# On the geometry of diffeological vector pseudobundles and infinite dimensional vector bundles: automorphisms, connections and covariant derivatives

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ABSTRACT. We consider the category of diffeological vector pseudo-bundles and study a possible extension of classical differential geometric tools on finite-dimensional vector bundles, namely, the group of automorphisms, the frame bundle, the space of connection 1-forms and the space of covariant derivatives. Substantial distinctions are highlighted in this generalized framework, including the non-isomorphism between connection 1-forms and covariant derivatives. Applications include not only finite-dimensional examples with singularities, but also infinite-dimensional vector bundles.

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