CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 555-567

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.19

Asymptotic modeling of non-linear viscopiezoelectric Kelvin-Voigt type plates via Trotter theory

THIBAUT WELLER, CHRISTIAN LICHT, SOMSAK ORANKITJAROEN, YOTSAWAT TERAPABKAJORNDED

ABSTRACT. We investigate the asymptotic behavior of the dynamic response of a thin viscopiezoelectric plate as its thickness, taken as a parameter, approaches zero. We use Trotter theory of convergence of semi-groups of operators acting on variable spaces. Depending on the kind of electrical loading and on the orders of magnitude of the density and of the viscosity, we highlight four different sensor and actuator behaviors.

1. INTRODUCTION

Technologies for creating advanced materials with specific physical couplings not easily find in nature have been the subject of intensive developments in recent times (see [7] in particular and the references therein). In most cases they are obtained through a complex internal structure design organized as a mixture of various compounds at micro and/or nanoscale. Predicting the mechanical properties of such materials requires precise and rigorous mathematical models which in turn help to steer research in the shaping of new materials with interesting properties. In this paper we focus our attention on non-linear viscoelastic of Kelvin-Voigt type materials exhibiting a piezoelectric coupling between the electric potential and the displacement. These materials are called viscopiezoelectric or electroviscoelastic (see [3] for a continuum mechanics approach). Some studies have been devoted to the static, quasi-static and even transient behavior of three-dimensional bodies made of such materials, with or without frictional contact and with no particular geometric properties (see [1, 2, 11] for example). Focusing on thin structures, one can find in [6] a mathematical analysis of a viscopiezoelectric plate involving a frictionless contact. This noteworthy article offers a rigorous asymptotic analysis carried out on a scaled domain in a quasi-static variational setting. It considers linear viscosity, isotropic elasticity but also homogeneous Neumann and Dirichlet electrical boundary conditions. Here, building on our previous works (see [14, 15, 16, 17, 19] in particular), our aim is to provide an asymptotic modeling of the *dynamic* behavior of thin viscopiezoelectric plates with frictional contact, non-linear viscosity and within a fully anisotropic setting. We also stress the critical influence of electrical boundary conditions on the behavior of the structure which may either act as a sensor or as an actuator. In addition, we establish our models via a convergence result of relative energetic gaps measured on the physical plate and not on a scaled domain. Recall that this kind of study relies on the study of the behavior of selected mathematical objects when some geometrical and mechanical data, understood as parameters, tend to their natural limits. To deal with the essential difficulty - a suitable definition of the limit of sequences of fields defined in variable domains - most of the studies first transform the genuine physical problem in an equivalent scaled one set in an abstract fixed domain and next proceed to a formal or rigorous analysis of convergence in standard function spaces. Although the aim is to propose an approximation of the real field of displacements, it is rare for a descaling to be subsequently carried out. But since even such a descaling does not provide a good approximation, here we present a method using an efficient and long-established notion of convergence well-suited to variable spaces, which has however remained rather curiously overlooked: the convergence in the sense of Trotter [18].

The variational formulation (P^s) describing the evolution of the thin plate Ω^ε subjected to electromechanical loading is set in Section 2. This problem is indexed by $s = (\varepsilon, \rho, b)$ containing the key physical data of the problem which respectively relates to the thickness of the plate as well as the density and the viscosity of its constitutive material.

In Section 3 we further formalize the fact that the generalized standard nature of this kind of physical systems implies that their dynamic response may be described by an equivalent differential inclusion

Received: 22.02.2024. In revised form: 05.11.2024. Accepted: 12.11.2024
2010 *Mathematics Subject Classification.* 74K20, 74F99, 74A05.
Key words and phrases. *Approximation of semi-groups in the sense of Trotter, asymptotic analysis, dimension reduction, maximal-monotone operators, piezoelectricity, thin plates, viscoelasticity of non-linear Kelvin-Voigt type.*
Corresponding author: Thibaut Weller; thibaut.weller@umontpellier.fr

$(\mathcal{P}^s)$ involving a maximal monotone operator $\mathcal{A}^s$ acting in a Hilbert space $\mathcal{Z}^s$ of possible electromechanical states $z = (y, v)$ with finite normalized energy. Here $y = (u, \varphi)$ and u, v and φ respectively stand for the fields of displacement, velocity and electric potential. Whenever possible, this kind of notation and choice of letters will be maintained. Existence and uniqueness of the solution z^s to $(\mathcal{P}^s)$ immediately follow. The true nature of this physical problem being quasi-static with respect to the electric potential and dynamic with respect to the displacement, it would be possible, as is done in [19] in the case of linear piezoelasticity, to eliminate the electric potential from $(\mathcal{P}^s)$. Our intent, however, among other things, is to illustrate how the introduction of additional variables, both mechanical and electrical, may refine the accuracy of the modeling, we chose to adopt an alternative method developed in [16].

Section 4 is therefore devoted to the study of the asymptotic behavior of z^s in its genuine form. The triplet s is then seen as a set of parameters going to some limit $\bar{s}$ in $\{0\} \times [0, +\infty) \times [0, +\infty]$. According to a non-linear extension [9] of Trotter theory of approximation of semi-groups of operators acting on variable spaces, this study reduces essentially to the behavior of the resolvents $(\mathbb{I} + \mathcal{A}^s)^{-1}$. Encoding both the nature of the electrical loading and the relative orders of magnitude of the physical data by way of an index I, the limit framework consisting of the limit spaces $\mathcal{Z}^{\mathrm{I}}$ and the limit operator $\mathcal{A}^{\mathrm{I}}$ are determined by studying the behavior of sequences of electromechanical states with uniformly bounded energy. This framework involves classical function spaces defined on an abstract fixed domain Ω. A crucial role is played by the linear operator $\mathrm{P}^{s\mathrm{I}}$ through which we associate to each element z of $\mathcal{Z}^{\mathrm{I}}$ a representative $\mathrm{P}^{s\mathrm{I}}$ in $\mathcal{Z}^s$ whose normalized energy converges toward $(|z|^s)^2$, the square of the norm of z, when s goes to $\bar{s}$. This means in particular that the "limit" problems $(\mathcal{P}^{\mathrm{I}})$ that arise should not be seen as describing our modeling as such but as a tool to build it: the unique solution z^{I} to $(\mathcal{P}^{\mathrm{I}})$ - which lives on Ω - has a representative $\mathrm{P}^{s\mathrm{I}}\, z^{\mathrm{I}}$ defined on the genuine physical structure Ω^ε and the relative energy gap between the genuine physical state z^s that solves $(\mathcal{P}^s)$ and $\mathrm{P}^{s\mathrm{I}}\, z^{\mathrm{I}}$ goes to zero! Because of the way this framework is built, very few proofs are needed. They have been gathered together at the end of this Section, in a dedicated part.

Section 5 presents the main properties of our various models. We highlight the discrepancies between sensors and actuators behaviors and show some of their characteristics. We also detail the way we construct our modeling and eventually recall that the strain of the real displacement is actually very far from a Kirchhoff-Love or even a Reissner-Mindlin field.

2. Setting the problem

Let $\{\, e_1, e_2, e_3 \,\}$ be an orthonormal basis of $\mathbb{R}^3$ equated with the Euclidean physical space. For all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\widehat{\xi}$ stands for (ξ_1, ξ_2). Throughout the paper Latin indices vary in the set $\{1, 2, 3\}$ while Greek ones take their values in $\{1, 2\}$. Let

$$\mathcal{H} := \mathbb{S}^3 \times \mathbb{R}^3 \tag{2.1}$$

where $\mathbb{S}^3$ denotes the space of all (3×3) real and symmetric matrices. For the sake of simplicity we will use the classical symbols $\cdot$ and $|\cdot|$ to respectively denote the inner product and norm in $\mathcal{H}$, $\mathbb{S}^3$, $\mathbb{R}^3$ and $\mathbb{R}^2$. For all ξ, ζ in $\mathbb{R}^3$, $\xi \otimes_s \zeta$ stands for the symmetrized tensor product of ξ by ζ. The set of all linear mappings from a space $\mathcal{V}$ into a space $\mathcal{W}$ is denoted by $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$. If $\mathcal{V} = \mathcal{W}$ we simply write $\mathrm{Lin}(\mathcal{V})$.

Here, within the context of small strains, we study the dynamic response of a viscopiezoelectric thin plate of non-linear Kelvin-Voigt type on which lives a physical state of the kind (u, φ) that is a couple (displacement field, electric potential) and subjected to a given electromechanical loading. The reference configuration of the plate of thickness 2ε is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$. Its middle surface ω is a bounded domain in $\mathbb{R}^2$ with a Lipchitz continuous boundary $\partial\omega$. The lateral part of the plate $\partial\omega \times [-\varepsilon, \varepsilon]$ is denoted by $\Gamma^\varepsilon_{\mathrm{lat}}$ while the set constituted by its lower and upper faces is $\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon\}$. Let $(\Gamma^\varepsilon_{\mathrm{mD}}, \Gamma^\varepsilon_{\mathrm{mN}})$, $(\Gamma^\varepsilon_{\mathrm{eD}}, \Gamma^\varepsilon_{\mathrm{eN}})$ two suitable partitions of $\partial\Omega^\varepsilon$ with both $\Gamma^\varepsilon_{\mathrm{mD}}$ and $\Gamma^\varepsilon_{\mathrm{eD}}$ of strictly positive Lebesgue measure. We assume that $\Gamma^\varepsilon_{\mathrm{mD}} = \gamma_0 \times (-\varepsilon, \varepsilon)$, with $\gamma_0 \subset \partial\omega$. The plate is, on one hand, clamped along $\Gamma^\varepsilon_{\mathrm{mD}}$ and at an electric potential φ^ε_a on $\Gamma^\varepsilon_{\mathrm{eD}}$ and, on the other hand, subjected to body forces f^ε and electrical loading F^ε in Ω^ε. Actually F^ε vanishes, the material being an insulator. Anyway, the following analysis remains valid if F^ε is different from 0. The plate is moreover subjected to surface forces g^ε and electrical loading d^ε on $\Gamma^\varepsilon_{\mathrm{mN}}$ and $\Gamma^\varepsilon_{\mathrm{eN}}$, respectively. The outward unit normal to $\partial\Omega^\varepsilon$ is denoted by n^ε.

The density $\rho\delta^\varepsilon$ of the plate, its piezoelectric coupling operator M^ε and the density of its viscous pseudo-potential of dissipation $b\mathcal{D}^\varepsilon$ satisfy:

$$\textbf{(H1)}\qquad \begin{cases} \rho > 0,\ b > 0, \\ \delta^\varepsilon \in L^\infty(\Omega^\varepsilon);\ \exists \alpha > 0 \text{ s.t. } \delta^\varepsilon(x^\varepsilon) \geq \alpha \text{ a.e. } x^\varepsilon \in \Omega^\varepsilon, \\ M^\varepsilon \in L^\infty\big(\Omega^\varepsilon; \mathrm{Lin}(\mathcal{H})\big);\ \alpha|h|^2 \leq M^\varepsilon(x^\varepsilon)h \cdot h \text{ a.e. } x^\varepsilon \in \Omega^\varepsilon, \forall h \in \mathcal{H}, \\ \exists q \in [1,2],\ \exists \beta > 0;\ -\alpha \leq \mathcal{D}^\varepsilon(x^\varepsilon, e) \leq \beta(1+|e|^q),\ \forall e \in \mathbb{S}^3,\ a.e.\ x^\varepsilon \in \Omega^\varepsilon, \end{cases}$$

Let $s := (\varepsilon, \rho, b)$ denote the key data of the structure. The fields of displacement u^s, electric potential φ^s and the velocity v^s living on the thin plate then satisfy:

$$(P^s)\qquad \begin{cases} \text{Find } z^s = (y^s, v^s), y^s = (u^s, \varphi^s),\ v^s = \dfrac{\partial u^s}{\partial t} \text{ sufficiently smooth on } \Omega^\varepsilon \times [0,T] \text{ such that:} \\ u^s = 0 \text{ on } \Gamma^\varepsilon_{\mathrm{mD}} \times [0,\mathrm{T}],\ \varphi^s = \varphi^\varepsilon_a \text{ on } \Gamma^\varepsilon_{\mathrm{eD}} \times [0,\mathrm{T}], \\ u^s(\cdot,0) = u^{s0}, v^s(\cdot,0) = v^{s0} \text{ in } \Omega^\varepsilon, \\ \displaystyle\int_{\Omega^\varepsilon} \rho\delta^\varepsilon \frac{\partial^2 u^s}{\partial t^2} \cdot w\, dx^\varepsilon + \int_{\Omega^\varepsilon} M^\varepsilon\, k^\varepsilon(y^s) \cdot k^\varepsilon(y)\, dx^\varepsilon + \int_{\Omega^\varepsilon} b\Big(\mathcal{D}^\varepsilon\big(x^\varepsilon, e^\varepsilon(v^s+w)\big) - \mathcal{D}^\varepsilon\big(x^\varepsilon, e^\varepsilon(v^s)\big)\Big)\, dx^\varepsilon \\ \qquad\qquad \displaystyle\geq \int_{\Omega^\varepsilon} f^\varepsilon \cdot w\, dx^\varepsilon + \int_{\Omega^\varepsilon} F^\varepsilon \psi\, dx^\varepsilon + \int_{\Gamma^\varepsilon_{\mathrm{mN}}} g^\varepsilon \cdot w\, d\mathfrak{h}_2 + \int_{\Gamma^\varepsilon_{\mathrm{eN}}} d^\varepsilon \psi\, d\mathfrak{h}_2, \\ \text{for all } y = (w,\psi) \text{ sufficiently smooth in } \Omega^\varepsilon \text{ such that } w = 0 \text{ on } \Gamma^\varepsilon_{\mathrm{mD}}, \psi = 0 \text{ on } \Gamma^\varepsilon_{\mathrm{eD}}, \end{cases}$$

where t is the time, $T > 0$, $\mathfrak{h}_n$ denotes the n-dimensional Hausdorff measure and

$$M^\varepsilon = \begin{bmatrix} M^\varepsilon_{\mathrm{mm}} & -M^\varepsilon_{\mathrm{me}} \\ (M^\varepsilon_{\mathrm{me}})^T & M^\varepsilon_{\mathrm{ee}} \end{bmatrix} \in \mathrm{Lin}(\mathcal{H}), \qquad k^\varepsilon(y) = k^\varepsilon(u,\varphi) := \begin{bmatrix} e^\varepsilon(u) \\ \nabla^\varepsilon \varphi \end{bmatrix} \in \mathcal{H}, \tag{2.2}$$

where $(M^\varepsilon_{\mathrm{mm}}, M^\varepsilon_{\mathrm{me}}, M^\varepsilon_{\mathrm{ee}}) \in \mathrm{Lin}(\mathbb{S}^3) \times \mathrm{Lin}(\mathbb{R}^3, \mathbb{S}^3) \times \mathrm{Lin}(\mathbb{R}^3)$ are respectively the elastic, piezoelectric and dielectric tensors and where $(M^\varepsilon_{\mathrm{me}})^T$ denotes the transpose of $M^\varepsilon_{\mathrm{me}}$, while $e^\varepsilon(u)$ is the linearized strain tensor associated with the displacement field u (the symmetric part of $\nabla^\varepsilon u$, the gradient of u with respect to x^ε-variable).

3. Existence and uniqueness

Classically, for all open set G of $\mathbb{R}^N$, $1 \leq N \leq 3$, $H^1_\gamma(G, \mathbb{R}^N)$ denotes the subset of the Sobolev space $H^1(G, \mathbb{R}^N)$ of elements with vanishing trace on γ included in ∂G. Moreover, for any Hilbert space H, $BV^1(0,\mathrm{T}\,; H)$ denotes the space of all elements in $BV(0,\mathrm{T}\,; H)$ with distributional time-derivatives in $BV(0,\mathrm{T}\,; H)$, the set of all elements in $L^1(0,\mathrm{T}\,; H)$ whose distributional time-derivatives is an H-valued bounded measure.

We assume that the electromechanical loading satisfies

$$\textbf{(H2)}\qquad \begin{cases} (f^\varepsilon, g^\varepsilon) \in BV^1\big(0,\mathrm{T}\,; L^2(\Omega^\varepsilon, \mathbb{R}^3) \times L^2(\Gamma^\varepsilon_{\mathrm{mN}}, \mathbb{R}^3)\big), \\ (F^\varepsilon, d^\varepsilon) \in BV^1\big(0,\mathrm{T}\,; L^2(\Omega^\varepsilon) \times L^2(\Gamma^\varepsilon_{\mathrm{eN}})\big), \\ \varphi^\varepsilon_a \text{ has a } BV^1\big(0,\mathrm{T}\,; H^1(\Omega^\varepsilon)\big) \text{ extension into } \Omega^\varepsilon \text{ still denoted by } \varphi^\varepsilon_a. \end{cases}$$

and seek $z^s = (u^s, \varphi^s, v^s)$ in the form

$$z^s = z^{se} + z^{sr}, \tag{3.1}$$

where the quasi-static electromechanical state field $z^{se}(t) := (y^{se}(t), 0) := (u^{se}(t), \varphi^{se}(t), 0)$ is given by

$$y^{se}(t) \in (0, \varphi^\varepsilon_a) + \mathcal{Y}^s; \quad \mathrm{m}^s(y^{se}, y) = L^s(t)(y), \quad \forall y \in \mathcal{Y}^s,\ \forall t \in [0,\mathrm{T}], \tag{3.2}$$

with

$$\begin{aligned} &\mathcal{Y}^s := H^1_{\Gamma^\varepsilon_{\mathrm{mD}}}(\Omega^\varepsilon, \mathbb{R}^3) \times H^1_{\Gamma^\varepsilon_{\mathrm{eD}}}(\Omega^\varepsilon), \\ &\mathrm{m}^s(y,z) := \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} M^\varepsilon\, k^\varepsilon(y) \cdot k^\varepsilon(z)\, dx^\varepsilon, \quad \forall y, z \in H^1(\Omega^\varepsilon, \mathbb{R}^3) \times H^1(\Omega^\varepsilon) \\ &L^s(t)(y) := \frac{1}{\varepsilon^3}\left(\int_{\Omega^\varepsilon} f^\varepsilon \cdot u\, dx^\varepsilon + \int_{\Omega^\varepsilon} F^\varepsilon \varphi\, dx^\varepsilon + \int_{\Gamma^\varepsilon_{\mathrm{mN}}} g^\varepsilon \cdot u\, d\mathfrak{h}_2 + \int_{\Gamma^\varepsilon_{\mathrm{eN}}} d^\varepsilon \varphi\, d\mathfrak{h}_2\right),\ \forall y = (u,\varphi) \in \mathcal{Y}^s. \end{aligned} \tag{3.3}$$

Due to assumptions **(H1)** and **(H2)**, y^{se} is well-defined and lives in $BV^1(0,\mathrm{T}\,; \mathcal{Y}^s)$.

Denoting the time derivative by an upper dot, the remaining part $z^{sr}(t) := \big(u^{sr}(t), \varphi^{sr}(t), v^{sr}(t) = \dot{u}^{sr}(t)\big)$ of z^s brings into play an evolution equation set in a Hilbert space $\mathcal{Z}^s$ of possible states with finite normalized total electromechanical energy governed by a maximal-monotone operator $\mathcal{A}^s$. To be more specific we introduce the bilinear forms

$$\mathbb{m}^s_{\text{sym}}(y,z) := \frac{1}{\varepsilon^3}\int_{\Omega^\varepsilon} M^\varepsilon_{\text{sym}}\, k^\varepsilon(y)\cdot k^\varepsilon(z)\, dx^\varepsilon, \quad \forall y,z\in\mathcal{Y}^s, \tag{3.4}$$

$$\mathbb{k}^s(v,w) := \frac{1}{\varepsilon^3}\int_{\Omega^\varepsilon} \rho\delta^\varepsilon\, v\cdot w\, dx^\varepsilon, \quad \forall v,w\in\mathcal{V}^s := L^2(\Omega^\varepsilon,\mathbb{R}^3), \tag{3.5}$$

with $M^\varepsilon_{\text{sym}} := \begin{bmatrix} M^\varepsilon_{\text{mm}} & 0 \\ 0 & M^\varepsilon_{\text{ee}} \end{bmatrix}$, and define the space $\mathcal{Z}^s := \mathcal{Y}^s\times\mathcal{V}^s$ endowed with the following inner product and norm

$$(z,z')^s := \mathbb{m}^s_{\text{sym}}(y,y') + \mathbb{k}^s(v,v'), \tag{3.6}$$

$$|z|^s := [(z,z)^s]^{1/2} = \left[\mathbb{m}^s(y,y)+\mathbb{k}^s(v,v)\right]^{\frac{1}{2}}, \quad \forall z=(y,v),\ z'=(y',v')\in\mathcal{Z}^s. \tag{3.7}$$

Introducing the normalized global viscous pseudo-potential of dissipation

$$\mathcal{D}^s(v) := \frac{b}{\varepsilon^3}\int_{\Omega^\varepsilon} \mathcal{D}^\varepsilon(x^\varepsilon, e^\varepsilon(v))\, dx^\varepsilon, \quad \forall v\in H^1_{\Gamma^\varepsilon_{\text{mD}}}(\Omega^\varepsilon,\mathbb{R}^3), \tag{3.8}$$

the multi-valued operator $\mathcal{A}^s$ in $\mathcal{Z}^s$ defined by

$$\begin{cases} \mathrm{D}(\mathcal{A}^s) := \Big\{ z=(u,\varphi,v)\in\mathcal{Z}^s;\ \text{i)}\ v\in H^1_{\Gamma^\varepsilon_{\text{mD}}}(\Omega^\varepsilon,\mathbb{R}^3), \\ \qquad\qquad \text{ii)}\ \exists(w,\psi)\in\mathcal{V}^s\times H^1_{\Gamma^\varepsilon_{\text{eD}}}(\Omega^\varepsilon)\ \text{s.t.} \\ \qquad \mathbb{k}^s(w,v') + \int_{\Omega^\varepsilon}\left(M^\varepsilon_{\text{mm}}\, e^\varepsilon(u) - M^\varepsilon_{\text{me}}\,\nabla^\varepsilon\varphi\right)\cdot e^\varepsilon(v')\, dx^\varepsilon + \mathcal{D}^s(v+v') - \mathcal{D}^s(v) \geq 0, \\ \qquad \int_{\Omega^\varepsilon}\left((M^\varepsilon_{\text{me}})^T e^\varepsilon(v) + M^\varepsilon_{\text{ee}}\,\nabla^\varepsilon\psi\right)\cdot\nabla^\varepsilon\varphi'\, dx^\varepsilon = 0, \quad \forall(v',\varphi')\in\mathcal{Y}^s \Big\}, \\ -\mathcal{A}^s z := \Big\{(v,\psi,w)\ \text{satisfying i) and ii) in the definition of}\ \mathrm{D}(\mathcal{A}^s)\Big\}, \end{cases}$$

obviously satisfies

Proposition 1. $\mathcal{A}^s$ is maximal-monotone and for all $\mathcal{X}^s = (\mathcal{X}^s_u,\mathcal{X}^s_\varphi,\mathcal{X}^s_v)\in\mathcal{Z}^s$

$$\begin{cases} \bar z^s = (\bar u^s,\bar\varphi^s,\bar v^s)\in\mathcal{Z}^s\ \text{s.t.} \\ \bar z^s + \mathcal{A}^s\,\bar z^s \ni \mathcal{X}^s \end{cases} \iff \begin{cases} \bar u^s = \bar v^s + \mathcal{X}^s_u, \\ (\bar v^s,\bar\varphi^s)\in\mathcal{Y}^s; \\ \mathbb{k}^s(\bar v^s,v') + \mathbb{m}^s\big((\bar v^s,\bar\varphi^s),y'\big) + \mathcal{D}^s(\bar v^s+v') - \mathcal{D}^s(\bar v^s) \\ \qquad\qquad \geq \mathbb{k}^s(\mathcal{X}^s_v,v') + \mathbb{m}^s_{\text{sym}}\big((-\mathcal{X}^s_u,\mathcal{X}^s_\varphi),y'\big), \\ \qquad\qquad\qquad \forall y'=(v',\varphi')\in\mathcal{Y}^s. \end{cases} \tag{3.9}$$

Observe that the very definition of z^{sr} implies

$$\int_{\Omega^\varepsilon}\left((M^\varepsilon_{\text{me}})^T e^\varepsilon(v^{sr}-\dot u^{se}) + M^\varepsilon_{\text{ee}}\,\nabla^\varepsilon\,\dot\varphi^{sr}\right)\cdot\nabla^\varepsilon\,\phi\, dx^\varepsilon = 0, \quad \forall\phi\in H^1_{\Gamma^\varepsilon_{\text{eD}}}(\Omega^\varepsilon). \tag{3.10}$$

Introducing ψ^{se} in $BV^1\big(0,T;H^1_{\Gamma^\varepsilon_{\text{eD}}}(\Omega^\varepsilon)\big)$ defined by

$$\psi^{se}(t)\in H^1_{\Gamma^\varepsilon_{\text{eD}}}(\Omega^\varepsilon)\ ;\ \int_{\Omega^\varepsilon} M^\varepsilon_{\text{ee}}\,\nabla^\varepsilon\,\psi^{se}(t)\cdot\nabla^\varepsilon\,\phi\, dx^\varepsilon = \int_{\Omega^\varepsilon}(M^\varepsilon_{\text{me}})^T e^\varepsilon\big(u^{se}(t)\big)\cdot\nabla^\varepsilon\,\phi\, dx^\varepsilon, \quad \forall\phi\in H^1_{\Gamma^\varepsilon_{\text{eD}}}(\Omega^\varepsilon), \tag{3.11}$$

the problem (P^s) is therefore formally equivalent to

$$(\mathcal{P}^s)\quad \begin{cases} \dfrac{dz^s}{dt} + \mathcal{A}^s\,(z^s - z^{se}) \ni (0,\dot\varphi^{se}+\dot\psi^{se},0), \\ z^s(0) = (u^{s0},\varphi^{s0},v^{s0}) =: z^{s0}, \end{cases}$$

as soon as initial conditions satisfy

$$\textbf{(H}^0\textbf{)}\qquad \int_{\Omega^\varepsilon}\left((M_{\text{me}})^T e^\varepsilon(u^{s0}) + M^\varepsilon_{\text{ee}}\,\nabla^\varepsilon\,\varphi^{s0}\right)\cdot\nabla^\varepsilon\,\phi\, dx^\varepsilon = L^s(0)(0,\phi), \quad \forall\phi\in H^1_{\Gamma^\varepsilon_{\text{eD}}}(\Omega^\varepsilon).$$

We therefore get

Theorem 1. Under assumptions **(H^0), (H1), (H2)** and

(H3) $\qquad z^{s0} \in z^{se}(0) + D(\mathcal{A}^s),$

the problem $(\mathcal{P}^s)$ has a unique solution z^s belonging to $W^{1,\infty}\big(0,T;\mathcal{Z}^s + (0,\varphi_a^\varepsilon,0)\big)$ and the first line of $(\mathcal{P}^s)$ is satisfied almost everywhere in $[0,T]$.

4. A MATHEMATICAL ANALYSIS OF THE ASYMPTOTIC BEHAVIOR OF z^s

Now we regard the key physical data $s = (\varepsilon,\rho,b)$ *as a triplet of parameters, taking values in a countable subset* $\mathfrak{S}$ of $(0,+\infty)^3$ with a unique cluster point $\bar{s}$ in $\{0\}\times[0,+\infty)\times[0,+\infty]$. Our work to date (see for examples [14, 15, 16, 17, 19]) leads us to consider eight several cases depending on the relative orders of magnitude of ε, ρ and b. We will index these cases by $\mathrm{I} = (\mathrm{I}_1,\mathrm{I}_2,\mathrm{I}_3)$ in $\{1,2\}\times\{1,2\}\times\{1,2\}$. More precisely we make the following hypothesis on the electrical loading:

(H4)
$$\begin{cases} \mathrm{I}_1 = 1 : \text{the extension of } \varphi_a^\varepsilon \text{ into } \Omega^\varepsilon \text{ does not depend on } x_3. \\ \mathrm{I}_1 = 2 : \text{the closure of the projection of } \Gamma_{\mathrm{eD}}^\varepsilon \text{ on } \omega \text{ coincides with } \overline{\omega}, \\ \qquad\qquad \text{and either } d^\varepsilon = 0 \text{ on } \Gamma_{\mathrm{eN}}^\varepsilon \cap \Gamma_{\mathrm{lat}}^\varepsilon \text{ or } \Gamma_{\mathrm{eN}}^\varepsilon \cap \Gamma_{\mathrm{lat}}^\varepsilon = \varnothing. \end{cases}$$

and let

$$\rho^{*\mathrm{I}_2} := \begin{cases} \rho\,\varepsilon^{-2} & \text{if } \mathrm{I}_2 = 1 \\ \rho & \text{if } \mathrm{I}_2 = 2 \end{cases}, \qquad b^{*\mathrm{I}_3} := b\varepsilon^{-(2-q)} \quad \text{for } \mathrm{I}_3 = 1,2.$$

We then make the following assumption relative to the orders of magnitude of density, thickness and viscosity:

(H5)
$$\begin{cases} \text{there exists } (\bar{\rho}^{\mathrm{I}_2},\bar{b}^{\mathrm{I}_3}) \text{ in } [0,+\infty)\times[0,+\infty] \text{ such that} \\ \bar{\rho}^{\mathrm{I}_2} := \lim\limits_{s\to\bar{s}} \rho^{*\mathrm{I}_2}, \\ \bar{b}^{\mathrm{I}_3} = \lim\limits_{s\to\bar{s}} b^{*\mathrm{I}_3} \ \text{ with } \bar{b}^1 < +\infty \text{ and } \bar{b}^2 = +\infty. \end{cases}$$

Classically we are led to consider the fixed open set $\Omega := \omega\times(-1,1)$ through the mapping π^ε:

$$x = (\widehat{x},x_3) \in \overline{\Omega} \mapsto \pi^\varepsilon\, x = (\widehat{x},\varepsilon x_3) \in \overline{\Omega}^\varepsilon. \tag{4.1}$$

In the sequel we will therefore systematically connect x^ε and x through $x^\varepsilon = \pi^\varepsilon x$ and, similarly, the index ε will be dropped for the notation of the inverse images of $\Gamma_\pm^\varepsilon$, $\Gamma_{\mathrm{lat}}^\varepsilon$, $\Gamma_{\mathrm{eD}}^\varepsilon$, $\Gamma_{\mathrm{mD}}^\varepsilon$, $\Gamma_{\mathrm{eN}}^\varepsilon$ and $\Gamma_{\mathrm{mN}}^\varepsilon$ by $(\pi^\varepsilon)^{-1}$. To build a meaningful mathematical connection between the fields living on the real plate Ω^ε and those living on Ω, we make the following assumption **(H6)** on the density of the material, its piezoelectric tensor, its viscous pseudo-potential of dissipation and the loading of the thin plate:

(H6)
$$\begin{cases} \exists\,(\delta,M) \in L^\infty\big(\Omega,\mathbb{R}\times\mathrm{Lin}(\mathcal{H})\big) \text{ s.t.} \\ \qquad \alpha \le \delta(x),\ \delta^\varepsilon(x^\varepsilon) = \delta(x) \text{ a.e. } x\in\Omega, \\ \qquad \beta|h|^2 \le M(x)\,h\cdot h,\ \forall h\in\mathcal{H}, \quad M^\varepsilon(x^\varepsilon) = M(x),\ \text{a.e. } x\in\Omega, \\ \exists\,\mathcal{D} \text{ measurable in } \Omega, \text{ convex on } \mathbb{S}^3 \text{ s.t.} \\ \qquad \left.\begin{array}{l} \exists\, q\in[1,2],\ -\alpha \le \mathcal{D}(x,e) \le \beta(1+|e|^q) \\ \mathcal{D}^\varepsilon(x^\varepsilon,e) = \mathcal{D}(x,e) \end{array}\right\} \forall e\in\mathbb{S}^3,\ \text{a.e. } x\in\Omega, \\ \exists(f,F,g,d,\varphi_a) \in BV^1\big(0,\mathrm{T}\,;L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega)\times L^2(\Gamma_{\mathrm{mN}},\mathbb{R}^3)\times L^2(\Gamma_{\mathrm{eN}})\times H^1(\Omega)\big) \text{ s.t.} \\ \qquad \widehat{f^\varepsilon}(x^\varepsilon) = \varepsilon\widehat{f}(x), \quad f_3^\varepsilon(x^\varepsilon) = \varepsilon^2 f_3(x), \quad F^\varepsilon(x^\varepsilon) = \varepsilon^{2-\mathrm{I}_1}F(x), \quad \forall x\in\Omega, \\ \qquad \widehat{g^\varepsilon}(x^\varepsilon) = \varepsilon^2\widehat{g}(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon^3 g_3(x), \quad \forall x\in\Gamma_{\mathrm{mN}}\cap\Gamma_\pm, \\ \qquad \widehat{g^\varepsilon}(x^\varepsilon) = \varepsilon\widehat{g}(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon^2 g_3(x), \quad \forall x\in\Gamma_{\mathrm{mN}}\cap\Gamma_{\mathrm{lat}}, \\ \qquad d^\varepsilon(x^\varepsilon) = \varepsilon^{3-\mathrm{I}_1}d(x),\ \forall x\in\Gamma_{\mathrm{eN}}\cap\Gamma_\pm, \\ \qquad d^\varepsilon(x^\varepsilon) = \varepsilon^{2-\mathrm{I}_1}d(x),\ \forall x\in\Gamma_{\mathrm{eN}}\cap\Gamma_{\mathrm{lat}}, \\ \qquad \varphi_a^\varepsilon(x^\varepsilon) = \varepsilon^{\mathrm{I}_1}\varphi_a(x),\ \forall x\in\Gamma_{\mathrm{eD}}. \end{cases}$$

It should be noted that $\mathrm{I}_1 = 1$ corresponds to sensor like electrical boundary conditions while $\mathrm{I}_1 = 2$ deals with viscopiezoelectric plates used as actuators.

From now on, C and c denote various constants which may vary from line to line and we use the convention $0 \times \infty = \infty \times 0 = 0$.

4.1. **A potential contender for the limit behavior.** Here we study the asymptotic behavior of sequences with bounded normalized total electromechanical energy.

4.1.1. *Building the limit functional framework.* Depending on the electrical loading, we introduce the scaling mapping

$$
\begin{aligned}
\mathcal{S}_\varepsilon^{\mathrm{I}_1} \quad : \quad L^2(\Omega^\varepsilon, \mathbb{R}^3 \times \mathbb{R}) &\longrightarrow L^2(\Omega, \mathbb{R}^3 \times \mathbb{R}) \\
(w, \psi) &\longmapsto \big(\mathcal{S}_{u\varepsilon}\, w(x), \mathcal{S}_{\varphi\varepsilon}^{\mathrm{I}_1}\, \psi(x)\big), \\
\mathcal{S}_{u\varepsilon}\, w(x) &:= \Big(\frac{1}{\varepsilon}\widehat{w}(x^\varepsilon), w_3(x^\varepsilon)\Big), \\
\mathcal{S}_{\varphi\varepsilon}^{\mathrm{I}_1}\, \psi(x) &:= \frac{1}{\varepsilon^{\mathrm{I}_1}}\psi(x^\varepsilon).
\end{aligned} \tag{4.2}
$$

This leads to

$$
\begin{aligned}
k^\varepsilon(w,\psi) &= \big(e^\varepsilon(w), \nabla^\varepsilon \psi\big)(x^\varepsilon) \\
&= \varepsilon k^{\mathrm{I}_1}\big(\varepsilon, \mathcal{S}_\varepsilon^{\mathrm{I}_1}\,(w,\psi)\big)(x) \\
&= \varepsilon\big(e(\varepsilon, \mathcal{S}_{u\varepsilon}\, w), g^{\mathrm{I}_1}(\varepsilon, \mathcal{S}_{\varphi\varepsilon}^{\mathrm{I}_1}\,\psi)\big), \quad \forall (w,\psi) \in H^1(\Omega^\varepsilon, \mathbb{R}^3 \times \mathbb{R}),
\end{aligned} \tag{4.3}
$$

where

$$
\begin{aligned}
e_{ij}(\varepsilon, w) &:= \begin{cases} e_{ij}(w) & \text{for } 1 \le i \le j \le 2, \\ \frac{1}{\varepsilon}\, e_{ij}(w) & \text{for } 1 \le i \le 2, j = 3, \\ e_{ji}(\varepsilon, w) & \text{for } 1 \le j < i \le 3, \\ \frac{1}{\varepsilon^2}\, e_{33}(w) & \text{for } i = j = 3, \end{cases}, \\
e_{ij}(w) &:= \frac{1}{2}(\partial_i w_j + \partial_j w_i), \\
g^1(\varepsilon, \psi) &:= \Big(\widehat{\nabla}\psi, \frac{1}{\varepsilon}\partial_3 \psi\Big), \\
g^2(\varepsilon, \psi) &:= (\varepsilon\widehat{\nabla}\psi, \partial_3\psi), \\
\widehat{\nabla}\psi &:= (\partial_1\psi, \partial_2\psi),\ \forall \psi \in H^1(\Omega).
\end{aligned} \tag{4.4}
$$

For all y, y' in $\mathcal{Y}^s$ and all v, v' in $\mathcal{V}^s$, the bilinear forms $\mathbb{m}^s, \mathbb{m}^s_{\text{sym}}, \mathbb{k}^s$ therefore read as:

$$
\begin{aligned}
\mathbb{m}^s(y, y') &= \int_\Omega M\, k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}_\varepsilon^{\mathrm{I}_1}\, y) \cdot k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}_\varepsilon^{\mathrm{I}_1}\, y')\, dx, \\
\mathbb{m}^s_{\text{sym}}(y, y') &= \int_\Omega M_{\text{sym}}\, k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}_\varepsilon^{\mathrm{I}_1}\, y) \cdot k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}_\varepsilon^{\mathrm{I}_1}\, y')\, dx, \\
\mathbb{k}^s(v, v') &= \rho \int_\Omega \delta(\widehat{\mathcal{S}_{u\varepsilon}\, v}) \cdot (\widehat{\mathcal{S}_{u\varepsilon}\, v'})\, dx + \frac{\rho}{\varepsilon^2}\int_\Omega \delta(\mathcal{S}_{u\varepsilon}\, v)_3 (\mathcal{S}_{u\varepsilon}\, v')_3\, dx,
\end{aligned} \tag{4.5}
$$

with of course $M = \begin{bmatrix} M_{\text{mm}} & -M_{\text{me}} \\ (M_{\text{me}})^T & M_{\text{ee}} \end{bmatrix}$ and $M_{\text{sym}} := \frac{1}{2}\big(M + M^T\big)$.

Let the space of "abstract Kirchhoff-Love displacement fields" be defined by

$$
V_{\text{KL}} := \big\{\, w \in H^1_{\Gamma_{\text{mD}}}(\Omega, \mathbb{R}^3);\ e_{i3}(w) = 0 \,\big\}. \tag{4.6}
$$

Before highlighting the appropriate space to describe the asymptotic behavior of the unique solution z^s to $(\mathcal{P}^s)$, we define:

$$
(4.7)\qquad
\begin{cases}
\mathcal{U}^0 := V_{\mathrm{KL}}, \quad \mathcal{U}^1 := H^1\big(-1,1;L^2(\omega,\mathbb{R}^3)\big)/L^2(\omega,\mathbb{R}^3),\\
\mathcal{U} := \mathcal{U}^0 \times \mathcal{U}^1,\\
E_u := e(u^0) + \partial_3 u^1 \otimes_s e_3, \quad \forall u = (u^0,u^1) \in \mathcal{U},\\
\mathcal{V}^1 := L^2(\Omega), \quad \mathcal{V}^2 := \{v \in L^2(\Omega,\mathbb{R}^3); v_3 = 0\},\\
\Phi^{1|0} := \{\varphi \in H^1_{\Gamma_{\mathrm{eD}}}(\Omega); \partial_3\varphi = 0\}, \qquad \Phi^{2|0} := H^1(-1,1;L^2(\omega)),\\
\Phi^{1|1} := \Phi^{2|0}, \qquad \Phi^{2|1} := \{0\},\\
\Phi^{\mathrm{I}_1} := \Phi^{\mathrm{I}_1|0} \times \Phi^{\mathrm{I}_1|1},\\
G^1_\varphi := (\widehat{\nabla}\varphi^0, \partial_3\varphi^1), \quad G^2_\varphi := (0,0,\partial_3\varphi^0), \quad \forall \varphi = (\varphi^0,\varphi^1) \in \Phi^{\mathrm{I}_1},\\
\mathcal{Y}^{\mathrm{I}} := \mathcal{U} \times \Phi^{\mathrm{I}_1},\\
K^{\mathrm{I}_1}_y := (E_u, G^{\mathrm{I}_1}_\varphi), \quad \forall y = (u,\varphi) \in \mathcal{Y}^{\mathrm{I}},\\
\mathcal{Z}^{\mathrm{I}} := \mathcal{Y}^{\mathrm{I}} \times \mathcal{V}^{\mathrm{I}_2},\\
\left.\begin{array}{l}
\mathbb{m}^{\mathrm{I}_1}(y,y') := \int_\Omega M\, K^{\mathrm{I}_1}_y \cdot K^{\mathrm{I}_1}_{y'}\, dx,\\
\mathbb{m}^{\mathrm{I}_1}_{\mathrm{sym}}(y,y') := \int_\Omega M_{\mathrm{sym}}\, K^{\mathrm{I}_1}_y \cdot K^{\mathrm{I}_1}_{y'}\, dx,
\end{array}\right\} \forall y,y' \in \mathcal{Y}^{\mathrm{I}},\\
\mathbb{k}^1(w,w') := \overline{\rho}^1 \displaystyle\int_\Omega \delta\, w_3\, w'_3\, dx, \quad \forall w,w' \in \mathcal{V}^1,\\
\mathbb{k}^2(w,w') := \overline{\rho}^2 \displaystyle\int_\Omega \delta\, \widehat{w} \cdot \widehat{w'}\, dx, \quad \forall w,w' \in \mathcal{V}^2,\\
\left.\begin{array}{l}
(z,z')^{\mathrm{I}} := \mathbb{m}^{\mathrm{I}_1}_{\mathrm{sym}}(y,y') + \mathbb{k}^{\mathrm{I}_2}(v,v'),\\
|z|^{\mathrm{I}} := \big[(z,z)^{\mathrm{I}}\big]^{\frac{1}{2}},
\end{array}\right\} \quad \forall z = (y,v), z' = (y',v') \in \mathcal{Z}^{\mathrm{I}}.
\end{cases}
$$

Clearly $\mathcal{Z}^{\mathrm{I}}$ equipped with the inner product $(\cdot,\cdot)^{\mathrm{I}}$ is a Hilbert space. The fact that this is the proper functional framework to describe the asymptotic behavior stems from the following two propositions whose proofs are to be found in subsection 4.3:

Proposition 2. For every $\mathcal{X}^s = (\mathcal{X}^s_y, \mathcal{X}^s_v) = (\mathcal{X}^s_u, \mathcal{X}^s_\varphi, \mathcal{X}^s_v)$ in $\mathcal{Z}^s$ such that $|\mathcal{X}^s|^s$ is uniformly bounded, there exists $\mathcal{X}^{\mathrm{I}} = (\mathcal{X}^{\mathrm{I}}_y, \mathcal{X}^{\mathrm{I}}_v) = (\mathcal{X}^{\mathrm{I}}_u, \mathcal{X}^{\mathrm{I}}_\varphi, \mathcal{X}^{\mathrm{I}}_v)$ in $\mathcal{Z}^{\mathrm{I}}$ and a not relabeled subsequence such that

i) $(K^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_y}, \mathcal{X}^{\mathrm{I}}_v)$ is the weak limit in $L^2\big(\Omega, \mathcal{H} \times \mathbb{R}^{2\mathrm{I}_2-1}\big)$ of
$\Big(k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}^{\mathrm{I}_1}_\varepsilon\, \mathcal{X}^s_y), \big(0,0,(\mathcal{S}_{u\varepsilon}\, \mathcal{X}^s_v)_3\big)\Big)$ when $\mathrm{I}_2 = 1$ (case of sensors) or of
$\Big(k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}^{\mathrm{I}_1}_\varepsilon\, \mathcal{X}^s_y), \mathcal{S}_{u\varepsilon}\, \mathcal{X}^s_v\Big)$ when $\mathrm{I}_2 = 2$ (case of actuators),

ii) $|\mathcal{X}^{\mathrm{I}}|^{\mathrm{I}} \le \underline{\lim}_{s\to\bar{s}} |\mathcal{X}^s|^s$.

Proposition 3. For all s in $\mathfrak{S}$ and all $z = (y,v)$ in $\mathcal{Z}^{\mathrm{I}}$, let $\mathrm{P}^{s\mathrm{I}}\, z := (\mathrm{P}^{s\mathrm{I}}_y\, y, \mathrm{P}^{s\mathrm{I}}_v\, v)$ in $\mathcal{Z}^s$ be defined by

$$
(4.8)\qquad \mathbb{m}^s(\mathrm{P}^{s\mathrm{I}}_y\, y, y') := \int_\Omega M\, K^{\mathrm{I}_1}_y \cdot k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}^{\mathrm{I}_1}_\varepsilon\, y')\, dx, \quad \forall y' \in \mathcal{Y}^s,
$$

$$
(4.9)\qquad \mathbb{k}^s(\mathrm{P}^{s\mathrm{I}}_v\, v, v') := \mathbb{k}^{\mathrm{I}_2}(v, \mathcal{S}_{u\varepsilon}\, v'), \quad \forall v' \in \mathcal{V}^s.
$$

There holds:

(P1) $\exists C > 0$ s.t. $|\mathrm{P}^{s\mathrm{I}}\, z|^s \le C\, |z|^{\mathrm{I}}, \quad \forall z \in \mathcal{Z}^{\mathrm{I}}, \quad \forall s \in \mathfrak{S}$,

(P2) $\lim_{s\to\bar{s}} |\mathrm{P}^{s\mathrm{I}}\, z|^s = |z|^{\mathrm{I}}, \quad \forall z \in \mathcal{Z}^{\mathrm{I}}$,

(P3) i. $\displaystyle\lim_{s\to\bar{s}} \frac{1}{\varepsilon^3} \int_{\Omega^\varepsilon} M^\varepsilon\, [k^\varepsilon(\mathrm{P}^{s\mathrm{I}}_y\, y) - K^{\varepsilon\mathrm{I}}_y] \cdot [k^\varepsilon(\mathrm{P}^{s\mathrm{I}}_y\, y) - K^{\varepsilon\mathrm{I}}_y]\, dx^\varepsilon = 0$,

with $K^{\varepsilon\mathrm{I}}_y(x^\varepsilon) := \varepsilon K^{\mathrm{I}_1}_y(x)$, a.e. $x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$, $\forall y \in \mathcal{Y}^{\mathrm{I}}$.

ii. $\mathrm{P}^{s\mathrm{I}}_v\, v = V^{\varepsilon\mathrm{I}}_v := \begin{cases} \dfrac{\overline{\rho}^{\mathrm{I}_2}}{\rho^{*\mathrm{I}_2}} (\mathcal{S}_{u\varepsilon})^{-1}\,(0,0,v_3), & \text{if } \mathrm{I}_2 = 1,\\[2mm] \dfrac{\overline{\rho}^{\mathrm{I}_2}}{\rho^{*\mathrm{I}_2}} (\mathcal{S}_{u\varepsilon})^{-1}\, v, & \text{if } \mathrm{I}_2 = 2. \end{cases}$

Remark 1. Property (P2) states that any element z of $\mathcal{Z}^{\mathrm{I}}$ has a representative $\mathrm{P}^{s\mathrm{I}}\, z$ in $\mathcal{Z}^s$ whose energy $(\mathrm{P}^{s\mathrm{I}}\, z, \mathrm{P}^{s\mathrm{I}}\, z)^s$ is arbitrarily close to the square of the norm of z in $\mathcal{Z}^{\mathrm{I}}$, ensuring that it is an appropriate space to describe the asymptotic behavior. Observe also that (4.7) implies that the "abstract velocities" living in the space $\mathcal{V}^{\mathrm{I}_2}$ involve their sole transverse component when $\mathrm{I}_2 = 1$ and only their in-plane components when $\mathrm{I}_2 = 2$.

4.1.2. *The limit operator* $\mathcal{A}^{\mathrm{I}}$. Denoting the identity operator in any space by $\mathbb{I}$, we now examine the asymptotic behavior of the resolvent $(\mathbb{I} + \mathcal{A}^s)^{-1}$ of $\mathcal{A}^s$ in order to complete the building of our framework and pinpoint the limit operator $\mathcal{A}^{\mathrm{I}}$. To this aim, relying on Proposition 1, we consider sequences z^s in $\mathcal{Z}^s$ with uniformly bounded global viscous pseudo-potentials of dissipation $\mathcal{D}^s(z^s)$ and total electromechanical energy functional $[|(z^s, z^s)|^s]^2$. Note that from a strictly mathematical point of view, z^s is such that $z^s + \mathcal{A}^s z^s$ is uniformly bounded in $\mathcal{Z}^s$. This suggests that the space $\widetilde{\mathcal{Y}^{\mathrm{I}}}$ of admissible virtual generalized "velocities" and the limit global pseudo-potential of dissipation $\mathcal{D}^{\mathrm{I}}$ write:

$$
\begin{aligned}
&\widetilde{\mathcal{Y}^{\mathrm{I}}} := \Big\{ y = (u, \varphi) \in \mathcal{Y}^{\mathrm{I}};\, (u^0)_3 = 0 \text{ if } \mathrm{I}_2 = 2 \Big\}, \\
&\mathcal{D}^{\mathrm{I}}(v) := \begin{cases} \bar{b}^1 \displaystyle\int_\Omega \mathcal{D}(E_v)\, dx, & \text{if } \mathrm{I}_3 = 1 \\ \boldsymbol{I}_{\{0\}}(v), & \text{if } \mathrm{I}_3 = 2 \end{cases}, \quad \forall v \in \mathcal{U},
\end{aligned}
\tag{4.10}
$$

where $\boldsymbol{I}_{\{0\}}$ is the indicator function defined by:

$$
\boldsymbol{I}_{\{0\}}(v) := \begin{cases} 0 & \text{if } v = 0, \\ +\infty & \text{if } v \neq 0. \end{cases}
\tag{4.11}
$$

A simple argument of lower semi-continuity together with Proposition 2 thus implies:

Proposition 4. For all sequences $\tilde{y}^s = (\tilde{u}^s, \tilde{\varphi}^s)$ in $\mathcal{Y}^s$ such that $\left[|(\tilde{y}^s, \tilde{u}^s)|^s\right]^2 + \mathcal{D}^s(\tilde{u}^s) \leq C$, there exists a not relabeled subsequence and $\tilde{y} = (\tilde{u}, \tilde{\varphi})$ in $\widetilde{\mathcal{Y}^{\mathrm{I}}}$ such that $k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}_\varepsilon^{\mathrm{I}_1} \tilde{y}^s)$ weakly converges in $L^2(\Omega, \mathbb{S}^3 \times \mathbb{R}^{5-2\mathrm{I}_1})$ toward $K^{\mathrm{I}}_{\tilde{y}}$ and:

$$
\left[|(\tilde{y}, (\mathring{\tilde{u}})^{\mathrm{I}})|^{\mathrm{I}}\right]^2 + \mathcal{D}^{\mathrm{I}}(\tilde{u}) \leq \varliminf_{s \to \bar{s}} \left(\left[|(\tilde{y}^s, \tilde{u}^s)|^s\right]^2 \right) + \mathcal{D}^s(\tilde{u}^s),
$$

where $(\mathring{\tilde{u}})^{\mathrm{I}} = (u^0)_3$ if $\mathrm{I}_2 = 1$, $(\mathring{\tilde{u}})^{\mathrm{I}} = u^0$ if $\mathrm{I}_2 = 2$, for all $u = (u^0, u^1)$ in $\mathcal{U}$.

Remark 2. It is already clear from **(H6)** that thin viscopiezoelectric plates may undergo electric potentials of different orders of magnitudes. The definition of $\mathcal{Y}^{\mathrm{I}}$ in (4.7) goes further and shows the influence this fact has on the limit equations: when the plate is used as a sensor, an additional 'microscale' electric potential appears whereas actuator behavior implies the transverse gradient of the electric potential. Proposition 4 highlights another property, this time related to displacements and velocities: when s goes to $\bar{s}$, the global pseudo-potential of dissipation $\mathcal{D}^{\mathrm{I}}$ and the kinetic energy do not involve the same variables anymore, $\mathcal{D}^{\mathrm{I}}$ being defined on $\mathcal{U}^0 \times \mathcal{U}^1$ while the kinetic energy is only defined through specific elements of $\mathcal{U}^0$.

Taking advantage of the concept of multi-valued operators, we introduce the operator $\mathcal{A}^{\mathrm{I}}$ defined by:

- When $\mathrm{I}_3 = 1$:

$$
\begin{cases}
\mathrm{D}(\mathcal{A}^{\mathrm{I}}) := \Big\{ z = (u, \varphi, v) = (y, v) \in \mathcal{Z}^{\mathrm{I}}; \quad \text{i) } \exists (\tilde{v}, 0) \in \widetilde{\mathcal{Y}^{\mathrm{I}}} \text{ s.t. } (\mathring{\tilde{v}})^{\mathrm{I}} = v, \\
\qquad\qquad \text{ii) } \exists (w, \psi) \in \mathcal{V}^{\mathrm{I}_2} \times \varPhi^{\mathrm{I}_1} \text{ s.t.} \\
\qquad \bullet)\ \Bbbk^{\mathrm{I}_2}\big(w, (\mathring{v}')^{\mathrm{I}}\big) + \displaystyle\int_\Omega (M_{\mathrm{mm}}\, E_u - M_{\mathrm{me}}\, G^{\mathrm{I}_1}_\varphi) \cdot E_{v'}\, dx + \mathcal{D}^{\mathrm{I}}(\tilde{v} + v') - \mathcal{D}^{\mathrm{I}}(\tilde{v}) \geq 0 \\
\qquad \bullet\bullet)\ \displaystyle\int_\Omega \big((M_{\mathrm{me}})^T E_{\tilde{v}} + M_{\mathrm{ee}}\, G^{\mathrm{I}_1}_\psi\big) \cdot G^{\mathrm{I}_1}_{\varphi'}\, dx = 0, \quad \forall (v', \varphi') \in \widetilde{\mathcal{Y}^{\mathrm{I}}} \Big\} \\
-\mathcal{A}^{\mathrm{I}}\, z := \{(\tilde{v}, \psi, w) \text{ satisfying i) and ii) of the definition of } \mathrm{D}(\mathcal{A}^{\mathrm{I}})\},
\end{cases}
$$

- When $\mathrm{I}_3 = 2$:

$$
\begin{cases}
D(\mathcal{A}^{\mathrm{I}}) := \mathcal{Y}^{\mathrm{I}} \times \{0\}, \\
-\mathcal{A}^{\mathrm{I}}\, z = \{0\} \times \mathcal{V}^{\mathrm{I}_2},
\end{cases}
$$

the very definition of which implies

Proposition 5. The operator $\mathcal{A}^{\mathrm{I}}$ is maximal-monotone and for all $\mathcal{X} = (\mathcal{X}_u, \mathcal{X}_\varphi, \mathcal{X}_v) \in \mathcal{Z}^{\mathrm{I}}$, when $\mathrm{I}_3 = 1$:

$$\begin{cases} \bar{z}^{\mathrm{I}} = (\bar{u}^{\mathrm{I}}, \bar{\varphi}^{\mathrm{I}}, \bar{v}^{\mathrm{I}}) \in \mathcal{Z}^{\mathrm{I}} \text{ s.t.} \\ \bar{z}^{\mathrm{I}} + \mathcal{A}^{\mathrm{I}}\bar{z}^{\mathrm{I}} \ni \mathcal{X} \end{cases} \iff \begin{cases} \bar{z}^{\mathrm{I}} = \big(\bar{v}^{\mathrm{I}} + \mathcal{X}_u, \bar{\varphi}^{\mathrm{I}}, (\mathring{\bar{v}})^{\mathrm{I}}\big) \text{ where} \\ \mathbb{k}^{\mathrm{I}_2}\big((\mathring{\bar{v}})^{\mathrm{I}}, v'\big) + \mathbb{m}^{\mathrm{I}_1}\big((\bar{v}^{\mathrm{I}}, \bar{\varphi}^{\mathrm{I}}), y'\big) + \mathcal{D}^{\mathrm{I}}(\bar{v}^{\mathrm{I}} + v') - \mathcal{D}^{\mathrm{I}}(\bar{v}^{\mathrm{I}}) \\ \qquad\qquad \geq \mathbb{k}^{\mathrm{I}_2}\big(\mathcal{X}_v, (\mathring{v}')^{\mathrm{I}}\big) + \mathbb{m}^{\mathrm{I}_1}_{\text{sym}}\big((-\mathcal{X}_u, \mathcal{X}_\varphi), y'\big), \\ \qquad\qquad\qquad \forall y' = (v', \varphi') \in \widetilde{\mathcal{Y}^{\mathrm{I}}}. \end{cases}$$

When $\mathrm{I}_3 = 2$ we have: $\bar{z}^{\mathrm{I}} + \mathcal{A}^{\mathrm{I}}\bar{z}^{\mathrm{I}} \ni \mathcal{X} \iff (\bar{u}^{\mathrm{I}}, \bar{\varphi}^{\mathrm{I}}, \bar{v}^{\mathrm{I}}) = (\mathcal{X}_u, \mathcal{X}_\varphi, 0)$.

Finally we consider $z^{\mathrm{Ie}} := (y^{\mathrm{Ie}}, 0)$ in $\mathcal{Z}^{\mathrm{I}} + \big(0, (\varphi_a, 0), 0\big)$ with $y^{\mathrm{Ie}} := (u^{\mathrm{Ie}}, \varphi^{\mathrm{Ie}})$ the solution to

$$\text{(4.12)} \quad \begin{aligned} &y^{\mathrm{Ie}}(t) \in \mathcal{Y}^{\mathrm{I}} + (0, (\varphi_a, 0))\,; \quad \mathbb{m}^{\mathrm{I}_1}(y^{\mathrm{Ie}}, y) = L(t)(y) := L^s(t)\Big(\big((\mathcal{S}_{u\varepsilon})^{-1} w^0, (\mathcal{S}^{\mathrm{I}_1}_{\varphi\varepsilon})^{-1} \psi^0\big)\Big), \\ &\qquad\qquad \forall y = \big((w^0, w^1), (\psi^0, \psi^1)\big) \in \mathcal{Y}^{\mathrm{I}}, \forall t \in [0, T], \end{aligned}$$

and introduce ψ^{Ie} in $BV^1(0, T; \Phi^{\mathrm{I}_1})$ defined for all t in $[0, T]$ by

$$\text{(4.13)} \qquad \psi^{\mathrm{Ie}}(t) \in \Phi^{\mathrm{I}_1}\,; \quad \int_\Omega M_{\mathrm{ee}}\, G^{\mathrm{I}_1}_{\psi^{\mathrm{Ie}}(t)} \cdot G^{\mathrm{I}_1}_\phi \,dx = \int_\Omega (M_{\mathrm{me}})^T E_{u^{\mathrm{Ie}}(t)} \cdot G^{\mathrm{I}_1}_\phi \,dx, \ \forall \phi \in \Phi^{\mathrm{I}_1}.$$

As with the operator $\mathcal{A}^s$, Propositions 3.2 and 3.3 in [4] then yield:

Theorem 2. Under assumptions **(H^0)**, **(H1)–(H6)** and

(H7) $\qquad z^{\mathrm{I}0} \in z^{\mathrm{Ie}}(0) + D(\mathcal{A}^{\mathrm{I}})$,

the differential inclusion

$$(\mathcal{P}^{\mathrm{I}}) \qquad \begin{cases} \dfrac{dz^{\mathrm{I}}}{dt} + \mathcal{A}^{\mathrm{I}}\,(z^{\mathrm{I}} - z^{\mathrm{Ie}}) \ni (0, \dot{\varphi}^{\mathrm{Ie}} + \dot{\psi}^{\mathrm{Ie}}, 0), \\ z^{\mathrm{I}}(0) = z^{\mathrm{I}0}, \end{cases}$$

has a unique solution z^{I} belonging to $W^{1,\infty}\big(0, T; \mathcal{Z}^{\mathrm{I}} + \big(0, (\varphi_a, 0), 0\big)\big)$ and the first line of $(\mathcal{P}^{\mathrm{I}})$ is satisfied almost everywhere in $[0, T]$.

4.2. **Convergence.** To prove the “convergence” of the solution z^s to $(\mathcal{P}^s)$ toward the solution z^{I} to $(\mathcal{P}^{\mathrm{I}})$, since z^s and z^{I} do not inhabit in the same space and by due account of Propositions 2 and 3, we use the theory of Trotter of approximation of semi-groups of linear operators acting on variable spaces [13, 18].

4.2.1. *Recaps on Trotter theory of approximation of semi-groups.* Let $(H_n)_{n\in\mathbb{N}}$, respectively H, be Hilbert spaces with norms $|\cdot|_n$, respectively $|\cdot|$, and a sequence of linear operators $(\mathrm{P}_n)_{n\in\mathbb{N}}$ from H into H_n satisfying:

(T1) There exists $C > 0$ such that $|\mathrm{P}_n X|_n \leq C|X|$ for all X in H and n in $\mathbb{N}$,
(T2) $\lim_{n\to\infty} |\mathrm{P}_n X|_n = |X|$ for all X in H.

A sequence $(X_n)_{n\in\mathbb{N}}$ in H_n is said *to converge in the sense of Trotter* toward X in H if

$$\lim_{n\to\infty} |\mathrm{P}_n X - X_n|_n = 0.$$

One has the following convergence result (see [9]):

Theorem 3. Let $A_n\colon H_n \rightrightarrows H_n$, $A : H \rightrightarrows H$ maximal-monotone operators, $F_n \in L^1(0, \mathrm{T}\,; H_n)$, $F \in L^1(0, \mathrm{T}\,; H)$, $X^0_n \in \overline{D(A_n)}$, $X^0 \in \overline{D(A)}$ and let X_n, X the weak solutions to

$$\begin{cases} \dfrac{dX_n}{dt} + A_n X_n \ni F_n, \\ X_n(0) = X^0_n, \end{cases} \qquad \begin{cases} \dfrac{dX}{dt} + AX \ni F, \\ X(0) = X^0. \end{cases}$$

If

i) $\displaystyle\lim_{n\to\infty} |(\mathbb{I} + A_n)^{-1}\, \mathrm{P}_n\, z - \mathrm{P}_n\, (\mathbb{I} + A)^{-1}\, z|_n = 0, \quad \forall z \in H,$

ii) $\displaystyle\lim_{n\to\infty} |\mathrm{P}_n\, X^0 - X^0_n|_n = 0, \quad \lim_{n\to\infty} \int_0^{\mathrm{T}} |\mathrm{P}_n\, F(t) - F_n(t)|_n \,dt = 0,$

then X_n converges in the sense of Trotter toward X uniformly on [0,T], namely:

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|\mathrm{P}_n X(t)-X_n(t)|_n=0$$

with moreover

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\big||X_n(t)|_n-|X(t)|\big|=0.$$

4.2.2. *Convergence results.* Propositions 2 and 3 immediately imply:

Proposition 6. The sequence $\mathcal{X}^s=(\mathcal{X}^s_y,\mathcal{X}^s_v)$ in $\mathcal{Z}^s$ converges in the sense of Trotter toward $\mathcal{X}=(\mathcal{X}_y,\mathcal{X}_v)$ in $\mathcal{Z}^{\mathrm{I}}$ if and only if both limits are satisfied:

i) $\displaystyle\lim_{s\to\bar{s}}\frac{1}{\varepsilon^3}\int_{\Omega^\varepsilon}M^\varepsilon(k^\varepsilon(\mathcal{X}^s_y)-K^{\varepsilon\mathrm{I}}_{\mathcal{X}_y})\cdot(k^\varepsilon(\mathcal{X}^s_y)-K^{\varepsilon\mathrm{I}}_{\mathcal{X}_y})\,dx^\varepsilon=0,$

ii) $\displaystyle\lim_{s\to\bar{s}}\mathbb{k}^s(\mathcal{X}^s_v-V^{\varepsilon\mathrm{I}}_{\mathcal{X}_v},\mathcal{X}^s_v-V^{\varepsilon\mathrm{I}}_{\mathcal{X}_v})=0.$

Remark 3. The convergence in the sense of Trotter is therefore the appropriate notion of approximation from a physical point of view: a convergence result of *relative* energetic gaps measured on the *physical plate* (the only one which has a meaning because the total mechanical energies are going to zero!) between the state $\mathcal{X}^s$ and the image on the genuine physical configuration Ω^ε of the limit state $\mathcal{X}$.

The following proposition is key in establishing the Trotter convergence of $z^s(t)$ toward $z^I(t)$ uniformly on $[0,T]$:

Proposition 7. There hold:

i) $\displaystyle\lim_{s\to\bar{s}}|\mathrm{P}^{s\mathrm{I}}\,(\mathbb{I}+\mathcal{A}^{\mathrm{I}})^{-1}\,z-(\mathbb{I}+\mathcal{A}^s)^{-1}\,\mathrm{P}^{s\mathrm{I}}\,z|^s=0,\quad\forall z\in\mathcal{Z}^{\mathrm{I}},$

ii) $\displaystyle\lim_{s\to\bar{s}}|\mathrm{P}^{s\mathrm{I}}\,z^{\mathrm{I}e}(t)-z^{se}(t)|^s=0,\quad\forall t\in[0,T].$

The reader will find the main argument of its proof in the next subsection. According to a non-linear extension of Trotter theory [9], the definitions (3.2) and (4.12) of z^{se} and $z^{\mathrm{I}e}$, their time regularities and the above proposition, we can now state our core convergence result:

Theorem 4. Under assumptions **(H1)–(H7)** and

(H8) $\quad\exists z^{\mathrm{I}0}=(u^{\mathrm{I}0},\varphi^{\mathrm{I}0},v^{\mathrm{I}0})=(y^{\mathrm{I}0},v^{\mathrm{I}0})\in z^{\mathrm{I}e}(0)+D(\mathcal{A}^{\mathrm{I}})$ s.t. $\displaystyle\lim_{s\to\bar{s}}|\mathrm{P}^{s\mathrm{I}}\,z^{\mathrm{I}0}-z^{s0}|^s=0,$

the solution z^s to $(\mathcal{P}^s)$ converges to the solution z^{I} to $(\mathcal{P}^{\mathrm{I}})$ in the sense that

$$\lim_{s\to\bar{s}}|\mathrm{P}^{s\mathrm{I}}\,z^{\mathrm{I}}(t)-z^s(t)|^s=0\text{ uniformly on }[0,T].$$

In addition, $\displaystyle\lim_{s\to\bar{s}}|z^s(t)|^s=|z^{\mathrm{I}}(t)|^{\mathrm{I}}$ uniformly on $[0,T]$ and

$$\int_\Omega\Big(\big(M_{\mathrm{me}}\big)^T E_{u^{\mathrm{I}0}}+M_{\mathrm{ee}}\,G^{\mathrm{I}_1}_{\varphi^{\mathrm{I}0}}\Big)\cdot G^{\mathrm{I}_1}_\phi\,dx=L(0)(0,\phi),\quad\forall\phi\in\Phi^{\mathrm{I}_1}.\tag{4.14}$$

4.3. **Proofs of Propositions 2, 3 and 7.** We shall focus here on Propositions 2, 3 and 7, the proofs of which require a little extra technical effort as compared their counterparts in [16, 17], which is due to the piezoelectric coupling. They may be easily derived from the two following lemmas.

Lemma 1. For every $\mathcal{X}^s_\varphi$ in $H^1_{\Gamma^\varepsilon_{\mathrm{eD}}}(\Omega^\varepsilon)$ such that $|(0,\mathcal{X}^s_\varphi)|^s$ is uniformly bounded, there exists $\mathcal{X}^{\mathrm{I}}_\varphi$ in Φ^{I_1} and a not relabeled subsequence such that

i) $g^{\mathrm{I}_1}(\varepsilon,\mathcal{S}^{\mathrm{I}_1}_{\varphi\varepsilon}\,\mathcal{X}^s_\varphi)$ weakly converges toward $G^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_\varphi}$ in $L^2(\Omega,\mathbb{R}^3)$,

ii) $\displaystyle\int_\Omega M_{\mathrm{ee}}\,G^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_\varphi}\cdot G^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_\varphi}\,dx\le\varliminf_{\varepsilon\to0}\frac{1}{\varepsilon^3}\int_{\Omega^\varepsilon}M^\varepsilon_{\mathrm{ee}}\,\nabla^\varepsilon\,\mathcal{X}^s_\varphi\cdot\nabla^\varepsilon\,\mathcal{X}^s_\varphi\,dx^\varepsilon.$

Lemma 2. Let $\mathcal{X}^s_y$ the unique solution to

$$\mathcal{X}^s_y\in\mathcal{Y}^s\quad;\quad\mathbb{m}^s(\mathcal{X}^s_y,y')=L_s(\mathcal{S}^{\mathrm{I}_1}_\varepsilon\,y'),\ \forall y'\in\mathcal{Y}^s,\tag{4.15}$$

with L_s a continuous linear form on $H^1_{\Gamma_{\mathrm{mD}}}(\Omega,\mathbb{R}^3)\times H^1_{\Gamma_{\mathrm{eD}}}(\Omega)$ converging strongly toward L, then there exists a unique $\mathcal{X}^{\mathrm{I}}_y$ in $\mathcal{Y}^{\mathrm{I}}$ solution to

$$\mathcal{X}^{\mathrm{I}}_y\in\mathcal{Y}^{\mathrm{I}}\quad;\quad\mathbb{m}^{\mathrm{I}_1}(\mathcal{X}^{\mathrm{I}}_y,y')=\int_\Omega M\,K^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_y}\cdot K^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_{y'}}=L(y'),\ \forall y'\in\mathcal{Y}^{\mathrm{I}},\tag{4.16}$$

with $K^{\mathrm{I}_1}_{\mathcal{X}^{\mathrm{I}}_y}$ the strong limit in $L^2(\Omega,\mathcal{H})$ of $k^{\mathrm{I}_1}(\varepsilon,\mathcal{S}^{\mathrm{I}_1}_\varepsilon\,\mathcal{X}^s_y)$ (see (4.3)).

Proof. The proof of Lemma 1 is obvious in the case $\mathrm{I}_1 = 1$ while, when $\mathrm{I}_1 = 2$, by due account of **(H4)**, Poincaré inequality implies that $\mathcal{S}^2_{\varphi\varepsilon}\,\mathcal{X}^s_\varphi$ is bounded in $L^2(\Omega)$ so that $\varepsilon\widehat{\nabla}\mathcal{S}^2_{\varphi\varepsilon}\,\mathcal{X}^s_\varphi$ converges to zero in the sense of distributions. Hence $g^2(\varepsilon, \mathcal{S}^2_{\varphi\varepsilon}\,\mathcal{X}^s_\varphi)$ converges weakly in $L^2(\Omega, \mathbb{R}^3)$ toward some $(0, 0, \partial_3\varphi^0)$ with φ^0 in $H^1(-1, 1; L^2(\omega))$.

As for the proof of Lemma 2, we first choose $y' = (0, \mathcal{X}^s_\varphi)$ in (4.15). Lemma 1 then implies that there exists some $\overline{y}^{\mathrm{I}}$ in $\mathcal{Y}^{\mathrm{I}}$ such that $k^{\mathrm{I}_1}(\varepsilon, \mathcal{S}^{\mathrm{I}_1}_\varepsilon\,\mathcal{X}^s_\varphi)$ weakly converges in $L^2(\Omega, \mathcal{H})$ toward some $K^{\mathrm{I}_1}_{\overline{y}^{\mathrm{I}}}$. To identify $\overline{y}^{\mathrm{I}}$ as $\mathcal{X}^{\mathrm{I}}_y$ the unique solution to (4.16), it remains to arbitrarily choose y'^0 in $\mathcal{U}^0 \times \Phi^{\mathrm{I}_1|0}$ and y'^1 in $\big(\mathcal{U}^1 \times \Phi^{\mathrm{I}_1|1}\big) \cap H^1\big(-1, 1; C^\infty_0(\omega, \mathbb{R}^4)\big)/\big(L^2(\omega, \mathbb{R}^3) \times \{0\}\big)$, which is dense in $\mathcal{U}^1 \times \Phi^{\mathrm{I}_1|1}$, and to go to the limit in (4.15) with $y' = (\mathcal{S}^{\mathrm{I}_1}_\varepsilon)^{-1}\big(\sum_{i=0}^{1} \varepsilon^i\, y'^i\big)$. The strong convergence of $k^{\mathrm{I}_1}(\varepsilon, \mathcal{X}^s_y)$ is obtained by choosing $y' = \mathcal{X}^s_y$.

□

5. Concluding remarks

According to each value of $\mathrm{I} = (\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$ in $\{1, 2\} \times \{1, 2\} \times \{1, 2\}$, we now write $(\mathcal{P}^{\mathrm{I}})$ in the form of variational equations. We recall that the space $\widetilde{\mathcal{Y}^{\mathrm{I}}}$ of "virtual limit admissible generalized velocities" and the limit global pseudo-potential of dissipation $\mathcal{D}^{\mathrm{I}}$ are defined in (4.10), while also remind that the linear form associated with the electromechanical loading reads as:

$$L(t)(y) := \int_\Omega f \cdot u\, dx + \int_\Omega F\varphi\, dx + \int_{\Gamma_{\mathrm{mN}}} g \cdot u\, d\mathfrak{h}_2 + \int_{\Gamma_{\mathrm{eN}}} d\varphi\, d\mathfrak{h}_2,\ \forall y = (u, \varphi) \in \mathcal{Y}^{\mathrm{I}}.$$

Let $\langle\delta\rangle := \int_{-1}^{1} \delta(\widehat{x}, x_3)\, dx_3$. For all I, we consider the following initial conditions:

$$y^{\mathrm{I}}(0) = \mathsf{y}^{\mathrm{I}0} := \big((\mathsf{u}^{\mathrm{I}0,0}, \mathsf{u}^{\mathrm{I}0,1}), (\varphi^{\mathrm{I}0,0}, \varphi^{\mathrm{I}0,1})\big),\ v^{\mathrm{I}}(0) = \mathsf{v}^{\mathrm{I}0}.$$

For the sake of simplicity, we write $u = (u^0, u^1)$ and $\varphi = (\varphi^0, \varphi^1)$ instead of $u^{\mathrm{I}} = (u^{\mathrm{I}0}, u^{\mathrm{I}1})$ and $\varphi^{\mathrm{I}} = (\varphi^{\mathrm{I}0}, \varphi^{\mathrm{I}1})$ and $u' = (u'^0, u'^1)$, $\varphi' = (\varphi'^0, \varphi'^1)$.

Taking into account (4.14), a time-integration leads to the following variational limit problem (P^{I}):

- $\mathrm{I} = (1, 1, 1)$:

(5.1)
$$\overline{\rho}^1 \int_\omega \langle\delta\rangle\, \ddot{u}^0_3\, u'^0_3\, d\widehat{x} + \int_\Omega M\, K^1_y \cdot K^1_{y'}\, dx + \int_\Omega \overline{b}^1\big(\mathcal{D}(E_{\dot{u}+u'}) - \mathcal{D}(E_{\dot{u}})\big)\, dx \geq L(t)(u'^0, \varphi'^0),\ \forall y' = (u', \varphi') \in \widetilde{\mathcal{Y}^{\mathrm{I}}},$$

- $\mathrm{I} = (1, 2, 1)$:

(5.2)
$$\overline{\rho}^1 \int_\omega \langle\delta\rangle\, \widehat{\ddot{u}^0} \cdot \widehat{u'^0}\, d\widehat{x} + \int_\Omega M\, K^1_y \cdot K^1_{y'}\, dx + \int_\Omega \overline{b}^1\big(\mathcal{D}(E_{\dot{u}+u'}) - \mathcal{D}(E_{\dot{u}})\big)\, dx \geq L(t)(u'^0, \varphi'^0),\ \forall y' = (u', \varphi') \in \widetilde{\mathcal{Y}^{\mathrm{I}}},$$
$$u^0_3(t) = \mathsf{u}^{0,0}_3\ , \quad \forall t \in [0, T],$$

- $\mathrm{I} = (2, 1, 1)$: $\quad \varphi = \varphi_a$ on $\Gamma_{\mathrm{eD}} \cap \Gamma_\pm,\ \forall t \in [0, T]$,

(5.3)
$$\overline{\rho}^1 \int_\omega \langle\delta\rangle\, \ddot{u}^0_3 u'^0_3\, d\widehat{x} + \int_\Omega M K^2_y \cdot K^2_{y'}\, dx + \int_\Omega \overline{b}^1\big(\mathcal{D}(E_{\dot{u}+u'}) - \mathcal{D}(E_{\dot{u}})\big)\, dx \geq L(t)(u'^0, \varphi'^0),\ \forall y' = (u', \varphi') \in \widetilde{\mathcal{Y}^{\mathrm{I}}},$$

- $\mathrm{I} = (2, 2, 1)$: $\quad \varphi = \varphi_a$ on $\Gamma_{\mathrm{eD}} \cap \Gamma_\pm,\ \forall t \in [0, T]$,

(5.4)
$$\overline{\rho}^1 \int_\omega \langle\delta\rangle\, \widehat{\ddot{u}^0} \cdot \widehat{u'^0}\, d\widehat{x} + \int_\Omega M K^2_y \cdot K^2_{y'}\, dx + \int_\Omega \overline{b}^1\big(\mathcal{D}(E_{\dot{u}+u'}) - \mathcal{D}(E_{\dot{u}})\big)\, dx \geq L(t)(u'^0, \varphi'^0),\ \forall y' = (u', \varphi') \in \widetilde{\mathcal{Y}^{\mathrm{I}}},$$
$$u^0_3(t) = \mathsf{u}^{0,0}_3\ , \quad \forall t \in [0, T],$$

- $\mathrm{I}_3 = 2$: $\quad y(t) = \mathsf{y}^0, v(t) = 0.$

Even though we discuss here the physical properties of fields living in an abstract domain, it is fairly easy to infer mechanical information from equations above.

First of all we note that the transient response of the displacement differs sharply from that of the electric potential. Indeed the electric potential response is quasi-static, whereas, depending on the relative magnitude of the density and the thickness, the feature of the evolution of the displacement field appears as a juxtaposition dynamic, quasi-static and possibly static effects, except in the case of a very high viscosity (*i.e.* when $\mathrm{I}_3 = 2$) in which the motion is frozen in the initial state. As can be seen above, dynamic evolution concerns the transverse part of the displacement for ρ of order ε^2 and the in-plane part for ρ of order 1. Note that the viscosity precludes the decoupling of membrane and flexural motions so that (P^{I}) is a three-dimensional problem.

Second, it is important to note that the additional fields $u^{\mathrm{I}1}$ and $\varphi^{\mathrm{I}1}$ - together with the generalized strain $K^{\mathrm{I}_1}_{y^{\mathrm{I}}}, y^{\mathrm{I}} = \big((u^{\mathrm{I}0}, u^{\mathrm{I}1}), (\varphi^{\mathrm{I}0}, \varphi^{\mathrm{I}1})\big)$ - lead to "limit variational equations" (P^{I}) exhibiting the same form as the genuine problem (P^s). It would not be the case if we merely considered the couple $(u^{\mathrm{I}0}, \varphi^{\mathrm{I}0})$: this would yield models featuring viscopiezoelectric behavior with delayed memory. Assuming furthermore that $\mathcal{D}$ is quadratic, it may be shown that the piezoelectric coupling then disappears in certain cases of crystal symmetry (see [19] for a detailed discussion on this subject).

Thirdly, the electrical boundary conditions being such as to exploit the piezoelectric coupling in two different ways, we shall observe that $\mathrm{I}_1 = 1$ corresponds to a thin plate used as a sensor while $\mathrm{I}_1 = 2$ is related to actuators. As done in [19], it is possible to propose alternative formulations to (P^{I}) related to these two types of behavior by introducing:

(5.5)
$$\mathcal{H}^1_- := \big\{(e,g) \in \mathcal{H}\ ;\ e = 0, \hat{g} = 0\big\},\ \mathcal{H}^1_0 := \big\{(e,g) \in \mathcal{H}\ ;\ g_3 = 0\big\},\ \mathcal{H}^1_+ := \big\{(e,g) \in \mathcal{H}\ ;\ (e,g) = (0,0)\big\},$$
$$\mathcal{H}^2_- := \big\{(e,g) \in \mathcal{H}\ ;\ (e,g) = (0,0)\big\},\ \mathcal{H}^2_0 := \big\{(e,g) \in \mathcal{H}\ ;\ \hat{g} = 0\big\},\ \mathcal{H}^2_+ := \big\{(e,g) \in \mathcal{H}\ ;\ e = 0, g_3 = 0\big\},$$

which implies two different decompositions of M into $M^{\mathrm{I}_1}_{\wedge\vee}$ with $\wedge$, $\vee$ in $\{-,0,+\}$. Noticing that for all y in $\mathcal{Y}^{\mathrm{I}}$ the respective projections of $M\,K^{\mathrm{I}_1}_{y^{\mathrm{I}}}$ and $K^{\mathrm{I}_1}_{y}$ on $\mathcal{H}^{\mathrm{I}_1}_-$ and $\mathcal{H}^{\mathrm{I}_1}_+$ vanish, we get that the Schur complement $\widetilde{M^1} := M^1_{00} - M^1_{0-}(M^{\mathrm{I}_1}_{--})^{-1}M^1_{-0}$ and $\widetilde{M^2} := M^2_{00}$ satisfy

(5.6)
$$M\,K^{\mathrm{I}_1}_{y^{\mathrm{I}}} \cdot K^{\mathrm{I}_1}_{y} = \widetilde{M^{\mathrm{I}_1}}\big(K^{\mathrm{I}_1}_{y^{\mathrm{I}}}\big)_0 \cdot \big(K^{\mathrm{I}_1}_{y}\big)_0, \big(K^1_y\big)_0 := (E_u, \widehat{\nabla}\varphi_0), \big(K^2_y\big)_0 := (E_u, \partial_3\varphi_0), \forall y = \big(u, (\varphi^0, \varphi^1)\big) \in \mathcal{Y}^{\mathrm{I}},$$

so that alternative versions to (5.1)-(5.4) easily ensue. Hence it is possible to eliminate $\varphi^{\mathrm{I}1}$ in the case of sensors, a situation in which $\widetilde{M^1}_{\mathrm{mm}}$ involves a mixture of elastic, piezoelectric and dielectric coefficients except for crystallographic classes 32, 422, $\bar{6}$, 622 and $\bar{6}m2$, while obviously $\widetilde{M^2}_{\mathrm{mm}} = M_{\mathrm{mm}}$.

Finally our modeling does not derive from the descaling $(\mathcal{S}^{\mathrm{I}_1}_\varepsilon)^{-1}\, y^{\mathrm{I}0}(t)$ (recall that $y^{\mathrm{I}0} = (u^{\mathrm{I}0}, \varphi^{\mathrm{I}0})$) but from our convergence result established in Theorem 4 and also from the crucial Proposition 6 which leads to

$$\lim_{s\to\bar{s}} \frac{1}{\varepsilon^3}\int_{\Omega^\varepsilon} M^\varepsilon(k^\varepsilon(y^s) - K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}}) \cdot (k^\varepsilon(y^s) - K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}})\, dx^\varepsilon = 0. \tag{5.7}$$

Hence as observed in [13, 8], $K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}}$ is a good approximation of the generalized strain $k^\varepsilon(y^s)$ in the sense that the *relative error* made by replacing $k^\varepsilon(y^s)$ by $K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}}$ tends to zero. However as $K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}}$ is not necessarily the generalized strain tensor of an electromechanical state living in $\mathcal{Y}^s$, we are led to consider

$$y^{\mathrm{I}s} := (\mathcal{S}^{\mathrm{I}_1}_\varepsilon)^{-1}\Big(\sum_{i=0}^{1} \varepsilon^i\, y^{\mathrm{I}i}_\varepsilon\Big),$$

where $y^{\mathrm{I}i}_\varepsilon$ is a smooth approximation of $y^{\mathrm{I}i}$, which leads to

$$\lim_{s\to\bar{s}} \frac{1}{\varepsilon^3}\int_{\Omega^\varepsilon} M^\varepsilon(k^\varepsilon(y^{\mathrm{I}s}) - K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}}) \cdot (k^\varepsilon(y^{\mathrm{I}s}) - K^{\varepsilon\mathrm{I}}_{y^{\mathrm{I}}})\, dx^\varepsilon = 0. \tag{5.8}$$

Thus $y^{\mathrm{I}s}$ is our proposal of approximation for y^s. We recall that its displacement component $u^{\mathrm{I}s}$ is far from a Kirchhoff-Love and even a Reissner-Mindlin field (see Remark 5.2 in [15]). It is obtained by first solving (P^{I}) which actually corresponds to a three-dimensional problem yet set on a "reasonable" fixed domain Ω and involving fields with simplified kinematics and second by means of $y^{\mathrm{I}s}$ which also involves the fixed domain Ω. It is therefore easy to implement a numerical method of approximation.

REFERENCES

[1] Bachmar, A.; Boutechebak, S.; Serrar, T. Variational analysis of a dynamic electroviscoelastic problem with friction, *Zh. Sib. Fed. Univ. Mat. Fiz.* **12** (2019), no. 1, 68–78.

[2] Barboteu, M.; Fernández, J. R.; Ouafik, Y. Numerical analysis of a frictionless viscoelastic piezoelectric contact problem, *M2AN Math. Model. Numer. Anal.* **42** (2008), no. 4, 667–682.

[3] Behera, S. K.; Kumar, D. ; Sarangi, S. Modeling of electro–viscoelastic dielectric elastomer: A continuum mechanics approach, *Eur. J. Mech. A Solids* **90** (2021), Paper no. 104369, 7 pp.

[4] Brezis, H. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland, Amsterdam, 1973.

[5] Ciarlet, P.G. *Mathematical Elasticity, Vol. II: Theory of Plates*. North-Holland, Amsterdam, 1997.

[6] Dilmi, M.; Dilmi, M.; Benseridi, H. Asymptotic analysis of quasistatic electro-viscoelastic problem with Tresca's friction Law, *Comput. Math. Methods* **1** (2019), no. 3, e1028, 15 pp.

[7] Erofeev, V. I.; Pavlov, I. S. *Structural Modeling of Metamaterials. Advanced Structured Materials*, 144. Springer, Cham, Switzerland, 2021.

[8] Gaudiello, A.; Monneau, R.; Mossino, J.; Murat, F.; Sili, A. Junction of elastic plates and beams, *ESAIM Control Optim. Calc. Var.* **13** (2007), no. 3, 419–457.

[9] Iosifescu, O.; Licht, C.; Michaille, G. Nonlinear boundary conditions in Kirchhoff-Love plate theory, *J. Elasticity* **96** (2009), no. 1, 57–79.

[10] Iosifescu, O.; Licht, C. Transient response of a thin linearly elastic plate with Norton or Tresca friction, *Asymptot. Anal.* **128** (2022), no. 4, 555–570.

[11] Lerguet, Z.; Shillor, M.; Sofonea, M. A frictional contact problem for an electro-viscoelastic body, *Electron. J. Differential Equations* (2007), no. 170, 16 pp.

[12] Licht, C. Thin linearly viscoelastic Kelvin-Voigt plates, *C. R. Mecanique* **341** (2013), no. 9-10, 697–700.

[13] Licht, C.; Weller, T. Approximation of semi-groups in the sense of Trotter and asymptotic mathematical modeling, *Discrete Contin. Dyn. Syst. Ser. S* **12** (2019), no. 6, 1709–1741.

[14] Licht, C.; Orankitjaroen, S.; Weller, T. Asymptotic analysis of the transient response of a thermoelastic assembly involving a thin layer, *C. R. Mecanique* **350** (2022), 27–45.

[15] Terapabkajorned, Y.; Orankitjaroen, S.; Licht, C. Asymptotic model of linearly viscoelastic Kelvin-Voigt type plates via Trotter theory, *Adv. Difference Equ.* (2019), Paper no. 186, 9 pp.

[16] Terapabkajorned, Y.; Orankitjaroen, S.; Licht, C.; Weller, T. Asymptotic modeling of the transient response of non-linear Kelvin-Voigt viscoelastic thin plates with Norton or Tresca friction by Trotter theory, *Appl. Math.* **69** (2024), no. 1, 25–48.

[17] Terapabkajorned, Y.; Orankitjaroen, S.; Licht, C.; Weller, T. Asymptotic modeling of thin plates and slender beams, a unifying approach, *C. R. Mecanique* **352** (2024), 201–222.

[18] Trotter, H. F. Approximation of semi-groups of operators, *Pacific J. Math.* **8** (1958), 887–919.

[19] Weller, T.; Licht, C. Asymptotic modeling of thin piezoelectric plates, *Ann. Solid Struct. Mech.* **1** (2010), 173–188.

LMGC, UNIVERSITÉ DE MONTPELLIER
CNRS, MONTPELLIER
FRANCE
Email address: christian.licht@umontpellier.fr, thibaut.weller@umontpellier.fr

CENTER OF EXCELLENCE IN MATHEMATICS
CHE, BANGKOK 10400, THAILAND
Email address: christian.licht@umontpellier.fr, somsak.ora@mahidol.ac.th

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
MAHIDOL UNIVERSITY, BANGKOK 10400, THAILAND
Email address: christian.licht@umontpellier.fr, somsak.ora@mahidol.ac.th, ongyosawat@gmail.com