

# Fixed points vs best proximity points and a contraction map sets with an external factor

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**ABSTRACT.** This research aims to introduce a new property, called the  $CD$  property as a less restrictive and more user-friendly variant of the  $UC$  one. This research then aims, by using the idea of noncyclic maps, to reveal a new method for generalizing Banach's fixed point theorem by introducing the concept of an external factor. Another goal of this study is to show a new way to obtain results for  $p$ -cyclic maps by using the results of noncyclic maps. Also, this research shows a relationship between fixed points and best proximity points similar to that between fixed points and coupled fixed points, as shown by A. Petrusel in 2018.

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