CARPATHIAN J. MATH.
Volume **41** (2025), No. 2,
Pages 569-595

Online version at https://www.carpathian.cunbm.utcluj.ro/
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2025.02.20

An inertial method for solving the split equality fixed point problem with multiple output sets

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ABSTRACT. In this paper, we introduce the split equality fixed point problem with multiple output sets in real Hilbert spaces and propose an iterative method for solving the problem. We then establish a strong convergence result under the assumption that the underlying mappings are uniformly continuous quasi-pseudocontractive. We give some specific cases of our main result and finally provide a numerical example to reveal the effectiveness of our method. Our result extends many of the results in the literature.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ and let C be a nonempty, closed and convex subset of H. Let $T\colon C \to H$ be a mapping. A point $p \in C$ is said to be a fixed point of T if $Tp = p$. The set of all fixed points of T is denoted by $F(T)$. A mapping $T\colon C \to H$ is said to be

i. *firmly quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2, \text{ for all } x \in C, \ p \in F(T);$$

ii. *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \text{ for all } x \in C, p \in F(T);$$

iii. *β-demicontractive* if $F(T) \neq \emptyset$ and there exists number $\beta \in (0, 1)$ with

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \beta\|Tx - x\|^2, \text{ for all } x \in H \text{ and } p \in F(T); \tag{1.1}$$

iv. *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C;$$

or equivalently

$$\langle (I - T)x - (I - T)y, \ x - y \rangle \geq 0, \text{ for all } x, y \in C; \tag{1.2}$$

v. *quasi-pseudocontractive* if $F(T) \neq \emptyset$, and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2;$$

or equivalently

$$\langle x - Tx, \ x - p \rangle \geq 0, \text{ for all } x \in C, \ p \in F(T);$$

vi. *Lipschitz continuous* if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

If in (vi), $L = 1$, then we say that T is *nonexpansive* and it is said to be a *contraction* if $L < 1$.

Received: 15.04.2024. In revised form: 05.10.2024. Accepted: 12.10.2024
2020 *Mathematics Subject Classification.* 39B12, 47B02, 47H10, 47J25, 47J26.
Key words and phrases. *Fixed points, quasi-pseudocontractive mappings, strong convergence, uniform continuity.*
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Remark 1.1. *We observe from the above definitions that the class of quasi-pseudocontractive mappings is a more general class of mappings that contains the classes of firmly quasi-nonexpansive, quasi-nonexpansive mappings, demicontractive mappings. It also contains the class of pseudocontractive mappings with nonempty set of fixed points.*

The mapping T is said to satisfy the demiclosedness property if $(I-T)$ is demiclosed at 0, that is, if $\{x_n\}$ is any sequence in C such that $x_n \rightharpoonup p$ and $\|(I-T)x_n\| \to 0$, then $Tp = p$.

The class of pseudocontractive mappings is closely related to the class of monotone mappings, where a mapping A: $D(A) \subset H \to H$ is said to be *monotone* if for all $x, y \in D(A)$, we have

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{1.3}$$

In fact, a mapping T: $H \to H$ is pseudocontractive if and only if the mapping $A = I - T$ is monotone. In this case, the set of fixed points of T is the same as the set of null points, $N(A)$, of A, where $N(A) = \{x \in H : Ax = 0\}$. Many physical problems can be modeled by initial value problems involving monotone mappings. One of such problems is the evolution equation which is given as

$$\frac{dx}{dt} = -Ax(t), \quad x(0) = x_0, \tag{1.4}$$

where A is a monotone mapping in an appropriate space [33]. If in (1.4), $x(t)$ is independent of the variable t, then (1.4) reduces to the problem $Ax = 0$, whose solutions correspond to the equilibrium points of the system (1.4).

The study of fixed point theory was motivated by the desire to study the existence and properties of boundary value problems for nonlinear partial differential equations [33]. Fixed point theory has also been applied in, for instance, biology, chemical reactions, chemistry, complementary problems, economics etc.

Due to these and other applications, the theory of fixed points has become an interesting area of research. Thus, several iterative algorithms have been proposed and studied for approximating fixed points of nonexpansive, strictly pseudocontractive, pseudocontractive, and quasi-pseudocontractive mappings (see, for instance, [5, 7, 13, 16, 17, 20, 23, 25, 37, 45, 46, 47, 48]).

Let C and Q be nonempty, closed, and convex subsets of the real Hilbert spaces H_1 and H_2, respectively, and let A: $H_1 \to H_2$ be a bounded linear mapping. The Split Feasibility Problem (SFP) is defined as finding a point

$$x^* \in C \bigcap A^{-1}(Q),$$

or equivalently, finding a point

$$x^* \in C \text{ such that } Ax^* \in Q. \tag{1.5}$$

The SFP which was initially introduced by Censor and Elfving [10] has got applications in certain inverse problems and later played a crucial role in real-life problems such as data compression, sensor networks, radiation therapy, antenna design, computerized tomography, immaterial science, medical image reconstruction, data denoising, (see, for instance, [8, 9, 35]).

If we assume that (1.5) has a solution, then it can be shown that p is a solution of (1.5) if and only if

$$p = P_C(p - \rho A^*(I - P_Q)Ap), \tag{1.6}$$

where ρ is a positive constant, P_C and P_Q are the metric projections of the Hilbert spaces H_1 onto C and H_2 onto Q, respectively, and I is the identity mapping on H_2 [35]. Thus, the split feasibility problem is closely related to fixed point problems, and thus fixed point methods can be applied in the split feasibility problems. Due to their wide applications, SFPs have attracted the attention of researchers, and thus several iterative algorithms have been introduced for approximating their solutions (see, for instance, [11, 12, 19]).

One of the famous methods for solving SFP is the $CQ-$algorithm which was introduced by Byrne [8] and is given as follows: For an arbitrary initial guess $x_0 \in H_1$, let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = P_C\left[x_n - \tau A^*\left(I - P_Q\right)Ax_n\right],\ n \geq 0, \tag{1.7}$$

where A^* is the adjoint of the bounded linear mapping A, I is the identity mapping on H_2, and $\tau > 0$ is a properly chosen step-size, P_C and P_Q are the metric projections of H_1 and H_2 onto C and Q, respectively. The author proved that the sequence generated by (1.7) converges strongly to a solution of (1.5) provided that H_1 is a finite dimensional space. Many other authors have also studied the $CQ-$algorithm (see, for instance, [39, 40, 41, 42, 44] and the references therein).

One of the problems which is more general than the SFP is the Split Feasibility Problem with Multiple Output Sets (SFPMOS) and it was introduced by Reich et al. [28] as follows: Let H, H_i, $i = 1, 2, \ldots, N$ be real Hilbert spaces and let $A_i : H \to H_i$ be bounded linear mappings. Let C and Q_i be nonempty, closed and convex subsets of H and H_i, respectively. The SFPMOS is defined as finding a point

$$x^* \in C \bigcap \left(\bigcap_{i=1}^{N} A_i^{-1}\left(Q_i\right)\right), \tag{1.8}$$

or equivalently, finding a point

$$x^* \in C \text{ such that } A_i x^* \in Q_i,\ \text{for } i = 1, 2, \ldots, N.$$

In 2020, Reich et al. [28] proposed the following iterative method for approximating SFPMOS (1.8): Starting with any $x_0 \in H$, let $\{x_n\}$ be the sequence generated arbitrarily by

$$x_{n+1} = P_C\left[x_n - \lambda \sum_{i=1}^{N} A_i^*\left(I - P_{Q_i}\right)A_i x_n\right], \tag{1.9}$$

where $0 < \lambda < \dfrac{1}{k \max\limits_{1 \leq i \leq k} \|A_i\|^2}$.

They obtained a weak convergence result of the sequence generated by (1.9). To obtain a strong convergence result, they modified (1.9) as follows (see, [28]): Starting from any initial guess $x_0 \in H$, their modified method produces the sequence $\{x_n\}$ by

$$x_{n+1} = \gamma_n f(x_n) + (1 - \gamma_n)P_C\left[x_n - \lambda \sum_{i=1}^{N} A_i^*\left(I - P_{Q_i}\right)A_i x_n\right], \tag{1.10}$$

where $\{\gamma_n\} \subseteq (0, 1)$ and f is a contraction function. They proved, under some appropriate conditions, that the sequence $\{x_n\}$ generated by (1.10) converges strongly to a solution of the problem (1.8).

If, in (1.8), the set C is replaced with $F(T)$ and Q_i with $F(S_i)$, where T: $H \to H$, S_i: $H_i \to H_i$, $i = 1, 2, \ldots, N$, are nonlinear mappings and A_i: $H \to H_i$, for $i = 1, 2, \ldots, N$, are bounded linear mapping with adjoints A_i^*, for $i = 1, 2, \ldots, N$, then we get the Split Fixed

Point Problem with Multiple Output Sets (SFPPMOS). The SFPPMOS was introduced by Wang [38] and is defined as finding a point $x^* \in H$ such that

$$x^* \in F(T) \bigcap \left(\bigcap_{i=1}^{N} A_i^{-1} \left(F(S_i) \right) \right), \tag{1.11}$$

or equivalently, finding a point

$$x^* \in F(T) \text{ such that } A_i x^* \in F(S_i), \text{ for } i = 1, 2, \ldots, N$$

In 2022, Wang [38] proposed the following iterative algorithm for solving SFPPMOS: Let S_i: $H_i \to H_i$ be κ_i-demicontractive mappings with $\kappa_i \in (0, 1)$, for each $i = 1, 2, 3, \ldots, N$. Let $\{x_n\}$ be the sequence generated from arbitrary $x_0 \in H$ by

$$x_{n+1} = T_\lambda \left[x_n - \tau \sum_{i=1}^{N} A_i^* \left(I - S_i \right) A_i x_n \right], \tag{1.12}$$

where $T_\lambda = (1 - \lambda)I + \lambda T$, τ and λ are properly chosen parameters. The author obtained a weak convergence result to a solution of (1.11) under the assumptions that S_i is demiclosed for each $i = 1, 2, \ldots, N$ and

$$0 < \tau < \frac{\min\limits_{1 \le i \le N} (1 - \kappa_i)}{\sum\limits_{i=1}^{N} \|A_i\|^2}, \ 0 < \lambda < 1 - \kappa_0. \tag{1.13}$$

In 2022, Reich et al. [31] introduced a more general problem called the *split common fixed point problem with multiple output sets* as follows: Let $H, H_i, i = 1, 2, ..., m$, be real Hilbert spaces. Let $A_i : H \to H_i$, $i = 1, 2, ..., m$, be bounded linear operators. Let $T_j : H \to H$, $j = 1, 2, ..., M,$, $S_k^i : H_i \to H_i$, $i = 1, 2, ..., N$, $k = 1, 2, ..., M_i$, be nonexpansive mappings. They defined the split common fixed point problem with multiple output sets as finding a point $x^* \in H$ such that

$$x^* \in \left(\bigcap_{j=1}^{M} F(S_j) \right) \bigcap \left(\bigcap_{i=1}^{N} A_i^{-1} \left(\bigcap_{k=1}^{M_i} F(S_k^i) \right) \right). \tag{1.14}$$

Moreover, they introduced an iterative algorithm (see Algorithm 3.1 of [31]) and established a strong convergence result to a solution of (1.14).

We have also another generalization of the split fixed point problems (see, [14]) known as the Split Equality Fixed Point Problem (SEFPP) which was introduced by Moudafi and Al-Shemas [26] and is defined as finding a point

$$(x^*, y^*) \in F(S_1) \times F(S_2) \text{ such that } Ax^* = By^*, \tag{1.15}$$

where H_1 and H_2 are real Hilbert spaces, S_1: $H_1 \to H_1$ and S_2: $H_2 \to H_2$ are nonlinear mappings, A: $H_1 \to H_3$ and B: $H_2 \to H_3$ are bounded linear mappings, where H_3 is another real Hilbert space.

Many authors have proposed and studied different iterative algorithms for approximating solutions of SEFPP (see, for instance, [2, 13, 14, 15, 26]).

In 2011, Moudafi and Al- Shemas [26] proposed the following algorithm which approximates a solution of SEFPP (1.15): Let H_1, H_2 and H_3 be real Hilbert spaces and let T:

$H_1 \to H_1$ and S: $H_2 \to H_2$ be firmly quasi-nonexpansive mappings. Let $\{(x_n, y_n)\}$ be the sequence obtained by the following iteration:

$$\begin{cases} x_{n+1} = T(x_n - \beta_n A^*(Ax_n - By_n)) \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_n - By_n)), \end{cases} \tag{1.16}$$

where $\{\beta_n\}$ is a real sequence satisfying some conditions and A: $H_1 \to H_3$ and B: $H_2 \to H_3$ are bounded linear mappings. Then they proved that the sequence $\{(x_n, y_n)\}$ converges weakly to a solution of the SEFPP (1.15) .

However, the calculation of the step size $\{\beta_n\}$ in (1.16) is dependent on the operator norms $\|A\|$ and $\|B\|$.

In 2015, Che and Li [15] proposed the following algorithm for solving SEFPP (1.15) which does not require prior information about the norms of the bounded linear mappings for calculating the step sizes: Let H_1, H_2 and H_3 be real Hilbert spaces and let T_1: $H_1 \to H_1$ and T_2: $H_2 \to H_2$ be quasi-nonexpansive mappings. Let $\{(x_n, y_n)\}$ be the sequence defined from arbitrary $(x_0, y_0) \in H_1 \times H_2$ as

$$\begin{cases} c_n = x_n - \theta_n A^*(Ax_n - By_n), \\ x_{n+1} = \varrho_n x_n + (1 - \varrho_n) T_1 c_n, \\ d_n = y_n - \theta_n B^*(By_n - Ax_n), \\ y_{n+1} = \varrho_n y_n + (1 - \varrho_n) T_2 d_n, \end{cases}$$

where A: $H_1 \to H_3$ and B: $H_2 \to H_3$ are bounded linear mappings. They established a weak convergence result under some appropriate conditions on the control sequences $\{\varrho_n\}$ and $\{\theta_n\}$.

In 2015, Chang et al. [13] proposed the following algorithm for solving SEFPP (1.15) involving more general mappings: Let H_1, H_2 and H_3 be real Hilbert spaces and let T: $H_1 \to H_1$ and S: $H_2 \to H_2$ be Lipschitz quasi-pseudocontractive mappings. Let A: $H_1 \to H_3$ and B: $H_2 \to H_3$ are bounded linear mappings with adjoints A^* and B^*, respectively. Let $\{(x_n, y_n)\}$ be the sequence obtained from the following scheme:

$$\begin{cases} c_n = x_n - \theta_n A^*(Ax_n - By_n), \\ x_{n+1} = \varrho_n x_n + (1 - \varrho_n) \left[(1 - \zeta_n) I + \zeta_n T \left((1 - \eta_n) I + \eta_n T\right)\right] c_n, \\ d_n = y_n - \theta_n B^*(By_n - Ax_n), \\ y_{n+1} = \varrho_n y_n + (1 - \varrho_n) \left[(1 - \zeta_n) I + \zeta_n S \left((1 - \eta_n) I + \eta_n S\right)\right] d_n, \end{cases} \tag{1.17}$$

Then they proved a weak convergence theorem under some appropriate conditions on the sequences $\{\varrho_n\}$, $\{\theta_n\}$, $\{\eta_n\}$ and $\{\zeta_n\}$. They have also obtained a strong convergence result if in addition T and S are semi-compact.

Recently, researchers have become interested in increasing the speed of convergence of iterative algorithms. One of the methods employed for accelerating iterative algorithms is the inertial method. The inertial method is a method where a specific term of the sequence of iterates depends on the combination of the immediate preceding two terms. Many authors have proposed a large number of inertial iterative algorithms for solving different problems (see, for instance, [2, 3, 4, 18, 22, 27, 36]).

We now raise the following important questions:

Question 1.1. Can we introduce a new problem which generalizes the aforementioned problems? Can we also propose an inertial iterative method for approximating a solution of the problem introduced?

Motivated and inspired by the results discussed above and the ongoing research interest in this direction, we introduce the *split equality fixed point problem with multiple output sets* (SEFPPMOS) which is defined as finding a point

(1.18) $(x^*, y_1^*, y_2^*, \dots, y_N^*) \in F(T) \times F(S_1) \times F(S_2) \times \dots F(S_N)$ such that $A_i x^* = B_i y_i^*$,

where C and Q_i are nonempty, closed and convex subsets of the real Hilbert spaces H and H_i for $i = 1, 2, \dots, N$; $T\colon C \to C$ and $S_i\colon Q_i \to Q_i$, for $i = 1, 2, \dots, N$, are nonlinear mappings; $A_i\colon H \to H_i$ and $B_i\colon H_i \to H_i$ are bounded linear mappings with adjoints A_i^* and B_i^*, respectively, for $i = 1, 2, \dots, N$. We also propose an inertial algorithm for solving the problem introduced under the assumption that the governing mappings are uniformly continuous quasi-pseudocontractive.

The SEFPPMOS (1.18) is a quite general problem which contains the split feasibility problem (SFP), split equality fixed point problem (SEFPP) and the split feasibility problem with multiple output sets (SFPMOS). Thus, it can be applied in numerous real-life problems such as data compression, sensor networks, radiation therapy, antenna design, computerized tomography, immaterial science, medical image reconstruction, data denoising with more complicated constraint sets.

2. Preliminaries

This section is devoted to present some basic definitions and important results that will be used in the sequel. The strong and weak convergence of a sequence $\{x_n\} \subseteq H$ to a point $x \in H$ will be denoted as $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let H be a real Hilbert space. Then we have the following relations:

(2.19) $$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle,$$

(2.20) $$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \text{ and}$$

(2.21) $$\|x + y\|^2 \leq \|x\|^2 + 2\langle x + y, y\rangle, \text{ for all } x, y \in H.$$

The following identities also hold for all $x, y, w, z \in H$:

(2.22) $$\|x - y\|^2 + \|y - w\|^2 - \|x - w\|^2 = 2\langle y - w, y - x\rangle,$$

(2.23) $$\|y - z\|^2 + \|x - w\|^2 - \|y - w\|^2 - \|x - z\|^2 = 2\langle w - z, y - x\rangle.$$

For a nonempty, closed and convex subset C in H, the metric projection of the point $x \in H$ onto C is defined as the unique point, $P_C x$, in C such that

$$\|P_C x - x\| = \inf\{\|x - y\| : y \in C\}.$$

The metric projection has the following important properties:

(2.24) $$z = P_C x \text{ if and only if } \langle x - z, y - z\rangle \leq 0, \text{ for all } y \in C, \text{ and}$$

(2.25) $$\|y - P_C x\|^2 + \|P_C x - x\|^2 \leq \|x - y\|^2, \text{ for all } x \in H, y \in C.$$

Lemma 2.1. [34] *Let H be a real Hilbert space and let $\alpha, \beta \in \mathbb{R}$. Then $\|\alpha x + \beta y\|^2 = \alpha(\alpha + \beta)\|x\|^2 + \beta(\alpha + \beta)\|y\|^2 - \alpha\beta\|x - y\|^2$, for any $x, y \in H$.*

Lemma 2.2. [49] *Let C be a nonempty, closed and convex subset of of a real Hilbert space H and let $T\colon C \to C$ be a continuous pseudocontractive mapping. Then*

(i) $F(T)$ is a closed and convex subset of C;
(ii) $(I - T)$ is demiclosed at zero.

Lemma 2.3. [43] *Let $\{x_n\}$ be a sequence of non-negative real numbers such that*

$$x_{n+1} \leq (1-\alpha_n)\, x_n + \alpha_n d_n,$$

where $\{\alpha_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ is a sequence of real numbers such that $\limsup_{n\to\infty} d_n \leq 0$. Then $\lim_{n\to\infty} x_n = 0$.

Lemma 2.4. [24] *Let $\{c_n\}$ be a sequence of non-negative real numbers. If $\{c_{n_i}\}$ is a sub-sequence of $\{c_n\}$ such that $c_{n_i} < c_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$c_{m_k} \leq c_{m_k+1} \quad \textit{and} \quad c_k \leq c_{m_k+1}.$$

In fact, m_k=$\max\{n \leq k : c_n < c_{n+1}\}$.

Lemma 2.5. [21] *Let H be a real Hilbert space and let C is a nonempty closed convex subset of H. For all $x \in H$ and $\alpha \geq \beta > 0$, the inequalities hold:*

$$\left\| \frac{x - P_C(x-\alpha Ax)}{\alpha} \right\| \leq \left\| \frac{x - P_C(x-\beta Ax)}{\beta} \right\|.$$

Lemma 2.6. [1] *If H_1, $H_2, \ldots, H_N$ are real Hilbert spaces, then $H = H_1 \times H_2 \times \ldots \times H_N$ is also a real Hilbert space with inner product*

$$\langle (x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle + \ldots + \langle x_N, y_N \rangle,$$

for all $(x_1, x_2, \ldots, x_N),\ (y_1, y_2, \ldots, y_N) \in H$ and

$$(x_{n,1}, x_{n,2}, \ldots, x_{n,N}) \rightharpoonup (x_1, x_2, \ldots, x_N) \textit{ implies } x_{n,i} \rightharpoonup x_i, \textit{ for each } i = 1, 2, \ldots, N.$$

.

Lemma 2.7. [6] *Let $H = H_1 \times H_2 \times \ldots \times H_N$, where H_1, H_2, ..., H_N are real Hilbert spaces, and let C be a nonempty, closed and convex subset of H. If $(u_1, u_2, \ldots, u_N) \in H$ and $(u_1^*, u_2^*, \ldots, u_N^*) = P_C(u_1, u_2, \ldots, u_N)$, then*

$$\langle (u_1, u_2, \ldots, u_N) - (u_1^*, u_2^*, \ldots, u_N^*), (x_1, x_2 \ldots, x_N) - (u_1^*, u_2^*, \ldots, u_N^*) \rangle \leq 0,$$

for all $(x_1, x_2 \ldots, x_N) \in C$.

3. Main Results

In this section, we state our algorithm and discuss its convergence analysis. Before introducing our main result, we prove the following lemma.

Lemma 3.8. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let T: $C \to C$ be a continuous quasi-pseudocontractive mapping. Then $F(T)$ is closed and convex.*

Proof. The closedness of $F(T)$ readily follows from the continuity of T. We now show that $F(T)$ is convex. Let $q_1, q_2 \in F(T)$ and put $q = tq_1 + (1-t)q_2$, $t \in (0,1)$. We show that $q \in F(T)$. Let $y_\beta = (1-\beta)q + \beta Tq$ for $\beta \in (0,1)$, that is, $q - y_\beta = \beta(q - Tq)$. Now, we have for all $p \in F(T)$ that

$$
\begin{aligned}
\|q - Tq\|^2 &= \langle q - Tq, q - Tq\rangle = \frac{1}{\beta}\langle q - y_\beta, q - Tq\rangle \\
&= \frac{1}{\beta}\big\langle q - y_\beta, q - Tq - (y_\beta - Ty_\beta)\big\rangle + \frac{1}{\beta}\big\langle q - y_\beta, y_\beta - Ty_\beta\big\rangle \\
&= \frac{1}{\beta}\big\langle q - y_\beta, q - Tq - (y_\beta - Ty_\beta)\big\rangle + \frac{1}{\beta}\big\langle q - p + p - y_\beta, y_\beta - Ty_\beta\big\rangle \\
&= \frac{1}{\beta}\big\langle q - y_\beta, q - Tq - (y_\beta - Ty_\beta)\big\rangle + \frac{1}{\beta}\big\langle q - p, y_\beta - Ty_\beta\big\rangle \\
&\quad + \frac{1}{\beta}\big\langle p - y_\beta, y_\beta - Ty_\beta\big\rangle \\
&= \frac{1}{\beta}\left[\|q - y_\beta\|^2 - \langle q - y_\beta, Tq - Ty_\beta\rangle\right] + \frac{1}{\beta}\big\langle q - p, y_\beta - Ty_\beta\big\rangle \\
&\quad + \frac{1}{\beta}\big\langle p - y_\beta, y_\beta - Ty_\beta\big\rangle.
\end{aligned} \tag{3.26}
$$

Since T is quasi-pseudocontractive, we have that $\langle p - y_\beta, y_\beta - Ty_\beta\rangle \leq 0$. Thus, it follows from (3.26) that

$$
\begin{aligned}
\|q - Tq\|^2 &\leq \frac{1}{\beta}\left[\|q - y_\beta\|^2 - \langle q - y_\beta, Tq - Ty_\beta\rangle\right] + \frac{1}{\beta}\big\langle q - p, y_\beta - Ty_\beta\big\rangle \\
&= \beta\|q - Tq\|^2 - \langle q - Tq, Tq - Ty_\beta\rangle + \frac{1}{\beta}\big\langle q - p, y_\beta - Ty_\beta\big\rangle,
\end{aligned}
$$

which upon substitution of $q - y_\beta$ with $\beta(q - Tq)$ and some rearrangement gives that

$$
(1 - \beta)\|q - Tq\|^2 \leq -\langle q - Tq, Tq - Ty_\beta\rangle + \frac{1}{\beta}\big\langle q - p, y_\beta - Ty_\beta\big\rangle. \tag{3.27}
$$

Taking $p = q_i$, for $i = 1, 2$, multiplying t and $(1 - t)$ on both sides of (3.27), respectively, and adding up we get

$$
(1 - \beta)\|q - Tq\|^2 \leq -\langle q - Tq, Tq - Ty_\beta\rangle. \tag{3.28}
$$

Since $y_\beta \to q$ as $\beta \to 0$, and T is continuous, it follows from (3.28) that $\|q - Tq\| = 0$, that is, $q \in F(T)$ and hence $F(T)$ is convex. □

Throughout the rest of the paper, we shall assume the following conditions.

Conditions

(C1) Let H be a real Hilbert space and let C and be nonempty, closed and convex subset of H;

(C2) Let Q_i be nonempty, closed and convex subsets of the real Hilbert spaces H_i, $i = 1, 2, \ldots, N$;

(C3) Let $T\colon C \to C$ and $S_i\colon Q_i \to Q_i$ be uniformly continuous quasi-pseudocontractive mappings with $(I - T)$ and $(I_i - S_i)$ being demiclosed at zero, for each $i = 1, 2, \ldots, N$;

(C4) Let $A_i\colon H \to H_i$ and $B_i\colon H_i \to H_i$, $i = 1, 2, \ldots, N$, be bounded linear mappings with adjoints A_i^* and B_i^*, respectively;

(C5) Let

$$
\Omega = \{(x^*, y_1^*, y_2^*, \ldots, y_N^*) \in F(T) \times F(S_1) \times F(S_2) \times \ldots \times F(S_N) :
A_i x^* = B_i y_i^*, i = 1, 2, \ldots, N\} \neq \emptyset;
$$

(C6) Let $\{\alpha_n\} \subset (0,1)$ be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C7) Let $\{\zeta_n\}$ be a sequence such that $\zeta_n \in \left(0, \frac{1}{2}\right)$ for all $n \geq 0$ and $\lim_{n\to\infty} \frac{\zeta_n}{\alpha_n} = 0$.

We now state our algorithm and discuss its convergence analysis.

Algorithm 3.1.

Initialization: *Let $x_0, x_1 \in H$, $y_{0,i},\ y_{1,i} \in H_i$, $i = 1, 2, \ldots, N$, $\theta \in [0,1)$. Let $\gamma, l, \mu, \delta \in (0,1)$. For arbitrary points $u \in C$ and $v_i \in Q_i$, $i = 1, 2, \ldots, N$, calculate $\{x_n\}$ and $\{y_{n,i}\}$ as follows:*

Step 1. *Given the iterates $x_{n-1}, x_n \in H$ and $y_{n-1,i},\ y_{n,i} \in H_i$, choose θ_n such that $0 \leq \theta_n \leq \sigma_n$, where*

(3.29)
$$\sigma_n = \begin{cases} \min\left\{\theta, \frac{\zeta_n}{\Theta_n}\right\}, & \text{if } \Theta_n \neq 0, \\ \theta, & \text{otherwise}, \end{cases}$$

where

$$\Theta_n = \|x_n - x_{n-1}\| + \max_{1\leq i\leq N} \{\|y_{n,i} - y_{n-1,i}\|\}.$$

Step 2. *Compute*

(3.30)
$$a_n = P_C\left[x_n + \theta_n\left(x_n - x_{n-1}\right)\right],$$

(3.31)
$$b_{n,i} = P_{Q_i}\left[y_{n,i} + \theta_n\left(y_{n,i} - y_{n-1,i}\right)\right].$$

Step 3. *Compute*

(3.32)
$$c_n = P_C\left(a_n - \gamma_n \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right)\right),$$

(3.33)
$$d_{n,i} = P_{Q_i}\left(b_{n,i} - \gamma_n B_i^*\left(B_i b_{n,i} - A_i a_n\right)\right),$$

where $0 < \rho \leq \gamma_n \leq \rho_n$ with

(3.34)
$$\rho_n = \min\left\{\rho + 1, \frac{\sum_{i=1}^{N}\|A_i a_n - B_i b_{n,i}\|^2}{2\left(\left[\sum_{i=1}^{N}\|A_i^*(A_i a_n - B_i b_{n,i})\|\right]^2 + \sum_{i=1}^{N}\left[\|B_i^*(B_i b_{n,i} - A_i a_n)\|\right]^2\right)}\right\},$$

for $m \in \Upsilon = \{m \in \mathbb{N} : A_i a_m - B_i b_{m,i} \neq 0\}$, otherwise $\gamma_n = \rho$.
Step 4. *Compute*

(3.35)
$$e_n = c_n - \lambda_n(I - T)c_n,$$
$$h_{n,i} = d_{n,i} - \eta_{n,i}(I_i - S_i)d_{n,i},$$

where $\lambda_n = \gamma l^{j_m}$ and j_m is the smallest nonnegative integer j satisfying

(3.36)
$$\|(I-T)\left(c_n - \gamma l^j (I-T)c_n\right) - (I-T)c_n\| \leq \mu\|(I-T)c_n\|,$$

and $\eta_{n,i} = \gamma l^{j_{m,i}}$, where $j_{m,i}$ is the smallest nonnegative integer j_i satisfying

(3.37)
$$\|(I_i - S_i)\left(d_{n,i} - \gamma l^{j_i}(I_i - S_i)d_{n,i}\right) - (I_i - S_i)d_{n,i}\| \leq \delta\|(I_i - S_i)d_{n,i}\|.$$

Step 5. *Compute*

$$p_n = P_C\left[e_n - \lambda_n((I-T)e_n - (I-T)c_n)\right], \tag{3.38}$$
$$q_{n,i} = P_{Q_i}\left[h_{n,i} - \eta_{n,i}((I_i - S_i)h_{n,i} - (I_i - S_i)d_{n,i})\right].$$

Step 6. *Compute*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)p_n, \tag{3.39}$$

$$y_{n+1,i} = \alpha_n v_i + (1-\alpha_n)q_{n,i}. \tag{3.40}$$

Set $n = n+1$ *and go to* ***Step 1***.

Remark 3.2. *The following are some of the novelties of our results:*

i. *The problem we introduced is more general than the SEFPP (1.15);*
ii. *It extends the nature of mappings considered in the results of Chang et al.* [13] *from Lipschitz continuous to uniformly continuous;*
iii. *The type of convergence in Chang et al.* [13] *is improved from weak to strong convergence;*

Remark 3.3. *We deduce from* (3.29) *and condition* $(C7)$ *that*

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}|\|x_n - x_{n-1}\| = 0,$$

and this together with the condition on α_n *gives that*

$$\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0. \tag{3.41}$$

Lemma 3.9. *Assume that Conditions* $(C1)-(C5)$ *hold. Then the Armijo line-search rules* (3.36) *and* (3.37) *are well defined.*

Proof. It is sufficient to show that (3.36) is well defined. If c_n is a fixed point of T, then obviously $j = 0$ satisfies the relation (3.36). Assume on the contrary that c_n is not a fixed point of T. Then the right hand side of (3.36) is always positive. On the other hand, we have from the continuity of T and the fact $l \in (0,1)$ that

$$\lim_{j\to\infty} \|(I-T)\left(c_n - \gamma l^j(I-T)c_n\right) - (I-T)c_n\| = 0.$$

Therefore, there exists a nonnegative integer j which satisfies the inequality (3.36) and hence (3.36) is well defined. Similarly, there exists a nonnegative integer j_i which satisfies the relation (3.37) for each $i = 1,2,3,\ldots,N$, and hence the proof is complete. □

Remark 3.4. *We note here that the line search rule (3.36) can be rewritten as*

$$\lambda_n\|(I-T)\left(c_n - \gamma l^j(I-T)c_n\right) - (I-T)c_n\| \le \mu\lambda_n\|(I-T)c_n\|,$$

or equivalently

$$\lambda_n\|(I-T)e_n - (I-T)c_n\| \le \mu\|e_n - c_n\|. \tag{3.42}$$

Similarly, (3.37) can be rewritten as

$$\eta_{n,i}\|(I_i - S_i)h_{n,i} - (I_i - S_i)d_{n,i}\| \le \delta\|h_{n,i} - d_{n,i}\|$$

Theorem 3.1. *Assume that conditions* $(C1)-(C7)$ *hold. Then the sequences* $\{x_n\}$ *and* $\{y_{n,i}\}$, $i = 1,2,\ldots,N$, *generated by Algorithm 3.1 are bounded.*

Proof. Let $(x^*, y_1^*, y_2^*, \dots, y_N^*) \in \Omega$. Then $x^* \in C$ and $y_i^* \in Q_i$, for each $i = 1, 2, \dots, N$. From (3.39) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1-\alpha_n)p_n - x^*\|^2 \\ &= \|\alpha_n(u - x^*) + (1-\alpha_n)(p_n - x^*)\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|p_n - x^*\|^2. \end{aligned} \tag{3.43}$$

From (3.38) and nonexpansivity of the metric projection, we obtain

$$\begin{aligned} \|p_n - x^*\|^2 &\le \|e_n - \lambda_n((I-T)e_n - (I-T)c_n) - x^*\|^2 \\ &= \|p_n\|^2 + \|x^*\|^2 - 2\langle e_n - \lambda_n\left((I-T)e_n - (I-T)c_n\right), x^*\rangle \\ &= \|p_n\|^2 + \|x^*\|^2 - 2\langle e_n, x^*\rangle + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle. \\ &= \|p_n\|^2 - \|e_n\|^2 + \|e_n\|^2 + \|x^*\|^2 - 2\langle e_n, x^*\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle. \\ &= \|e_n - x^*\|^2 - \|e_n\|^2 + \|p_n\|^2 + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle \\ &= \|e_n - x^*\|^2 - \|e_n\|^2 + 2\|p_n\|^2 - \|p_n\|^2 - 2\langle e_n, p_n\rangle + 2\langle e_n, p_n\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle \\ &= \|e_n - x^*\|^2 - \|p_n - e_n\|^2 + 2\|p_n\|^2 - 2\langle e_n, p_n\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle \\ &= \|e_n - x^*\|^2 - \|p_n - e_n\|^2 + 2\langle p_n, p_n\rangle - 2\langle e_n, p_n\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle \\ &= \|e_n - x^*\|^2 - \|p_n - e_n\|^2 + 2\langle p_n, p_n - e_n\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle \\ &= \|e_n - x^*\|^2 - \|p_n - e_n\|^2 - 2\langle p_n, \lambda_n\left[(I-T)e_n - (I-T)c_n\right]\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^*\rangle \\ &= \|e_n - x^*\|^2 - \|p_n - e_n\|^2 + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^* - p_n\rangle. \end{aligned} \tag{3.44}$$

But, we have from (2.23) that

$$\|e_n - x^*\|^2 - \|e_n - p_n\|^2 = \|c_n - x^*\|^2 - \|c_n - p_n\|^2 + 2\langle e_n - c_n, p_n - x^*\rangle. \tag{3.45}$$

Substituting (3.45) into (3.44), we obtain

$$\begin{aligned} \|p_n - x^*\|^2 \le \|c_n - x^*\|^2 - \|p_n - c_n\|^2 + 2\langle e_n - c_n, p_n - x^*\rangle \\ + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^* - p_n\rangle. \end{aligned} \tag{3.46}$$

We also have from (2.22) that

$$\|p_n - c_n\|^2 = \|p_n - e_n\|^2 + \|e_n - c_n\|^2 - 2\langle e_n - c_n, e_n - p_n\rangle. \tag{3.47}$$

Substituting (3.47) into (3.46), we obtain

$$\begin{aligned} \|p_n - x^*\|^2 \le \|c_n - x^*\|^2 - \|p_n - e_n\|^2 - \|e_n - c_n\|^2 + 2\langle e_n - c_n, e_n - p_n\rangle \\ + 2\langle e_n - c_n, p_n - x^*\rangle + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^* - p_n\rangle. \end{aligned} \tag{3.48}$$

We have from (3.32), nonexpansivity of the metric projection and property (2.21) that

$$\begin{aligned}\|c_n - x^*\|^2 &= \Big\| P_C\big[a_n - \gamma_n \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right)\big] - x^* \Big\|^2 \\ &\le \Big\| a_n - \gamma_n \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right) - x^* \Big\|^2 \\ &\le \|a_n - x^*\|^2 - 2\gamma_n \Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - x^* \Big\rangle, \end{aligned} \tag{3.49}$$

where $s_n = a_n - \gamma_n \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right)$. Combining (3.48) and (3.49), we get

$$\begin{aligned}\|p_n - x^*\|^2 \le \|a_n - x^*\|^2 &- \|p_n - e_n\|^2 - \|e_n - c_n\|^2 + 2\langle e_n - c_n, e_n - p_n\rangle \\ &+ 2\langle e_n - c_n, p_n - x^*\rangle + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^* - p_n\rangle \\ &- 2\gamma_n \Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - x^* \Big\rangle. \end{aligned} \tag{3.50}$$

Moreover, we have from (3.30), nonexpansivity of the metric projection and (2.20) that

$$\begin{aligned}\|a_n - x^*\|^2 &= \|P_C\left(x_n + \theta_n(x_n - x_{n-1})\right) - x^*\|^2 \\ &\le \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|x^*\|^2 - 2\langle x_n + \theta_n(x_n - x_{n-1}), x^*\rangle + \|t_n\|^2, \end{aligned} \tag{3.51}$$

where $t_n = x_n + \theta_n(x_n - x_{n-1})$. We now obtain from (3.51) that

$$\begin{aligned}\|a_n - x^*\|^2 &\le \|x^*\|^2 - 2\langle x_n, x^*\rangle - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle + \|t_n\|^2 \\ &= \|x^*\|^2 - \|x_n\|^2 + \|x_n\|^2 - 2\langle x_n, x^*\rangle - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle + \|t_n\|^2 \\ &= \|x_n - x^*\|^2 - \|x_n\|^2 - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle + \|t_n\|^2 \\ &= \|x_n - x^*\|^2 - \|x_n\|^2 - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle - \|t_n\|^2 \\ &\quad + 2\|t_n\|^2 - 2\langle x_n, t_n\rangle + 2\langle x_n, t_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle + 2\|t_n\|^2 - 2\langle x_n, t_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle + 2\langle t_n, t_n - x_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - 2\theta_n\langle x_n - x_{n-1}, x^*\rangle + 2\theta_n\langle t_n, x_n - x_{n-1}\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 + 2\theta_n\langle t_n - x^*, x_n - x_{n-1}\rangle. \end{aligned} \tag{3.52}$$

Substituting (3.52) into (3.50), we obtain

(3.53)

$$\begin{aligned}\|p_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - \|p_n - e_n\|^2 - \|e_n - c_n\|^2 + 2\langle e_n - c_n, e_n - p_n\rangle \\ &\quad + 2\langle e_n - c_n, p_n - x^*\rangle + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^* - p_n\rangle \\ &\quad - 2\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - x^*\Big\rangle - 2\theta_n\langle x^* - t_n, x_n - x_{n-1}\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - \|p_n - e_n\|^2 - \|e_n - c_n\|^2 \\ &\quad + 2\langle e_n - c_n, e_n - x^*\rangle + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, x^* - e_n\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, e_n - p_n\rangle \\ &\quad - 2\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - x^*\Big\rangle - 2\theta_n\langle x^* - t_n, x_n - x_{n-1}\rangle. \\ &= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - \|p_n - e_n\|^2 - \|e_n - c_n\|^2 \\ &\quad + 2\langle \lambda_n\left[(I-T)e_n - (I-T)c_n\right] - (e_n - c_n), x^* - e_n\rangle \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, e_n - p_n\rangle - 2\theta_n\langle x^* - t_n, x_n - x_{n-1}\rangle \\ &\quad - 2\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - x^*\Big\rangle.\end{aligned}$$

But, $e_n = c_n - \lambda_n(I-T)c_n$. Now let $\tau_n = (I-T)e_n$. Then we have

$$e_n + \lambda_n\tau_n = c_n + \lambda_n\left[(I-T)e_n - (I-T)c_n\right],$$

which implies that

$$\tau_n = \frac{1}{\lambda_n}\left[\lambda_n\left((I-T)e_n - (I-T)c_n\right) - (e_n - c_n)\right]. \tag{3.54}$$

Since T is quasi-pseudocontractive, we have that

$$\langle \tau_n, e_n - x^*\rangle \geq 0. \tag{3.55}$$

Thus, substituting (3.54) into (3.55), we obtain

$$\langle \lambda_n\left((I-T)e_n - (I-T)c_n\right) - (e_n - c_n), e_n - x^*\rangle \geq 0. \tag{3.56}$$

Thus, we obtain from (3.53) and (3.56) that

$$\begin{aligned}\|p_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - \|p_n - e_n\|^2 - \|e_n - c_n\|^2 \\ &\quad + 2\lambda_n\langle (I-T)e_n - (I-T)c_n, e_n - p_n\rangle \\ &\quad - 2\theta_n\langle x^* - t_n, x_n - x_{n-1}\rangle \\ &\quad - 2\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - x^*\Big\rangle.\end{aligned} \tag{3.57}$$

We also have from the Cauchy Schwarz Inequality that

(3.58)

$$\begin{aligned}-\langle\theta_n(x_n-x_{n-1}),x^*-t_n\rangle &\leq \theta_n\|x_n-x_{n-1}\|\ \|x^*-t_n\|\\ &\leq \frac{\theta_n}{2}\|x_n-x_{n-1}\|\left[\|x^*-t_n\|^2+1\right]\\ &= \frac{\theta_n}{2}\|x_n-x_{n-1}\|\left[\|x^*-x_n+x_n-t_n\|^2+1\right]\\ &\leq \frac{\theta_n}{2}\|x_n-x_{n-1}\|\left[2\|x^*-x_n\|^2+2\|x_n-t_n\|^2+1\right]\\ &= \theta_n\|x_n-x_{n-1}\|\|x_n-x^*\|^2+\theta_n\|x_n-x_{n-1}\|\|x_n-t_n\|^2\\ &\quad+\frac{\theta_n}{2}\|x_n-x_{n-1}\|.\end{aligned}$$

From (3.57), (3.58) and the Cauchy Schwarz inequality, we obtain that

$$\begin{aligned}\|p_n-x^*\|^2 &\leq \|x_n-x^*\|^2-\|x_n-t_n\|^2-\|p_n-e_n\|^2-\|e_n-c_n\|^2\\ &\quad+2\lambda_n\langle(I-T)e_n-(I-T)c_n,e_n-p_n\rangle+2\zeta_n\|x_n-x^*\|^2\\ &\quad+2\zeta_n\|x_n-t_n\|^2+\zeta_n-2\gamma_n\Big\langle\sum_{i=1}^N A_i^*\left(A_ia_n-B_ib_{n,i}\right),s_n-x^*\Big\rangle\\ &\leq \|x_n-x^*\|^2-\|x_n-t_n\|^2-\|p_n-e_n\|^2-\|e_n-c_n\|^2\\ &\quad+2\lambda_n\|(I-T)e_n-(I-T)c_n\|\|e_n-p_n\|+2\zeta_n\|x_n-x^*\|^2\\ &\quad+2\zeta_n\|x_n-t_n\|^2+\zeta_n-2\gamma_n\Big\langle\sum_{i=1}^N A_i^*\left(A_ia_n-B_ib_{n,i}\right),s_n-x^*\Big\rangle,\end{aligned}$$

which implies by (3.42) that

(3.59)

$$\begin{aligned}\|p_n-x^*\|^2 &\leq \|x_n-x^*\|^2-\|x_n-t_n\|^2-\|p_n-e_n\|^2-\|e_n-c_n\|^2\\ &\quad+2\mu\|e_n-c_n\|\|e_n-p_n\|+2\zeta_n\|x_n-x^*\|^2+2\zeta_n\|x_n-t_n\|^2\\ &\quad+\zeta_n-2\gamma_n\Big\langle\sum_{i=1}^N A_i^*\left(A_ia_n-B_ib_{n,i}\right),s_n-x^*\Big\rangle\\ &\leq (1+2\zeta_n)\|x_n-x^*\|^2-(1-2\zeta_n)\|x_n-t_n\|^2-\|p_n-e_n\|^2-\|e_n-c_n\|^2\\ &\quad+2\mu\left[\frac{\|e_n-c_n\|^2+\|e_n-p_n\|^2}{2}\right]+\zeta_n\\ &\quad-2\gamma_n\Big\langle\sum_{i=1}^N A_i^*\left(A_ia_n-B_ib_{n,i}\right),s_n-x^*\Big\rangle\\ &\leq (1+2\zeta_n)\|x_n-x^*\|^2-(1-2\zeta_n)\|x_n-t_n\|^2-(1-\mu)\|p_n-e_n\|^2\\ &\quad-(1-\mu)\|e_n-c_n\|^2+\zeta_n-2\gamma_n\Big\langle\sum_{i=1}^N A_i^*\left(A_ia_n-B_ib_{n,i}\right),s_n-x^*\Big\rangle.\end{aligned}$$

As $\zeta_n \in \left(0, \frac{1}{2}\right)$ and $\mu \in (0,1)$, we obtain from (3.59) that

$$\|p_n - x^*\|^2 \le (1+2\zeta_n)\|x_n - x^*\|^2 + \zeta_n - 2\gamma_n \Big\langle \sum_{i=1}^{N} A_i^* \left(A_i a_n - B_i b_{n,i}\right), s_n - x^* \Big\rangle. \tag{3.60}$$

Substituting (3.59) back into (3.43), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le &+\alpha_n \|u - x^*\|^2 + (1-\alpha_n)(1+2\zeta_n)\|x_n - x^*\|^2 \\ &- (1-\alpha_n)(1-2\zeta_n)\|x_n - t_n\|^2 - (1-\alpha_n)(1-\mu)\|p_n - e_n\|^2 \\ &- (1-\alpha_n)(1-\mu)\|e_n - c_n\|^2 + (1-\alpha_n)\zeta_n \\ &- 2(1-\alpha_n)\gamma_n \Big\langle \sum_{i=1}^{N} A_i^* \left(A_i a_n - B_i b_{n,i}\right), s_n - x^* \Big\rangle. \end{aligned} \tag{3.61}$$

Similarly, we have

$$\begin{aligned} \|y_{n+1,i} - y_i^*\|^2 \le \alpha_n \|v_i - y_i^*\|^2 &+ (1-\alpha_n)(1+2\zeta_n)\|y_{n,i} - y_i^*\|^2 \\ &- (1-\alpha_n)(1-2\zeta_n)\|y_{n,i} - u_{n,i}\|^2 - (1-\alpha_n)(1-\delta)\|q_{n,i} - h_{n,i}\|^2 \\ &- (1-\alpha_n)(1-\delta)\|h_{n,i} - d_{n,i}\|^2 + (1-\alpha_n)\zeta_n \\ &- 2(1-\alpha_n)\gamma_n \langle B_i^* \left(B_i b_{n,i} - A_i a_n\right), r_{n,i} - y_i^* \rangle, \end{aligned} \tag{3.62}$$

for each $i \in \{1, 2, 3, \dots, N\}$. Since $\frac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$, for any $\varepsilon \in \left(0, \frac{1}{2}\right)$ there exits $n_0 \in \mathbb{N}$ such that $\zeta_n < \varepsilon\alpha_n$, for all $n \ge n_0$. Thus, we obtain from (3.61) and (3.62), respectively, that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le \alpha_n \|u - x^*\|^2 + (1-\alpha_n)(1+2\varepsilon\alpha_n)\|x_n - x^*\|^2 + \varepsilon\alpha_n \\ - 2(1-\alpha_n)\gamma_n \Big\langle \sum_{i=1}^{N} A_i^* \left(A_i a_n - B_i b_{n,i}\right), s_n - x^* \Big\rangle, \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} \|y_{n+1,i} - y_i^*\|^2 \le \alpha_n \|v_i - y_i^*\|^2 + (1-\alpha_n)(1+2\varepsilon\alpha_n)\|y_{n,i} - y_i^*\|^2 + \varepsilon\alpha_n \\ - 2(1-\alpha_n)\gamma_n \langle B_i^* \left(B_i b_{n,i} - A_i a_n\right), r_{n,i} - y_i^* \rangle \text{ for all } n \ge n_0. \end{aligned} \tag{3.64}$$

Taking the summation of (3.64), we obtain

(3.65)

$$\begin{aligned} \sum_{i=1}^{N} \|y_{n+1,i} - y_i^*\|^2 \le \alpha_n \sum_{i=1}^{N} \|v_i - y_i^*\|^2 + (1-\alpha_n)(1+2\varepsilon\alpha_n) \sum_{i=1}^{N} \|y_{n,i} - y_i^*\|^2 + N\varepsilon\alpha_n \\ - 2(1-\alpha_n)\gamma_n \sum_{i=1}^{N} \langle B_i^* \left(B_i b_{n,i} - A_i a_n\right), r_{n,i} - y_i^* \rangle. \end{aligned}$$

Denote $\Delta_n = \|x_n - x^*\|^2 + \sum_{i=1}^{N} \|y_{n,i} - y_i^*\|^2$ and $\Gamma = \|u - x^*\|^2 + \sum_{i=1}^{N} \|v_i - y_i^*\|^2$, for all $n \ge n_0$. Then combining (3.63) and (3.65), we obtain

$$\begin{aligned} \Delta_{n+1} \le \alpha_n \Gamma + (1-\alpha_n)(1+2\varepsilon\alpha_n)\Delta_n + (N+1)\varepsilon\alpha_n \\ - 2(1-\alpha_n)\gamma_n \sum_{i=1}^{N} \langle A_i a_n - B_i b_{n,i},\ A_i s_n - B_i r_{n,i} \rangle, \text{ for all } n \ge n_0. \end{aligned} \tag{3.66}$$

We have from the Cauchy Schwarz inequality that

$$
\begin{aligned}
&-\sum_{i=1}^{N}\langle A_ia_n - B_ib_{n,i},\ A_is_n - B_ir_{n,i}\rangle \\
&= -\sum_{i=1}^{N}\langle A_ia_n - B_ib_{n,i},\ A_ia_n - B_ib_{n,i}\rangle - \sum_{i=1}^{N}\langle A_ia_n - B_ib_{n,i},\ A_is_n - A_ia_n\rangle \\
&\quad -\sum_{i=1}^{N}\langle A_ia_n - B_ib_{n,i},\ B_ib_{n,i} - B_ir_{n,i}\rangle \\
&\leq -\sum_{i=1}^{N}\|A_ia_n - B_ib_{n,i}\|^2 + \sum_{i=1}^{N}\|A_i^*(A_ia_n - B_ib_{n,i})\|\|s_n - a_n\| \\
&\quad +\sum_{i=1}^{N}\|B_i^*(A_ia_n - B_ib_{n,i})\|\|b_{n,i} - r_{n,i}\|.
\end{aligned} \tag{3.67}
$$

Moreover, we have

$$
\|s_n - a_n\| = \|a_n - \gamma_n\sum_{i=1}^{N}A_i^*(A_ia_n - B_ib_{n,i}) - a_n\| = \gamma_n\|\sum_{i=1}^{N}A_i^*(A_ia_n - B_ib_{n,i})\|, \tag{3.68}
$$

and

$$
\|r_{n,i} - b_{n,i}\| = \|b_{n,i} - \gamma_nB_i^*(B_ib_{n,i} - A_ia_n) - b_{n,i}\| = \gamma_n\|B_i^*(B_ib_{n,i} - A_ia_n)\|. \tag{3.69}
$$

Combining (3.67), (3.68) and (3.69) and using (3.34), we obtain

$$
\begin{aligned}
&-2\gamma_n\sum_{i=1}^{N}\langle A_ia_n - B_ib_{n,i}, A_is_n - B_ir_{n,i}\rangle \\
&\leq -2\gamma_n\sum_{i=1}^{N}\|A_ia_n - B_ib_{n,i}\|^2 + 2\gamma_n^2\left(\sum_{i=1}^{N}\|A_i^*(A_ia_n - B_ib_{n,i})\|\right)^2 \\
&\quad +2\gamma_n^2\sum_{i=1}^{N}(\|B_i^*(A_ia_n - B_ib_{n,i})\|)^2 \\
&\leq -\rho\sum_{i=1}^{N}\|A_ia_n - B_ib_{n,i}\|^2 - \gamma_n\sum_{i=1}^{N}\|A_ia_n - B_ib_{n,i}\|^2 \\
&\quad +2\gamma_n^2\left(\sum_{i=1}^{N}\|A_i^*(A_ia_n - B_ib_{n,i})\|\right)^2 + 2\gamma_n^2\sum_{i=1}^{N}(\|B_i^*(A_ia_n - B_ib_{n,i})\|)^2 \\
&\leq -\rho\sum_{i=1}^{N}\|A_ia_n - B_ib_{n,i}\|^2.
\end{aligned} \tag{3.70}
$$

Combining (3.66) and (3.70), we obtain

(3.71)

$$\begin{aligned}\Delta_{n+1} &\leq \alpha_n \Gamma + (1-\alpha_n)(1+2\varepsilon\alpha_n)\Delta_n + (N+1)\varepsilon\alpha_n - (1-\alpha_n)\rho\sum_{i=1}^{N}\|A_i a_n - B_i b_{n,i}\|^2 \\ &\leq \alpha_n \Gamma + (1-\alpha_n)(1+2\varepsilon\alpha_n)\Delta_n + (N+1)\varepsilon\alpha_n \\ &\leq [1-\alpha_n(1-2\varepsilon)]\Delta_n + \alpha_n[\Gamma + (1+N)\varepsilon] \\ &= [1-\alpha_n(1-2\varepsilon)]\Delta_n + \alpha_n(1-2\varepsilon)\left[\frac{\Gamma+(1+N)\varepsilon}{(1-2\varepsilon)}\right] \leq \max\left\{\Delta_n, \frac{\Gamma+(1+N)\varepsilon}{(1-2\varepsilon)}\right\}.\end{aligned}$$

We conclude by induction that $\Delta_n \leq \max\left\{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_{n_0-1}, \frac{\Gamma+(1+N)\varepsilon}{(1-2\varepsilon)}\right\}$. Thus, $\{\Delta_n\}$ is bounded. This implies that the sequences $\{\|x_n - x^*\|\}$ and $\{\|y_{n,i} - y_i^*\|\}$ are bounded which in turn implies that $\{x_n\}$ and $\{y_{n,i}\}$ are bounded for each $i = 1, 2, 3, \ldots, N$, and hence the proof is complete. □

Theorem 3.2. *Let* $(C1) - (C7)$ *hold. Then the sequence* $\{(x_n,\ y_{n,1},\ y_{n,1}, \ldots,\ y_{n,N})\}$ *generated by Algorithm 3.1 converges strongly to* $(p, q_1, q_2, \ldots, q_N)$*, where* $(p, q_1, q_2, \ldots, q_N) = P_\Omega(u, v_1, v_2, \ldots, v_N)$.

Proof. We can easily conclude from Lemma 3.8 that the set Ω is closed and convex and thus projections onto Ω are well defined. Let $(p, q_1, q_2, \ldots, q_N) = P_\Omega(u, v_1, v_2, \ldots, v_N)$. Then it follows by Lemma 2.7 that

$$\big\langle (u, v_1, v_2, \ldots, v_N) - (p, q_1, q_2, \ldots, q_N), (w, w_1, w_2, \ldots, w_N) - (p, q_1, q_2, \ldots, q_N)\big\rangle \leq 0, \tag{3.72}$$

for all $(w, w_1, w_2, \ldots, w_N) \in \Omega$. Now, we obtain from (3.39) and (2.21) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n u + (1-\alpha_n)p_n - p\|^2 = \|\alpha_n(u-p) + (1-\alpha_n)(p_n - p)\|^2 \\ &\leq (1-\alpha_n)\|(p_n - p)\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle.\end{aligned} \tag{3.73}$$

Substituting (3.60) into (3.73) with $x^* = p$, we obtain

(3.74)

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\left[(1+2\zeta_n)\|x_n - p\|^2 + \zeta_n\right] \\ &\quad - 2(1-\alpha_n)\gamma_n\left[\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n - p\Big\rangle\right] + 2\alpha_n\langle u - p, x_{n+1} - p\rangle.\end{aligned}$$

Since $\frac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$, for any $\varepsilon \in \left(0, \frac{1}{2}\right)$ there exits $n_1 \in \mathbb{N}$ such that $\zeta_n < \varepsilon\alpha_n,$ for all $n \geq n_1$. Thus, we obtain from (3.74) and the Cauchy Schwarz inequality that

(3.75)

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\|x_n-p\|^2 + 2\varepsilon\alpha_n\|x_n-p\|^2 \\
&\quad - 2(1-\alpha_n)\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n-p\Big\rangle \\
&\quad + 2\alpha_n\|u-p\|\,\|x_{n+1}-x_n\| + \langle u-p, x_n-p\rangle + \zeta_n \\
&= [1-\alpha_n(1-2\varepsilon)]\|x_n-p\|^2 - 2(1-\alpha_n)\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n-p\Big\rangle \\
&\quad + 2\alpha_n\|u-p\|\,\|x_{n+1}-x_n\| + \langle u-p, x_n-p\rangle + \alpha_n\frac{\zeta_n}{\alpha_n} \\
&= [1-\alpha_n(1-2\varepsilon)]\|x_n-p\|^2 - 2(1-\alpha_n)\gamma_n\Big\langle \sum_{i=1}^{N} A_i^*\left(A_i a_n - B_i b_{n,i}\right), s_n-p\Big\rangle \\
&\quad + (1-2\varepsilon)\alpha_n\left[\frac{2\|u-p\|\,\|x_{n+1}-x_n\| + 2\langle u-p, x_n-p\rangle + \dfrac{\zeta_n}{\alpha_n}}{1-2\varepsilon}\right],
\end{aligned}$$

for all $n \ge n_1$. Similarly, we have for each $i \in \{1,2,\ldots,N\}$ that

$$\begin{aligned}
\|y_{n+1,i}-q_i\|^2 &\le [1-\alpha_n(1-2\varepsilon)]\|y_{n,i}-q_i\|^2 - 2(1-\alpha_n)\gamma_n\big\langle B_i^*\left(B_i b_{n,i} - A_i a_n\right), r_{n,i}-q_i\big\rangle \\
&\quad + (1-2\varepsilon)\alpha_n\left[\frac{2\|v_i-q_i\|\,\|y_{n+1,i}-y_{n,i}\| + 2\langle v_i-q_i, y_{n,i}-q_i\rangle + \dfrac{\zeta_n}{\alpha_n}}{1-2\varepsilon}\right],
\end{aligned}$$

which gives upon summation that that

(3.76)

$$\begin{aligned}
&\sum_{i=1}^{N}\|y_{n+1,i}-q_i\|^2 \\
&\quad\le [1-\alpha_n(1-2\varepsilon)]\sum_{i=1}^{N}\|y_{n,i}-q_i\|^2 - 2(1-\alpha_n)\gamma_n\sum_{i=1}^{N}\big\langle B_i^*\left(B_i b_{n,i} - A_i a_n\right), r_{n,i}-q_i\big\rangle \\
&\quad + (1-2\varepsilon)\alpha_n\left[\frac{2\displaystyle\sum_{i=1}^{N}\|v_i-q_i\|\,\|y_{n+1,i}-y_{n,i}\| + 2\sum_{i=1}^{N}\langle v_i-q_i, y_{n,i}-q_i\rangle + N\dfrac{\zeta_n}{\alpha_n}}{1-2\varepsilon}\right],
\end{aligned}$$

for all $n \ge n_1$. Then combining (3.75) and (3.76), we obtain

$$\Delta_{n+1} \le [1-\alpha_n(1-2\varepsilon)]\,\Delta_n + \alpha_n(1-2\varepsilon)(\Upsilon_n + \Xi_n), \text{ for all } n \ge n_1, \tag{3.77}$$

where

$$\Upsilon_n = \frac{2\|u-p\|\,\|x_{n+1}-x_n\| + 2\langle u-p, x_n-p\rangle + \alpha_n\dfrac{\zeta_n}{\alpha_n}}{1-2\varepsilon}, \tag{3.78}$$

and

$$\Xi_n = \frac{2\sum_{i=1}^{N}\|v_i - q_i\|\,\|y_{n+1,i} - y_{n,i}\| + 2\sum_{i=1}^{N}\langle v_i - q_i, y_{n,i} - q_i\rangle + N\alpha_n\frac{\zeta_n}{\alpha_n}}{1-2\varepsilon}. \tag{3.79}$$

Combining (3.61) and the summation over $i = 1, 2, 3, \ldots, N$ of (3.62) and using the relation (3.70), we obtain upon rearrangement that

(3.80)

$$\begin{aligned}
&(1-\alpha_n)(1-2\zeta_n)\|x_n - t_n\|^2 + (1-\alpha_n)(1-\mu)\|p_n - e_n\|^2 + (1-\alpha_n)(1-\mu)\|e_n - c_n\|^2\\
&+ (1-\alpha_n)(1-2\zeta_n)\sum_{i=1}^{N}\|y_{n,i} - u_{n,i}\|^2 + (1-\alpha_n)(1-\delta)\sum_{i=1}^{N}\|q_{n,i} - h_{n,i}\|^2\\
&+ (1-\alpha_n)(1-\delta)\sum_{i=1}^{N}\|h_{n,i} - d_{n,i}\|^2 + \rho(1-\alpha_n)\sum_{i=1}^{N}\|A_i a_n - B_i b_{n,i}\|^2\\
&\qquad\le \Delta_n - \Delta_{n+1} + \alpha_n\left[\Gamma + (2\varepsilon - 1)\Delta_n + \frac{(N+1)\zeta_n}{\alpha_n}\right].
\end{aligned}$$

Now, we consider two cases on the sequence $\{\Delta_n\}$ of nonnegative real numbers.

Case I: Suppose the sequence $\{\Delta_n\}$ is nonincreasing. Then we have by the Monotone Convergence Theorem that $\{\Delta_n\}$ is convergent. Taking the limit as $n \to \infty$ of (3.80), we get

$$\lim_{n\to\infty}\sum_{i=1}^{N}\|A_i a_n - B_i b_{n,i}\| = 0, \tag{3.81}$$

$$\lim_{n\to\infty}\|x_n - t_n\| = \lim_{n\to\infty}\|p_n - e_n\| = \lim_{n\to\infty}\|e_n - c_n\| = 0, \tag{3.82}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{N}\|y_{n,i} - u_{n,i}\| = \lim_{n\to\infty}\sum_{i=1}^{N}\|q_{n,i} - h_{n,i}\| = \lim_{n\to\infty}\sum_{i=1}^{N}\|h_{n,i} - d_{n,i}\|. \tag{3.83}$$

We also have from (3.32) and the condition on γ_n that

(3.84)

$$\begin{aligned}
&\|c_n - a_n\| = \|a_n - P_C(a_n - \gamma_n\sum_{i=1}^{N}A_i^*\left(A_i a_n - B_i b_{n,i}\right))\|\\
&\le \gamma_n\sum_{i=1}^{N}\|A_i^*\left(A_i a_n - B_i b_{n,i}\right)\| \le (\rho+1)\sum_{i=1}^{N}\|A_i^*\left(A_i a_n - B_i b_{n,i}\right)\| \to 0, \text{ as } n \to \infty.
\end{aligned}$$

From (3.39), boundedness of $\{p_n\}$ and the condition on α_n, we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - p_n\| = \lim_{n\to\infty}\|\alpha_n u + (1-\alpha_n)p_n - p_n\| = \lim_{n\to\infty}\alpha_n\|u - p_n\| = 0. \tag{3.85}$$

Moreover, we have from the nonexpansivity of the metric projection and (3.41) that

$$\begin{aligned}
\lim_{n\to\infty}\|a_n - x_n\| &= \lim_{n\to\infty}\|P_C\left(x_n + \theta_n(x_n - x_{n-1})\right) - P_C x_n\|\\
&\le \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0.
\end{aligned} \tag{3.86}$$

From (3.82), (3.84), (3.85) and (3.86), we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 \tag{3.87}$$

Similarly, we obtain that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\|y_{n+1,i}-y_n\|=0. \tag{3.88}$$

Since the sequence $\{(x_n, y_{n,1},\ y_{n,2}, \dots y_{n,N})\}$ is a bounded sequence, there exists a subsequence
$\{(x_{n_k}, y_{n_k,1}, y_{n_k,2}, \dots, y_{n_k,N})\}$ of $\{(x_n, y_{n,1},\ y_{n,2}, \dots y_{n,N})\}$ and a point $(\bar{x}, \bar{y_1}, \bar{y_2}, \dots, \bar{y_N})$ such that
$(x_{n_k}, y_{n_k,1}, y_{n_k,2}, \dots, y_{n_k,N}) \rightharpoonup (\bar{x}, \bar{y_1}, \bar{y_2}, \dots, \bar{y_N})$ and

$$\begin{aligned}
&\limsup_{n\to\infty}\Big\langle (u, v_1, v_2, \dots, v_N) - (p, q_1, q_2, \dots, q_N),\\
&\qquad (x_n, y_{n,1}, y_{n,2}, \dots, y_{n,N}) - (p, q_1, q_2, \dots, q_N)\Big\rangle\\
&= \lim_{k\to\infty}\Big\langle (u, v_1, v_2, \dots, v_N) - (p, q_1, q_2, \dots, q_N),\\
&\qquad (x_{n_k}, y_{n_k,1}, y_{n_k,2}, \dots, y_{n,N}) - (p, q_1, q_2, \dots, q_N)\Big\rangle.
\end{aligned} \tag{3.89}$$

As a consequence, we have $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k,i} \rightharpoonup \bar{y_i}$, for each $i \in \{1, 2, \dots, N\}$. Now, we show that $(\bar{x}, \bar{y_1}, \bar{y_2}, \dots, \bar{y_N}) \in \Omega$.
Put $z_{n_k} = P_C\left(c_{n_k} - \lambda_{n_k} l^{-1}(I-T)c_{n_k}\right)$. From (3.84) and (3.86), we obtain $c_{n_k} \rightharpoonup \bar{x}$. By Lemma 2.5 and (3.82), we have

$$\|c_{n_k} - z_{n_k}\| \le \frac{1}{l}\|c_{n_k} - e_{n_k}\| \to 0, \text{ as } n \to \infty. \tag{3.90}$$

Therefore, $z_{n_k} \rightharpoonup \bar{x}$. Thus, we have that $\{z_{n_k}\}$ is bounded. Since $I - T$ is uniformly continuous, we have

$$\|(I-T)c_{n_k} - (I-T)z_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{3.91}$$

By the Armijo line-search rule (3.36), we have

$$\lambda_{n_k} l^{-1}\|(I-T)(c_{n_k} - \lambda_{n_k} l^{-1}(I-T)c_{n_k}) - (I-T)c_{n_k}\| > \mu\|\lambda_{n_k} l^{-1}(I-T)c_{n_k})\|,$$

which implies

$$\frac{1}{\mu}\|(I-T)(c_{n_k} - \lambda_{n_k} l^{-1}(I-T)c_{n_k}) - (I-T)c_{n_k}\| > \|(I-T)c_{n_k})\|. \tag{3.92}$$

We conclude from (3.91) and (3.92) that $\lim_{k\to\infty}(I-T)c_{n_k} = 0$. This together with the fact $c_{n_k} \rightharpoonup \bar{x}$ and demiclosedness of T implies that $(I-T)\bar{x} = 0$, that is, $\bar{x} \in F(T)$. It can be similarly shown that $\bar{y_i} \in F(S_i)$.
Moreover, we have by (2.21) that

$$\begin{aligned}
\|A_i\bar{x} - B_i\bar{y_i}\|^2 &= \|A_i a_{n_k} - B_i b_{n_k,i} + A_i\bar{x} - A_i a_{n_k} + B_i b_{n_k,i} - B_i\bar{y_i}\|^2\\
&\le \|A_i a_{n_k} - B_i b_{n_k,i}\|^2 + 2\langle A_i\bar{x} - B_i\bar{y_i},\ A_i\bar{x} - A_i a_{n_k} + B_i b_{n_k,i} - B_i\bar{y_i}\rangle.
\end{aligned} \tag{3.93}$$

Since $a_{n_k} \rightharpoonup \bar{x}$, we obtain from (3.81) and (3.93) that $A_i\bar{x} = B_i\bar{y_i}$, and thus we conclude that $(\bar{x}, \bar{y_i}) \in \Omega$.

Thus, we have from (3.89) and Lemma 2.6 that

(3.94)

$$\begin{aligned}&\limsup_{n\to\infty}\Big\langle (u,v_1,v_2,\ldots,v_N)-(p,q_1,q_2,\ldots,q_N),\\&\qquad\qquad (x_n,y_{n,1},y_{n,2},\ldots,y_{n,N})-(p,q_1,q_2,\ldots,q_N)\Big\rangle\\&=\lim_{k\to\infty}\Big\langle (u,v_1,v_2,\ldots,v_N)-(p,q_1,q_2,\ldots,q_N),\\&\qquad (x_{n_k},y_{n_k,1},y_{n_k,2},\ldots,y_{n,N})-(p,q_1,q_2,\ldots,q_N)\Big\rangle\\&=\langle (u,v_1,v_2,\ldots,v_N)-(p,q_1,q_2,\ldots,q_N),(\bar{x},\bar{y_1},\bar{y_2},\ldots,\bar{y_N})-(p,q_1,q_2,\ldots,q_N)\rangle\le 0.\end{aligned}$$

From (3.78), (3.79), (3.87), (3.88) and (3.94), we conclude that

$$\limsup_{n\to\infty}(\Upsilon_n+\Xi_n)\le 0. \tag{3.95}$$

From (3.77), (3.95) and Lemma 2.3, we obtain that $\lim_{n\to\infty}\Delta_n=0$, which implies that $\lim_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}\|y_{n,i}-q_i\|=0$ and hence $\lim_{n\to\infty}x_n=p$ and $\lim_{n\to\infty}y_{n,i}=q_i$ for $i=1,2,\ldots,N$.

Case II. Suppose there exists a subsequence $\{\Delta_{n_k}\}$ of $\{\Delta_n\}$ with $\Delta_{n_k}<\Delta_{n_k+1}$ for all $k\ge 0$. Then, by Lemma 2.4, there exists a nondecreasing sequence $\{m_j\}$ of positive integers such that $\lim_{j\to\infty}m_j=\infty$ and

$$\Delta_{m_j}\le\Delta_{m_j+1}\ \text{ and }\ \Delta_j\le\Delta_{m_j+1}, \tag{3.96}$$

for all positive integers j. We have from (3.80) and (3.96) that

$$\begin{aligned}&(1-\alpha_{m_k})(1-2\zeta_{m_k})\|x_{m_k}-t_{m_k}\|^2+(1-\alpha_{m_k})(1-\mu)\|p_{m_k}-e_{m_k}\|^2\\&+(1-\alpha_{m_k})(1-\mu)\|e_{m_k}-c_{m_k}\|^2+(1-\alpha_{m_k})(1-2\zeta_{m_k})\sum_{i=1}^{N}\|y_{m_k,i}-u_{m_k,i}\|^2\\&+(1-\alpha_{m_k})(1-\delta)\sum_{i=1}^{N}\|q_{m_k,i}-h_{m_k,i}\|^2+(1-\alpha_{m_k})(1-\delta)\sum_{i=1}^{N}\|h_{m_k,i}-d_{m_k}\|^2\\&+\rho(1-\alpha_{m_k})\sum_{i=1}^{N}\|A_ia_{m_k}-B_ib_{m_k,i}\|^2\le\alpha_{m_k}\left[\Gamma+(2\varepsilon-1)\Delta_{m_k}+\frac{(N+1)\zeta_{m_k}}{\alpha_{m_k}}\right].\end{aligned} \tag{3.97}$$

Taking the limit as $k\to\infty$ and following similar methods used in Case I, we obtain

$$\limsup_{j\to\infty}(\Upsilon_{m_j}+\Xi_{m_j})\le 0. \tag{3.98}$$

We obtain from (3.77) and (3.96) that $\alpha_{m_j}(1-2\varepsilon)\Delta_{m_j+1}\le\alpha_{m_j}(1-2\varepsilon)(\Upsilon_{m_j}+\Xi_{m_j})$, which implies that

$$\Delta_{m_j+1}\le\Upsilon_{m_j}+\Xi_{m_j}. \tag{3.99}$$

Taking the limit as $j\to\infty$ of (3.99) and using (3.98), we obtain that $\lim_{j\to\infty}\Delta_{m_j+1}=0$. This together with (3.96) implies that $\lim_{j\to\infty}\Delta_j=0$. Thus, we have $\lim_{j\to\infty}\|x_j-p\|=\lim_{j\to\infty}\|y_{j,i}-q_i\|=0$ and hence we have $\lim_{j\to\infty}x_j=p$ and $\lim_{j\to\infty}y_{j,i}=q_i$, for $i=1,2,\ldots,N$.

Thus, we conclude from Case I and Case II that the sequence $\{(x_n,\ y_{n,1},\ y_{n,2},\ldots,\ y_{n,N})\}$ generated by Algorithm 3.1 converges strongly to a point $(p,q_1,q_2,\ldots,q_N)$, where $(p,q_1,q_2,\ldots,q_N)=P_\Omega(u,v_1,v_2,\ldots,v_N)$ and hence the proof is complete. □

We now deduce the following corollaries from our main result.

Corollary 3.1. *Assume that conditions* $(C1)-(C2)$ *and* $(C4)-(C6)$ *hold. If* $T: C \to C$ *and* $S_i: Q_i \to Q_i$ *are uniformly continuous pseudocontractive mappings, then the sequence* $\{(x_n,\ y_{n,1},\ y_{n,2}, \ldots,\ y_{n,N})\}$ *generated by Algorithm 3.1 converges strongly to* $(p, q_1, q_2, \ldots, q_N)$, *where* $(p, q_1, q_2, \ldots, q_N) = P_{\Omega}(u, v_1, v_2, \ldots, v_N)$.

3.1. **Some Particular Cases of the Main Result.** In this subsection, we draw some special cases of our main result.

3.1.1. ***Split Feasibility Problem***. If we take $i = 1$ and $B_1 = I_1$, $F(T) = C$ and $F(S_1) = Q_1$ in (1.18), then the SEFPPMOS reduces to the problem of finding a point

$$x^* \in C : A_1 x^* \in Q_1$$

which is the SFP (1.5). Denote $\Pi = \{x^* \in C : A_1 x^* \in Q_1\}$. Then the following corollary follows from Theorem 3.2.

Corollary 3.2. *Assume that conditions* $(C1)-(C4)$ *and* $(C6)-(C7)$, *with* $i = 1$ *and* $B_1 = I_1$, *hold. Assume also that* $\Pi \neq \emptyset$. *Then the sequence* $\{(x_n,\ y_{n,1})\}$ *generated by Algorithm 3.1, converges strongly to* (p, q_1), *where* $(p, q_1) = P_{\Pi}(u, v_1)$.

3.1.2. ***Split Fixed Point Problem with Multiple Output Sets***. If we take $B_i = I_i$, where I_i is the identity mapping on H_i, then (1.18) reduces to the *split fixed point problem with multiple output sets* defined as finding a point

$$x^* \in F(T) \bigcap \left(\bigcap_{i=1}^{N} A_i^{-1}\left(F(S_i)\right) \right), \text{ for } i = 1, 2, \ldots, N.$$

Denote $\Gamma^* = \left\{ x^* \in F(T) \bigcap \left(\bigcap_{i=1}^{N} A_i^{-1}\left(F(S_i)\right) \right) \right\}$. Thus, we have the following corollaries.

Corollary 3.3. *Assume that conditions* $(C1)-(C4)$ *and* $(C6)-(C7)$, *with* $B_i = I_i$, *hold. Assume also that* $\Gamma^* \neq \emptyset$. *Then the sequence* $\{(x_n,\ y_{n,1},\ y_{n,2}, \ldots,\ y_{n,N})\}$ *generated by Algorithm 3.1, converges strongly to* $(p, q_1, q_2, \ldots, q_N)$, *where* $(p, q_1, q_2, \ldots, q_N) = P_{\Gamma^*}(u, v_1, v_2, \ldots, v_N)$.

Corollary 3.4. *Assume that conditions* $(C1)-(C2)$, $(C4)$ *and* $(C6)-(C7)$, *with* $B_i = I_i$ *for* $i = 1, 2, \ldots, N$, *hold. Let* T: $C \to C$ *and* S_i: $Q_i \to Q_i$, $i = 1, 2, \ldots, N$, *be uniformly continuous pseudocontractive mappings. Assume also that* $\Gamma^* \neq \emptyset$. *Then the sequence* $\{(x_n,\ y_{n,1},\ y_{n,2}, \ldots,\ y_{n,N})\}$ *generated by Algorithm 3.1, converges strongly to* $(p, q_1, q_2, \ldots, q_N)$, *where* $(p, q_1, q_2, \ldots, q_N) = P_{\Gamma^*}(u, v_1, v_2, \ldots, v_N)$.

3.1.3. ***Split Equality Fixed Point Problems***. If we take $i = 1$ in Algorithm 3.1, then (1.18) reduces to *split equality fixed point problem* of finding a point $(p, q_1) \in F(T) \times F(S_1)$ such that $A_1 p = B_1 q_1$. Denote $\Omega^* = \{(p, q_1) \in F(T) \times F(S_1) : A_1 p = B_1 q_1\}$. Then we obtain the following corollaries.

Corollary 3.5. *Assume that conditions* $(C1)-(C4)$ *and* $(C6)-(C7)$, *with* $i = 1$, *hold. Assume also that* $\Omega^* \neq \emptyset$. *Then the sequence* $\{(x_n,\ y_{n,1})\}$ *generated by Algorithm 3.1, converges strongly to* (p, q_1), *where* $(p, q_1) = P_{\Omega^*}(u, v_1)$.

Corollary 3.6. *Assume that conditions* $(C1)-(C2)$, $(C4)$ *and* $(C6)-(C7)$, *with* $i = 1$, *hold. Let* T: $C \to C$ *and* S_1: $Q_1 \to Q_1$ *be uniformly continuous pseudocontractive mappings. Assume also that* $\Omega^* \neq \emptyset$. *Then the sequence* $\{(x_n, y_{n,1})\}$ *generated by Algorithm 3.1, converges strongly to* (p, q_1), *where* $(p, q_1) = P_{\Omega^*}(u, v_1)$.

3.1.4. ***The Split Equality Null Point Problem with Multiple Output Sets.*** Let H, H_i for $i = 1, 2, \ldots, N$ be real Hilbert spaces. Let K: $H \to H$ and K_i: $H_i \to H_i$ be nonlinear mappings and let A_i: $H \to H_i$ and B_i: $H_i \to H_i$ be bounded linear mappings. We define the split equality null point problem with multiple output sets (SENPPMOS) as finding a point $(x^*, y_1^*, y_2^*, \ldots, y_N^*) \in N(K) \times N(K_1) \times N(K_2) \times \ldots \times N(K_N)$ such that $A_i x^* = B_i y_i^*$. Denote

$$\Phi = \{(x^*, y_1^*, y_2^*, \ldots, y_N^*) \in N(K) \times N(K_1) \times N(K_2) \times \ldots \times N(K_N) : A_i x^* = B_i y_i^*\}.$$

A mapping K: $H \to H$ is called quasi-monotone if the mapping $T = I - K$ is quasi-pseudocontractive and it is called monotone if $T = I - K$ is pseudocontractive. In this case, one observes that the set of fixed points T is the same as the set of null points, $N(K)$, of K where $N(K) = \{z \in H : Kz = 0\}$. With these monotonicty properties, we now have the following corollaries from our main results:

Corollary 3.7. *Let $H, H_1, \ldots, H_N$ be real Hilbert spaces and let $K : H \to H$ and $K_i : H_i \to H_i$, $i = 1, 2, \ldots, N$ be uniformly continuous quasi-monotone mappings with K and K_i, for $i = 1, 2, \ldots N$, demiclosed at zero. Let A_i: $H \to H_i$ and B_i: $H_i \to H_i$ be bounded linear mappings. Assume that the set $\Phi \neq \emptyset$. If the conditions $(C6)$ and $(C7)$ hold, then the sequence $\{(x_n,\ y_{n,1},\ y_{n,2}, \ldots,\ y_{n,N})\}$ generated by Algorithm 3.1, with $T = I - K$ and $S_i = I_i - K_i$, converges strongly to an element $(p, q_1, q_2, \ldots, q_N)$, where $(p, q_1, q_2, \ldots, q_N) = P_\Phi(u, v_1, v_2, \ldots, v_N)$.*

Proof. Taking $K = I - T$ and $K_i = I_i - S_i$, the proof follows from Theorem 3.2. □

4. Numerical examples

In this section, we provide examples of quasi-pseudocontractive mappings and conduct numerical experiments.

Example 4.1. *Let $H = H_i = \mathbb{R}$ and $C = Q_i = [0, \infty)$, for $i = 1, 2, 3$ with the usual metric. Let T: $C \to C$ and S_i: $Q_i \to Q_i$, $i = 1, 2, 3$, be defined by $T(x) = x - \sqrt{x} + \sqrt{2}$, $S_1(y) = y - \sqrt{y} + \frac{3}{\sqrt{7}}$, $S_2(y) = y - \sqrt[3]{y} + \sqrt[3]{\frac{5}{18}}$ and $S_3(y) = y - \sqrt[3]{y} + \sqrt[3]{3}$. The mapping T is uniformly continuous quasi-pseudocontractive on C which is not Lipschitz continuous. In fact, let $M > 0$ be given and choose $y = 0$ and $0 < x < \frac{1}{M^2}$ so that $M < \frac{1}{\sqrt{x}}$. Now,*

$$\frac{|T(x) - T(y)|}{|x - y|} = \frac{|x - \sqrt{x}|}{|x|} = \Big|\frac{1}{\sqrt{x}} - 1\Big| > M - 1.$$

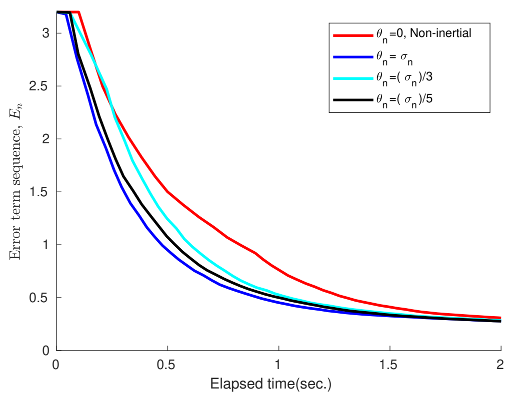
Since M is arbitrary, one concludes that T is not Lipschitz continuous. Similarly, it can be shown that S_1, S_2 and S_3 are uniformly continuous mappings on Q which are not Lipschitz continuous. We also have that $p = 2 \in F(T)$, $q_1 = \frac{9}{7} \in F(S_1)$, $q_2 = \frac{5}{18} \in F(S_2)$ and $q_3 = 3 \in F(S_3)$. Since

$$\begin{aligned}\langle x - Tx, x - p\rangle &= \langle x - Tx, x - 2\rangle = \langle x - x + \sqrt{x} - \sqrt{2}, x - 2\rangle \\ &= (\sqrt{x} + \sqrt{2})|\sqrt{x} - \sqrt{2}|^2 \geq 0,\end{aligned}$$

we have that the mapping T is quasi-pseudocontractive. Similarly, we can show that the mappings S_1, S_2 and S_3 are quasi-pseudocontractive. Now, define the mappings A_i, B_i: $H \to H$ as

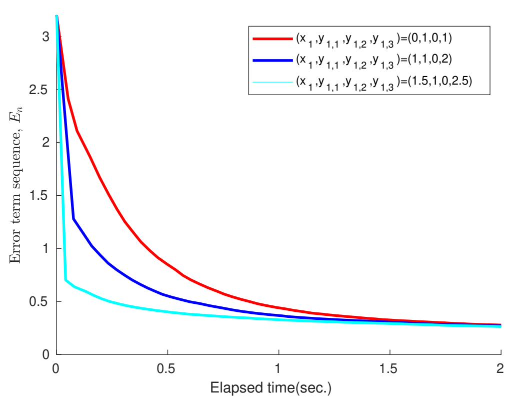
$$A_i(x) = \frac{x}{i}, \text{for } i = 1,\ 2,\ 3,\ B_1(y) = \frac{14}{9}y,\ B_2(y) = \frac{18}{5}y \text{ and } B_3(y) = \frac{2}{9}y.$$

Clearly, A_i *and* B_i *are bounded linear mappings with adjoints* $A_i^*(x) = \frac{x}{i}$*, for* $i = 1, 2, 3$*,* $B_1^*(y) = \frac{14}{9}y$*,* $B_2^*(y) = \frac{18}{5}y$ *and* $B_3^*(y) = \frac{2}{9}y$*. Moreover, we have* $A_1(p) = 2 = B_1(q_1)$*,* $A_2(p) = 1 = B_2(q_2)$*,* $A_3(p) = \frac{2}{3} = B_3(q_3)$*. Thus,* $(p, q_1, q_2, q_3) = \left(2, \frac{9}{7}, \frac{5}{18}, 3\right) \in \Omega$*. Let* $(x_0,\ y_{0,1},\ y_{0,2},\ y_{0,3}) = (0, 0, 0, 0)$*,* $\zeta_n = \frac{1}{n^2+5}$*,* $\alpha_n = \frac{1}{n+100}$*,* $n \geq 1$*,* $\gamma = 0.5$*,* $l = 0.5$*,* $\theta = 0.5$*,* $\mu = 0.4$*,* $\delta = 0.4$*. Thus, conditions* $(C1) - (C7)$ *are satisfied. We obtained the following numerical experiment results which demonstrate that the error term sequence* $E_n = \{(x_n,\ y_{n,1},\ y_{n,2},\ y_{n,3}) - (p, q_1, q_2, q_3)\}$*,* $n \geq 1$*, converges strongly to zero for different values of the inertial parameter* θ_n *and different choices of initial points* $(x_1,\ y_{1,1},\ y_{1,2},\ y_{1,3})$*.*



$\theta = 0.5,\quad l = 0.5,\quad \gamma = 0.5,\quad \delta = 0.4,\quad \mu = 0.4,\ (x_0,\ y_{0,1},\ y_{0,2},\ y_{0,3}) = (0, 0, 0, 0),$
$(x_1,\ y_{1,1},\ y_{1,2},\ y_{1,3}) = (1, 0, 1, 0).$

FIGURE 1. Convergence rate for different values of the inertial parameter θ_n.



$\theta = 0.5,\quad l = 0.5,\quad \gamma = 0.5,\quad \delta = 0.4,\quad \mu = 0.4,\ (x_0, y_{0,1}, y_{0,2}, y_{0,3}) = (0, 0, 0, 0).$

FIGURE 2. Convergence rate for different initial points $(x_1,\ y_{1,1},\ y_{1,2},\ y_{1,3})$.

Remark 4.5. *One can observe from FIGURE 1 that the inertial version* ($\theta_n \neq 0$) *of the algorithm converges at a faster rate than that of the non-inertial version* ($\theta_n = 0$)*. FIGURE 2 reveals convergence of the method for different values of initial points and its seems that the convergence gets faster as the initial point* $(x_1,\ y_{1,1},\ y_{1,2},\ y_{1,3})$ *gets closer to the solution* $\left(2, \frac{9}{7}, \frac{5}{18}, 3\right)$.

5. Conclusions

In this paper, we introduced the split equality fixed point problem with multiple output sets and proposed an inertial algorithm for approximating its solution. A strong convergence theorem was proved under some conditions, where the underlying mappings are uniformly continuous quasi-pseudocontractive and demiclosed at zero. A numerical example is also provided to demonstrate effectiveness of the algorithm. The main result in this paper extends the results of [10, 13, 15, 26, 28, 29, 30, 31, 38] in the sense that: (i) the introduced problem is a more general problem that contains all the problems in the literature; (ii) it extends all the mappings discussed in the literature to more general uniformly continuous quasi-pseudocontractive mappings. It can also be observed from Corollary 3.9 and Corollary 3.10 that the Lipschitz continuity and inverse strong monotonicity properties of the mappings considered [32] have been extended in our result to a more general class of uniformly continuous monotone mappings. Thus, the result in [32] is special case of our result.

Acknowledgments

The authors sincerely thank the anonymous referees for their careful reading, comments and useful suggestions. The first author gratefully acknowledges the funding received from Simons Foundation based at Botswana International University of Science and Technology (BIUST). He is also grateful to Aksum University for the partial financial support provided during his study.

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