[3] Bauschke, H.H. The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space. *J. Math. Anal. Appl.* **202** (1996), no. 1, 150–159 .
[4] Bauschke, H.H.; Borwein, J.M. On projection algorithms for solving convex feasibility problems. *SIAM Rev.* **38** (1996), no. 3, 367–426.
[5] Bauschke, H.H.; Combettes, P.L. Construction of best Bregman approximations in reflexive Banach spaces. *Proc. Amer. Math. Soc.* **131** (2003), no. 12, 3757–3766.
[6] Bauschke, H.H.; Combettes, P.L. *Convex analysis and monotone operator theory in Hilbert spaces*. Springer, New York, 2011.
[7] Cui, H. An approach for solving split common fixed point problems with multiple output sets that uses dynamic step sizes. *Numer. Funct. Anal. Optim.* **44** (2023), no. 15-16, 1669–1683 .
[8] Cui, H.; Wang, F. The split common fixed point problem with multiple output sets for demicontractive mappings. *Optimization.* **73** (2024), no. 6, 1933–1947, .
[9] Dong, Q.L.; Huang, J.Z.; Li, X.H.; Cho, Y.J.; Rassias, Th.M. MiKM: multi-step inertial Krasnosel'skii–Mann algorithm and its applications. *J. Global Optim.* **73** (2019), 801—824
[10] Ha, N.S.; Tuyen, T.M.; Huyen, P.T.V. Inertial proximal point algorithm for the split common solution problem of monotone operator equations. *Comp. Appl. Math.* **42** (2023), Art. ID 303.
[11] Hao, Y.; Zhao, J. Two-step inertial Bregman projection iterative algorithm for solving the split feasibility problem. *Applicable Nonlinear Analysis*. **1** (2024), 64–78.
[12] Jolaoso, L.O.; Shehu, Y.; Yao, J.C. Strongly convergent inertial proximal point algorithm without on-line rule. *J. Optim. Theory Appl.* **200** (2024), no. 2, 555–584.
[13] Kopecká, E.; Reich, S. A note on alternating projections in Hilbert space. *J. Fixed Point Theory Appl.* **12** (2012), 41–47.
[14] Okeke, C.C.; Jolaoso, L.O.; Shehu, Y. Inertial accelerated algorithms for solving split feasibility with multiple output sets in Hilbert spaces. *Int. J. Nonlinear Sci. Numer. Simul.* **24** (2023), no. 2, 769–790.
[15] Reich, S.; Tuyen, T.M.; Ha, M.T.N. The split feasibility problem with multiple output sets in Hilbert spaces. *Optim. Lett.* **14** (2020), 2335–2353 .
[16] Reich, S.; Tuyen, T.M.; Ha, N.S. Variational inequalities over the solution sets of split variational inclusion problems. *Appl. Numer. Math.* **192** (2023), 319–336 .

[17] Taddele, G.H.; Kumam, P.; Sunthrayuth, P.; Gebrie, A.G. Self-adaptive algorithms for solving split feasibility problem with multiple output sets. *Numer. Algorithms.* **92** (2023), no. 2, 1335–1366.
[18] Taiwo, A.; Reich, S.; Izuchukwu, C. Strong convergence of two regularized relaxed extragradient schemes for solving the split feasibility and fixed point problem with multiple output sets. *Appl. Anal.* **102** (2023), no. 18, 5132–5156.
[19] Tong, X.; Ling, T.; Shi, L. Self-adaptive relaxed CQ algorithms for solving split feasibility problem with multiple output sets. *J. Appl. Math. Comput.* **70** (2024), 1441–1469.
[20] Wang, F. The split feasibility problem with multiple output sets for demicontractive mappings. *J. Optim. Theory Appl.* **195** (2022), 837–853.
[21] Xu, H-K. A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. *Inverse Problems.* **22** (2006), 2021–2034 .

[1] THAINGUYEN UNIVERSITY OF TECHNOLOGY, THAINGUYEN, VIETNAM
Email address: phanthivanhuyen@tnut.edu.vn

[3] THAINGUYEN UNIVERSITY OF SCIENCES, THAINGUYEN, VIETNAM
Email address: hans@tnus.edu.vn

[3] THAINGUYEN UNIVERSITY OF SCIENCES, THAINGUYEN, VIETNAM
Email address: tuyentm@tnus.edu.vn