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Naturally ordered transformation semigroups preserving an equivalence relation on an invariant set

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ABSTRACT. For a nonempty set X, let $T(X)$ represent the full transformation semigroup on X. Given a nonempty subset Y of X and an equivalence relation E defined on X, we consider the set

$$\overline{S}_E(X,Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x,y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

Then $\overline{S}_E(X,Y)$ is a subsemigroup of $T(X)$. In this paper, we provide a characterization of the natural partial order on $\overline{S}_E(X,Y)$. Moreover, we investigate the elements within $\overline{S}_E(X,Y)$ which are minimal, maximal right-compatible, and left-compatible with respect to such order.

1. INTRODUCTION AND PRELIMINARIES

The concept of a natural partial order on a semigroup originates from the examination of the set of idempotent elements, as explained in [1, Section 2.7]. This idea has undergone successive expansions, as detailed in [12, 2, 6]. Mitsch [5] played a significant role in this development by revealing the natural partial order on any semigroup in 1986, as defined below:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

where S^1 is the semigroup obtained from S by adding an identity element if S lacks one.

Consider an arbitrary nonempty set X. The full transformation semigroup on X, denoted by $T(X)$, comprises all mappings from X to X under the operation of function composition. It is established in the literature, as discussed in [3], that $T(X)$ is a regular semigroup. Furthermore, it is noteworthy that every semigroup can be embedded in $T(X)$ for a suitably chosen set X. This general result underscores the significance of studying transformation semigroups. In 1986, Kowol and Mitsch [4] conducted an investigation into the natural partial order within $T(X)$. Their analysis included characterizing this order based on images and kernels, providing a comprehensive understanding. Additionally, they presented descriptions of the maximal, minimal, and covering elements inherent in this order. Furthermore, their inquiry extended to the examination of lower and upper bounds concerning pairs of transformations.

For a fixed nonempty subset Y of X, define

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

The set $S(X,Y)$ constitutes a semigroup of total transformations on X that preserves the subset Y. This semigroup serves as both a subsemigroup and a generalization of $T(X)$. In the investigation outlined in [9], Sun and Wang delved into the natural partial order within $S(X,Y)$. They ascertained conditions for the relationship between two elements in $S(X,Y)$, identified compatible elements, and provided characterizations of maximal

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and minimal elements, as well as the greatest lower bound of two elements. In a complementary study presented in [8], Sun L. and Sun J. explored all elements in the semigroup $S(X, Y)$ that exhibit compatibility with respect to the natural partial order.

Consider an equivalence relation E on a set X. Define the subset $T_E(X)$ of $T(X)$ as follows:

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X,\ (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}.$$

It is evident that if E is a non-trivial equivalence relation, then $T_E(X)$ forms a proper subsemigroup of $T(X)$; and if E is the identity or universal relation, then $T_E(X)$ is identical to $T(X)$. In their work [10], Sun, Pei, and Cheng provided a characterization of the natural partial order on the semigroup $T_E(X)$. They explored the compatibility of multiplication and investigated all compatible elements. Additionally, they identified maximal, minimal, and covering elements with respect to the order. In another study [11], Sun delved into left and right compatibility with respect to the natural partial order on $T_E(X)$.

In this paper, we investigate a subsemigroup of $T(X)$ resulting from the amalgamation of $S(X, Y)$ and $T_E(X)$, as initially introduced in [7]. Specifically, this subsemigroup, denoted as $\overline{S}_E(X, Y)$, is defined with Y serving as a fixed nonempty subset of X, and E is an equivalence relation on X. Formally, it is expressed as:

$$\overline{S}_E(X, Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

This semigroup serves as a generalization of all previously mentioned semigroups. This becomes evident when we consider two key cases. First, when $X = Y$, $\overline{S}_E(X, Y)$ reduces to the semigroup $T_E(X)$. Second, if E is the identity or the universal relation, $\overline{S}_E(X, Y)$ coincides with $S(X, Y)$. Combining these cases, where $X = Y$ and E is either the identity or the universal relation, $\overline{S}_E(X, Y)$ simplifies further to $T(X)$.

Section 2 provides a characterization of the natural partial order within the semigroup $\overline{S}_E(X, Y)$. Moving forward to Section 3, we thoroughly investigate both minimal and maximal elements within $\overline{S}_E(X, Y)$, considering the natural partial order. Section 4 is dedicated to the identification of elements exhibiting compatibility with the natural partial order.

We present essential preliminary concepts in this paper. Our notation adheres to the convention of right-to-left function application. Specifically, in the composition $\alpha\beta$, the transformation α is applied first. It is pertinent to recall that for collections $\mathcal{A}$ and $\mathcal{B}$ of nonempty subsets of X, $\mathcal{A}$ is considered to refine $\mathcal{B}$ if, for each $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

For any equivalence relation E on X, we denote the quotient set of X by E as X/E, representing the set of all E-classes. Furthermore, for any $\alpha \in \overline{S}_E(X, Y)$, we define $\pi(\alpha)$ as the partition of the set X induced by α, given by:

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\},$$

where $x\alpha^{-1}$ signifies the inverse image of x under α. In particular,

$$E(\alpha) = \{A\alpha^{-1} : A \in X/E \text{ and } A \cap X\alpha \neq \emptyset\},$$

where $A\alpha^{-1} = \bigcup_{x \in A \cap X\alpha} x\alpha^{-1}$. Notably, $E(\alpha)$ also constitutes a partition of X, with $\pi(\alpha)$ refining $E(\alpha)$. For a nonempty subset Y of X, the restrictions of $\pi(\alpha)$ and $E(\alpha)$ to Y are defined as follows:

$$\pi_Y(\alpha) = \{P \in \pi(\alpha) : P \cap Y \neq \emptyset\} \text{ and}$$

$$E_Y(\alpha) = \{(A \cap Y)\alpha^{-1} : A \in X/E \text{ and } A \cap X\alpha \cap Y \neq \emptyset\}.$$

Consistent with the previous result, both $\pi_Y(\alpha)$ and $E_Y(\alpha)$ emerge as partitions of $Y\alpha^{-1}$, with $\pi_Y(\alpha)$ refining $E_Y(\alpha)$.

2. CHARACTERIZATION

In this section, we investigate the condition under which $\alpha \leq \beta$ holds for two elements $\alpha, \beta \in \overline{S}_E(X, Y)$. The following theorem provides insight into the characterization of the natural partial order on the semigroup $\overline{S}_E(X, Y)$.

Theorem 2.1. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then, $\alpha \leq \beta$ if and only if the following conditions are satisfied:*

(1) *$\pi(\beta)$ refines $\pi(\alpha)$ and $E_Y(\beta)$ refines $E_Y(\alpha)$.*
(2) *$X\alpha \subseteq X\beta$.*
(3) *For any $x \in X$, if $x\beta \in X\alpha$, then $x\alpha = x\beta$.*
(4) *For every $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$.*

Proof. Assume that $\alpha \leq \beta$. In this case, there exist $\gamma, \delta \in \overline{S}_E(X, Y)$ such that $\alpha = \beta\gamma = \delta\beta$ and $\alpha = \alpha\gamma$. The relationship $\alpha = \beta\gamma$ implies that $\pi(\beta)$ refines $\pi(\alpha)$. Let $U \in E_Y(\beta)$ be expressed as $U = (A \cap Y)\beta^{-1}$, where $A \in X/E$ and $A \cap Y \cap X\beta \neq \emptyset$. This implies $U\alpha = U\beta\gamma \subseteq (A \cap Y)\gamma$. Since $\gamma \in \overline{S}_E(X, Y)$, we have $(A \cap Y)\gamma \subseteq B \cap Y$ for some $B \in X/E$. Consequently, $U \subseteq (B \cap Y)\alpha^{-1}$, indicating that $E_Y(\beta)$ refines $E_Y(\alpha)$, and condition (1) holds. To demonstrate the satisfaction of condition (2), let $x \in X\alpha$. Then, $x = x'\alpha$ for some $x' \in X$. Thus, $x = x'\alpha = x'\delta\beta = (x'\delta)\beta \in X\beta$. Therefore, $X\alpha \subseteq X\beta$. Assume $z\beta \in X\alpha$. Then $z\beta = z'\alpha$ for some $z' \in X$. Hence, $z\alpha = z\beta\gamma = z'\alpha\gamma = z'\alpha = z\beta$, confirming the validity of condition (3). To establish condition (4), let $A \in X/E$. Since $\delta \in \overline{S}_E(X, Y)$, we have $(A \cap Y)\delta \subseteq B \cap Y$ for some $B \in X/E$. Consequently, $(A \cap Y)\alpha = (A \cap Y)\delta\beta \subseteq (B \cap Y)\beta$.

Conversely, suppose conditions (1)-(4) are satisfied. For each $x \in X\beta$, there exists $x' \in X$ such that $x = x'\beta$. Additionally, for each $A \in X/E$ with $A \cap Y \cap X\beta \neq \emptyset$, we choose $x_A \in A \cap Y \cap X\beta$. Now, define γ on X by

$$x\gamma = \begin{cases} x'\alpha & \text{if } x \in X\beta, \\ x'_A\alpha & \text{if } x \in (A \cap Y) \setminus X\beta, \text{ where } A \cap Y \cap X\beta \neq \emptyset, \\ x & \text{otherwise.} \end{cases}$$

By condition (1), we have $\pi(\beta)$ refines $\pi(\alpha)$ and this implies $\gamma \in T(X)$. To demonstrate that $\gamma \in \overline{S}_E(X, Y)$, consider $x, y \in Y$ with $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. If $A \cap Y \cap X\beta = \emptyset$, then $(x\gamma, y\gamma) = (x, y) \in E$. Assume that $A \cap Y \cap X\beta \neq \emptyset$. Thus $(x, x_A), (y, x_A) \in E$. Two cases arise.

Case 1: Both x and y are in $X\beta$. Then $x\beta^{-1}, y\beta^{-1} \subseteq (A \cap Y)\beta^{-1} \in E_Y(\beta)$. Since $x' \in x\beta^{-1}$ and $y' \in y\beta^{-1}$, it follows that $x', y' \in (A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ due to the refinement property of $E_Y(\beta)$ over $E_Y(\alpha)$. Consequently, $x'\alpha, y'\alpha \in B \cap Y$. This implies that $(x\gamma, y\gamma) = (x'\alpha, y'\alpha) \in E$, and $x\gamma, y\gamma \in Y$.

Case 2: Either x or y is not in $X\beta$. Without loss of generality, assume that $x \in X\beta$ and $y \notin X\beta$. Then $y\gamma = x'_A\alpha = x_A\gamma$. Since $x, x_A \in X\beta$, by Case 1, we get $(x\gamma, y\gamma) = (x\gamma, x_A\gamma) \in E$ and $x\gamma, y\gamma \in Y$.

From both cases, we conclude that $\gamma \in \overline{S}_E(X, Y)$. Let $x \in X$. Then $x\beta \in X\beta$ and thus $x\beta\gamma = (x\beta)\gamma = x\alpha$, resulting in $\alpha = \beta\gamma$. Furthermore, by condition (2), we have $x\alpha \in X\alpha \subseteq X\beta$ and then, by our setting, $(x\alpha)'\beta = x\alpha \in X\alpha$. By condition (3), we obtain $x\alpha\gamma = (x\alpha)'\alpha = (x\alpha)'\beta = x\alpha$. This implies $\alpha = \alpha\gamma$.

To define δ, for each $A \in X/E$, by condition (4), we choose $A' \in X/E$ such that $(A \cap Y)\alpha \subseteq (A' \cap Y)\beta$. For any $x \in X$, if $x \in Y$, then there exists $A \in X/E$ such that $x \in A \cap Y$. In this case, we choose $x' \in A' \cap Y$ such that $x\alpha = x'\beta$. If $x \in X \setminus Y$, given that $X\alpha \subseteq X\beta$, we choose $x' \in X$ such that $x\alpha = x'\beta$. Define $\delta \in T(X)$ by $x\delta = x'$. It is clear that

$x\delta \in Y$ for all $x \in Y$. To establish that $\delta \in \overline{S}_E(X,Y)$, consider $x, y \in Y$ with $(x,y) \in E$. Consequently, $x, y \in A \cap Y$ for some $A \in X/E$. This implies $x', y' \in A' \cap Y$, indicating that $(x\delta, y\delta) = (x', y') \in E$. Finally, consider any $x \in X$. Then $x\delta\beta = x'\beta = x\alpha$, and thus $\alpha = \delta\beta$. Therefore, $\alpha \leq \beta$, as required. □

Remark 2.1. *If $X = Y$, then $\overline{S}_E(X,Y) = T_E(X)$, and we obtain the characterization of $\leq$ on $T_E(X)$, as initially presented in* [10, Theorem 2.1].

As a direct consequence of Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let $\alpha, \beta \in \overline{S}_E(X,Y)$ with $\alpha \leq \beta$. If $X\alpha = X\beta$, then $\alpha = \beta$.*

3. Minimal and maximal elements

In this section, we discuss the minimal and maximal elements in $\overline{S}_E(X,Y)$. For an element $z \in X$, the notation χ_z is utilized to represent the constant map with the range $\{z\}$. Clearly, for each $a \in Y$, χ_a is an element of $\overline{S}_E(X,Y)$, and $\pi(\chi_a) = \{X\} = E_Y(\chi_a)$. Furthermore, we use the notation $\alpha < \beta$ instead of $\alpha \leq \beta$ unless α is not equal to β. The following theorem describes the minimal elements with respect to the natural partial order.

Theorem 3.1. *Let $\alpha \in \overline{S}_E(X,Y)$. Then α is minimal if and only if α is a constant map.*

Proof. Assume that α is minimal. Suppose, to the contrary, that $|X\alpha| \geq 2$. Let $a \in Y\alpha$. It can be readily seen that $\chi_a < \alpha$, and this contradicts the minimality of α.

Conversely, assume $\alpha = \chi_a$, for some $a \in Y$. Let $\beta \in \overline{S}_E(X,Y)$ be such that $\beta \leq \alpha$. Consequently, $X\beta \subseteq X\alpha = \{a\}$ and this implies that $X\beta = X\alpha$. Hence, by Corollary 2.2, $\beta = \alpha$, and thus, α is minimal. □

Definition 3.1. Let $\alpha \in \overline{S}_E(X,Y)$ and $U \in E_Y(\alpha)$. Then U is termed *divisible* if there exist C and K such that $C \in X/E$ with $C \cap Y \cap X\alpha = \emptyset$, and $\emptyset \neq K \subseteq U$ with $|K\alpha| \leq |C \cap Y|$ and satisfying one of the following conditions:

(1) $K \subseteq U \setminus Y$ and $(U - K)\alpha = U\alpha$.
(2) $K = A \cap Y$ for some $A \in X/E$ and there exists $B \in X/E$ such that $B \neq A$ with $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha$.

Initiating the examination, we identify the necessary conditions for elements in $\overline{S}_E(X,Y)$ to be maximal with respect to the natural partial order.

Lemma 3.2. *Let $\alpha \in \overline{S}_E(X,Y)$ be maximal. Then U is not divisible for all $U \in E_Y(\alpha)$.*

Proof. Assume, to the contrary, that there exists $U \in E_Y(\alpha)$ such that U is divisible. Then there exist C and K satisfying the properties in Definition 3.1. Let $\varphi : K\alpha \to C \cap Y$ be injective. Define $\beta \in \overline{S}_E(X,Y)$ as follows:

$$x\beta = \begin{cases} x\alpha\varphi & \text{if } x \in K, \\ x\alpha & \text{otherwise.} \end{cases}$$

Then $\pi(\beta) = \{P \setminus K : P \in \pi(\alpha) \text{ and } P \nsubseteq K\} \cup \{P \cap K : P \in \pi(\alpha) \text{ and } P \cap K \neq \emptyset\}$ refines $\pi(\alpha)$, and $E_Y(\beta) = (E_Y(\alpha) \setminus \{U\}) \cup \{U \setminus K, K\}$ refines $E_Y(\alpha)$. So $\beta \in \overline{S}_E(X,Y)$. Furthermore, by properties of K, we have $X\beta = X\alpha \cup K\alpha\varphi$, resulting in $X\alpha \subsetneq X\beta$ and $x\alpha = x\beta$ for all $x\beta \in X\alpha$. To verify that α and β satisfy condition (4) of Theorem 2.1, we consider two cases.

Case 1: $K \subseteq U \setminus Y$ and $(U - K)\alpha = U\alpha$. Then $(D \cap Y)\alpha = (D \cap Y)\beta$ for all $D \in X/E$.

Case 2: $K = A \cap Y$ for some $A \in X/E$ and there exists $B \in X/E$ such that $B \neq A$ and $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha$. In this case, we have $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha = (B \cap Y)\beta$ and $(D \cap Y)\alpha = (D \cap Y)\beta$ for all $D \in X/E$ with $D \neq A$.
Hence, α and β satisfy all conditions in Theorem 2.1 and $\alpha \neq \beta$. Therefore, $\alpha < \beta$ which contradicts to the maximality of α. □

Lemma 3.3. *Let $\alpha \in \overline{S}_E(X, Y)$ be maximal. If $X \setminus Y \nsubseteq X\alpha$, then $\{z\} \in \pi(\alpha)$ for all $z \in X \setminus Y$.*

Proof. Suppose that $X \setminus Y \nsubseteq X\alpha$. Then there exists $c \in (X \setminus Y) \setminus X\alpha$. Assume, to the contrary, that there exists $z \in X \setminus Y$ such that $\{z\} \notin \pi(\alpha)$. Then $z \in P$ for some $P \in \pi(\alpha)$ and $|P| > 1$. Define $\beta : X \to X$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = z, \\ x\alpha & \text{otherwise.} \end{cases}$$

Since $z, c \notin Y$ and $|P| > 1$, we get $\beta \in \overline{S}_E(X, Y)$ and $X\beta = X\alpha \cup \{c\}$, implying that $\alpha \neq \beta$. Note that $\pi(\beta) = (\pi(\alpha) \setminus \{P\}) \cup \{P \setminus \{z\}, \{z\}\}$ refines $\pi(\alpha)$ and $E_Y(\alpha) = E_Y(\beta)$. Clearly, $x\beta \in X\alpha$ implying $x\alpha = x\beta$, and $(A \cap Y)\alpha = (A \cap Y)\beta$ for all $A \in X/E$. Thus $\alpha < \beta$, contradicting the maximality of α. □

Lemma 3.4. *Let $\alpha \in \overline{S}_E(X, Y)$ be maximal and let $U \in E_Y(\alpha)$ be such that $U\alpha \notin X/E$.*

(1) *If $A, B \in X/E$ with $A \cap Y \neq \emptyset$ and $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha \subseteq U\alpha$, then $A = B$.*

(2) *If $P \in \pi(\alpha)$ with $|P| > 1$, then $P \cap (U \setminus Y) = \emptyset$ and $|P \cap U \cap (A \cap Y)| \leq 1$ for all $A \in X/E$.*

Proof. Since $U \in E_Y(\alpha)$, we can express $U = (C \cap Y)\alpha^{-1}$ for some $C \in X/E$. Due to $U\alpha \notin X/E$, we have $U\alpha \subsetneq C \cap Y$. Choose $c \in (C \cap Y) \setminus U\alpha$. Note that $c \notin X\alpha$.

(1) Let $A, B \in X/E$ be such that $A \cap Y \neq \emptyset$ and $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha \subseteq U\alpha$. Choose $a \in A \cap Y$ and define $\beta \in \overline{S}_E(X, Y)$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = a, \\ x\alpha & \text{otherwise.} \end{cases}$$

If $A \neq B$, then $X\beta = X\alpha \cup \{c\}$, implying $X\alpha \subsetneq X\beta$. As before, we can see that α and β satisfy all conditions in Theorem 2.1. Hence, $\alpha < \beta$, contradicting the maximality of α. Therefore, $A = B$, as required.

(2) Let $P \in \pi(\alpha)$ be such that $|P| > 1$. Assume, to the contrary, that $P \cap (U \setminus Y) \neq \emptyset$ or $|P \cap U \cap (A \cap Y)| > 1$ for some $A \in X/E$.

Case 1: $P \cap (U \setminus Y) \neq \emptyset$. In this situation, there exists $z \in P \cap (U \setminus Y)$. Define $\beta \in \overline{S}_E(X, Y)$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = z, \\ x\alpha & \text{otherwise.} \end{cases}$$

Since $|P| > 1$, we have $X\beta = X\alpha \cup \{c\}$; furthermore, α and β satisfy all conditions in Theorem 2.1. This implies that $\alpha < \beta$, contradicting the maximality of α.

Case 2: $|P \cap U \cap (A \cap Y)| > 1$ for some $A \in X/E$. Let $a \in P \cap U \cap (A \cap Y)$. Define $\beta : X \to X$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = a, \\ x\alpha & \text{otherwise.} \end{cases}$$

Similar to Case 1, $\beta \in \overline{S}_E(X,Y)$ and $\alpha < \beta$, a contradiction. □

Theorem 3.5. *Let $\alpha \in \overline{S}_E(X,Y)$. Then α is maximal if and only if α satisfies all of the following conditions:*

(1) *U is not divisible for all $U \in E_Y(\alpha)$.*
(2) *If $X \setminus Y \nsubseteq X\alpha$, then $\{z\} \in \pi(\alpha)$ for all $z \in X \setminus Y$.*
(3) *If $U \in E_Y(\alpha)$ with $U\alpha \notin X/E$, then $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha \subseteq U\alpha$ implies $A = B$ for all $A, B \in X/E$ with $A \cap Y \neq \emptyset$.*
(4) *If $U \in E_Y(\alpha)$ with $U\alpha \notin X/E$, then $P \cap (U \setminus Y) = \emptyset$ and $|P \cap U \cap (A \cap Y)| < 1$ for all $P \in \pi(\alpha)$ with $|P| > 1$, and for all $A \in X/E$.*

Proof. By Lemmas 3.2 - 3.4 collectively imply the truth of conditions (1) through (4).

Conversely, assume that conditions (1) through (4) hold. Let $\beta \in \overline{S}_E(X,Y)$ be such that $\alpha \leq \beta$. To demonstrate that $\alpha = \beta$, consider an arbitrary element $x \in X$. Then, $x\alpha \in X\alpha \subseteq X\beta$. Consequently, $x\alpha = z\beta$ for some $z \in X$. Note that $x \in P$ for some $P \in \pi(\alpha)$. Since $z\beta \in X\alpha$, we deduce $x\alpha = z\beta = z\alpha$, and thus, $z \in P$. If $|P| = 1$, then $x = z$, leading to $x\alpha = x\beta$. For the case where $|P| > 1$, we consider two cases.

Case 1: $x\alpha \notin Y$. Then $x \in X \setminus Y$. According to condition (2) and the fact that $|P| > 1$, it follows that $X \setminus Y \subseteq X\alpha$. If $x\beta \in Y$, then $x \in U$ for some $U \in E_Y(\beta)$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $x \in U \subseteq V$ for some $V \in E_Y(\alpha)$. This implies that $x\alpha \in V\alpha \subseteq Y$, which is impossible. Thus, $x\beta \in X \setminus Y \subseteq X\alpha$, and so $x\beta = x\alpha$.

Case 2: $x\alpha \in Y$. Then $x \in U = (A \cap Y)\alpha^{-1}$ for some $A \in X/E$. Let $V = (A \cap Y)\beta^{-1} \in E_Y(\beta)$. If $x\beta \notin Y$, then $x \notin Y$. Since $|P| > 1$, by (2), we obtain that $X \setminus Y \subseteq X\alpha$. This implies that $x\beta \in X\alpha$, and so $x\beta = x\alpha \in Y$, which is a contradiction. Hence, $x\beta \in Y$.

Subcase 2.1: $U\alpha \in X/E$. Then $U\alpha = A \cap Y$. From $x\alpha = z\beta = z\alpha$, we have $z \in U$ and $z \in V$. As $E_Y(\beta)$ refines $E_Y(\alpha)$, we conclude that $V \subseteq U$. Since $A \cap Y = U\alpha \subseteq X\alpha \subseteq X\beta$, we deduce that $V\beta = A \cap Y \subseteq X\alpha$, implying $V\beta = V\alpha$. To show $x \in V$, we assume, to the contrary, that $x \notin V$. Since $x\beta \in Y$, it follows that $x \in W = (C \cap Y)\beta^{-1}$ for some $C \in X/E$, where $C \neq A$. Consequently, $\emptyset \neq W \subseteq U$. We claim that U is divisible, and so we will examine properties of C while choosing a suitable K. If $C \cap Y \cap X\alpha \neq \emptyset$, then $(C \cap Y)\alpha^{-1} \in E_Y(\alpha)$. Let $c \in (C \cap Y)\alpha^{-1}$. Thus, $c\alpha \in C \cap Y \cap X\beta$, so $c\alpha = c'\beta = c'\alpha$ for some $c' \in X$. Hence, $c, c' \in (C \cap Y)\alpha^{-1}$ and $c' \in (C \cap Y)\beta^{-1} = W$. Thus, $W \subseteq (C \cap Y)\alpha^{-1}$. Since $W \subseteq U = (A \cap Y)\alpha^{-1}$, it follows that $A = C$, which is a contradiction. Therefore, $C \cap Y \cap X\alpha = \emptyset$.

If $W \cap Y = \emptyset$, then we choose $K = W$. Since $\pi(\beta)$ refines $\pi(\alpha)$, we obtain $|K\alpha| = |W\alpha| \leq |W\beta| \leq |C \cap Y|$. Furthermore, $A \cap Y = V\alpha \subseteq (U - K)\alpha \subseteq A \cap Y$ since $V, K \subseteq U$ and $V \cap K = \emptyset$. Thus $K \subseteq U \setminus Y$ such that $(U - K)\alpha = U\alpha$. Hence, C and K satisfy the properties in Definition 3.1.

If $W \cap Y \neq \emptyset$, then there exists $D \in X/E$ such that $D \cap Y \subseteq W$. Choose $K = D \cap Y$. Since $\alpha \leq \beta$, $(D \cap Y)\alpha \subseteq (B \cap Y)\beta$ for some $B \in X/E$. If $D = B$, then let $d \in D \cap Y$. Hence, $d\alpha = b\beta$ for some $b \in B \cap Y = D \cap Y \subseteq W$. Thus, $d\alpha \in U\alpha = A \cap Y$ and $b\beta \in W\beta \subseteq C \cap Y$. Therefore, $A = C$, which is a contradiction. It follows that $D \neq B$ and $(D \cap Y)\alpha \subseteq (B \cap Y)\alpha$ since $a\beta = a\alpha$ for all $a\beta \in (D \cap Y)\alpha$. Since $\pi(\beta)$ refines $\pi(\alpha)$, we get $|(D \cap Y)\alpha| \leq |(D \cap Y)\beta| \leq |W\beta| \leq |C \cap Y|$. Hence, C and K satisfy the properties in Definition 3.1.

Thus, U being divisible leads to a contradiction. This implies that $x \in V$, and subsequently, $x\beta \in V\beta = A \cap Y \subseteq X\alpha$. This results in $x\beta = x\alpha$.

Subcase 2.2: $U\alpha \notin X/E$. Since $|P| > 1$, according to (4), we can conclude that $x \in Y$. Let $B \in X/E$ be such that $x \in B \cap Y$. Then $(B \cap Y)\alpha \subseteq (D \cap Y)\beta$ for some $D \in X/E$.

Furthermore, since $a\beta = a\alpha$ for all $a\beta \in (B \cap Y)\alpha$, we obtain $(B \cap Y)\alpha \subseteq (D \cap Y)\alpha \subseteq U\alpha$. It follows from (3) that $B = D$, and thus $x\alpha = d\beta = d\alpha$ for some $d \in D \cap Y = B \cap Y$. Hence, $x, d \in P \cap U \cap B \cap Y$. It follows from (4) that $|P \cap U \cap B \cap Y| < 1$, that is, $x = d$, and so $x\beta = x\alpha$. □

Corollary 3.6. *Let $\gamma \in \overline{S}_E(X, Y)$. Then the following statements hold:*

(1) *If α is surjective, then α is a maximal element.*
(2) *If α is injective, then α is a maximal element.*

4. Compatibility

Recall that, in the context of any semigroup S, an element $a \in S$ is referred to as being *right (left) compatible* with respect to the partial order $\leq$ if, for every $b \leq c$, the implication holds that $ba \leq ca$ ($ab \leq ac$).

For any a within the set Y, it is evident that χ_a possesses the property of right-compatibility. Here, id_X is used to denote the identity map on X. It is clear that $\pi(\mathrm{id}_X) = \{\{x\} : x \in X\}$ and $E_Y(\mathrm{id}_X) = \{A \cap Y : A \in X/E\}$, consequently, $\pi(\mathrm{id}_X)$ refines $\pi(\alpha)$ while $E_Y(\mathrm{id}_X)$ refines $E_Y(\alpha)$ for all $\alpha \in \overline{S}_E(X, Y)$. Subsequently, we proceed to establish a necessary condition for elements in $\overline{S}_E(X, Y)$ that are not constant maps, ensuring their right-compatibility.

Lemma 4.1. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible such that $|X\gamma| > 1$. Then, for all $z \in X \setminus Y$, $\{z\} \in \pi(\gamma)$.*

Proof. Assume, to the contrary, that there exist $z \in X \setminus Y$ and $w \in X \setminus \{z\}$ such that $z\gamma = w\gamma$. Since $|X\gamma| > 1$, there exists $k \in X$ such that $k\gamma \neq z\gamma$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} k & \text{if } x = z, \\ x & \text{otherwise.} \end{cases}$$

It is evident that $\alpha \in \overline{S}_E(X, Y)$, and $\alpha \leq \mathrm{id}_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \mathrm{id}_X\gamma = \gamma$. However, $w\alpha\gamma = w\gamma = z\gamma \neq k\gamma = z\alpha\gamma$ while $z\gamma = w\gamma$, leading to the conclusion that $\pi(\gamma)$ does not refine $\pi(\alpha\gamma)$, resulting in a contradiction. □

Lemma 4.2. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible. Then $(X \setminus Y)\gamma \subseteq Y$ or $(X \setminus Y)\gamma \subseteq X \setminus Y$.*

Proof. Assume, to the contrary, that $(X \setminus Y)\gamma \nsubseteq Y$ and $(X \setminus Y)\gamma \nsubseteq X \setminus Y$. Then there exist $w, z \in X \setminus Y$ such that $z\gamma \in Y$ and $w\gamma \in X \setminus Y$. Thus, $z \in (A \cap Y)\gamma^{-1}$ for some $A \in X/E$. Fix $a \in A \cap Y$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in Y, \\ w & \text{otherwise.} \end{cases}$$

It is evident that $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq \mathrm{id}_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \mathrm{id}_X\gamma = \gamma$. However, $z\alpha\gamma = w\gamma \notin Y$. Therefore $z \notin (B \cap Y)(\alpha\gamma)^{-1}$ for all $B \in X/E$. We conclude that $E_Y(\gamma)$ does not refine $E_Y(\alpha\gamma)$, which leads to a contradiction. □

Lemma 4.3. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible, and let $U \in E_Y(\gamma)$. If $U \cap (X \setminus Y) \neq \emptyset$ and $|U| > 1$, then $X \setminus Y \subseteq U$.*

Proof. Assume that there exist $z \in X \setminus Y$ and $z \neq w \in X$ such that $z\gamma, w\gamma \in A \cap Y$ for some $A \in X/E$. Let $a \in X \setminus Y$. According to Lemma 4.2, $a\gamma \in B \cap Y$ for some $B \in X/E$. Define $\alpha : X \to X$ as follows:

$$x\alpha = \begin{cases} a & \text{if } x = z, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X,Y)$ and $\alpha \leq \mathrm{id}_X$. As γ is right-compatible, it follows that $\alpha\gamma \leq \gamma$. According to Theorem 2.1, we conclude that $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$. Since $w\alpha\gamma = w\gamma \in A \cap Y$ and $z\alpha\gamma = a\gamma \in B \cap Y$, it follows that $w \in (A \cap Y)(\alpha\gamma)^{-1}$ and $z \in (B \cap Y)(\alpha\gamma)^{-1}$. Furthermore, since w and z belong to $(A \cap Y)\gamma^{-1}$ and $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$, we deduce that $(A \cap Y)\gamma^{-1} \subseteq (A \cap Y)(\alpha\gamma)^{-1}$ and $(A \cap Y)\gamma^{-1} \subseteq (B \cap Y)(\alpha\gamma)^{-1}$. Consequently, $(A \cap Y)(\alpha\gamma)^{-1} = (B \cap Y)(\alpha\gamma)^{-1}$, establishing $A = B$. This result implies $a\gamma \in A \cap Y$, as required. □

Lemma 4.4. *Let $\gamma \in \overline{S}_E(X,Y)$ be right-compatible. If there exist distinct $A, B \in X/E$ such that $(A \cap Y)\gamma, (B \cap Y)\gamma \subseteq C \cap Y$ for some $C \in X/E$, then $Y\gamma \subseteq C \cap Y$.*

Proof. Assume that there exist distinct $A, B \in X/E$ such that $(A \cap Y)\gamma, (B \cap Y)\gamma \subseteq C \cap Y$ for some $C \in X/E$. Let $y \in Y$. Then there exists $D \in X/E$ such that $y\gamma \in D \cap Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} y & \text{if } x \in B \cap Y, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X,Y)$ and $\alpha \leq \mathrm{id}_X$. Since γ is right-compatible, it follows that $\alpha\gamma \leq \gamma$. According to Theorem 2.1, we deduce that $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$. For every $a \in A \cap Y$ and $b \in B \cap Y$, it holds that $a, b \in (C \cap Y)\gamma^{-1}$. This implies $a\alpha\gamma = a\gamma \in C \cap Y$ and $b\alpha\gamma = y\gamma \in D \cap Y$. Consequently, $a \in (C \cap Y)(\alpha\gamma)^{-1}$ and $b \in (D \cap Y)(\alpha\gamma)^{-1}$. As $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$, we conclude that $(C \cap Y)\gamma^{-1} \subseteq (C \cap Y)(\alpha\gamma)^{-1}$ and $(C \cap Y)\gamma^{-1} \subseteq (D \cap Y)(\alpha\gamma)^{-1}$. Thus, $(C \cap Y)(\alpha\gamma)^{-1} = (D \cap Y)(\alpha\gamma)^{-1}$ leads to $C = D$. Consequently, $y\gamma \in C \cap Y$, and thus, $Y\gamma \subseteq C \cap Y$, as stipulated. □

Lemma 4.5. *Let $\gamma \in \overline{S}_E(X,Y)$ be right-compatible, and let $U \in E_Y(\gamma)$. If $|U \cap (X \setminus Y)| > 1$, then $E_Y(\gamma) = \{X\}$.*

Proof. Assume the existence of distinct z and w in $X \setminus Y$ such that $z\gamma$ and $w\gamma$ both belong to $A \cap Y$ for some $A \in X/E$. Utilizing Lemma 4.3, we deduce that $a\gamma \in A \cap Y$ for all $a \in X \setminus Y$. To demonstrate $Y\gamma \subseteq A \cap Y$, we assume, to the contrary, that there exists $a \in Y$ such that $a\gamma \in B \cap Y$ for some $A \neq B \in X/E$. Define the function $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in Y \cup \{z\}, \\ w & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X,Y)$ and $\alpha \leq \mathrm{id}_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \gamma$. By Theorem 2.1, we deduce that $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$. Considering $X\alpha\gamma = \{a\gamma, w\gamma\}$, we find $E_Y(\alpha\gamma) = \{Y \cup \{z\}, X \setminus (Y \cup \{z\})\}$. However, $X \setminus Y \subseteq (A \cap Y)\gamma^{-1} \in E_Y(\gamma)$, which implies that $E_Y(\gamma)$ does not refine $E_Y(\alpha\gamma)$, leading to a contradiction. Hence, $Y\gamma \subseteq A \cap Y$, allowing us to conclude that $X\gamma \subseteq A \cap Y$. □

Lemma 4.6. *Let $\gamma \in \overline{S}_E(X,Y)$ be right-compatible, with $|Y\gamma| > 1$, and let $A \in X/E$. Then $\gamma|_{A \cap Y}$ is constant or injection.*

Proof. Assume, to the contrary, that $\gamma|_{A \cap Y}$ is neither constant nor an injection. Then there exist distinct $a, b, c \in A \cap Y$ such that $a\gamma = b\gamma \neq c\gamma$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} c & \text{if } x = a, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq \mathrm{id}_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \gamma$. However, $a\gamma = b\gamma$ while $a\alpha\gamma = c\gamma \neq b\gamma = b\alpha\gamma$. This contradiction arises from the fact that $\pi(\gamma)$ refines $\pi(\alpha\gamma)$. This completes the proof. $\square$

Lemma 4.7. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible such that $|X\gamma| > 1$. Then $\pi(\gamma)$ is in one of the following forms.*

(1) $\pi(\gamma) = \{\{x\} : x \in X\}$.
(2) $\pi(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$.
(3) $\pi(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$.

Proof. Since γ is right-compatible and $|X\gamma| > 1$, according to Lemma 4.1, we have $\{\{z\} : z \in X \setminus Y\} \subseteq \pi(\gamma)$. If γ is injective, it is evident that $\pi(\gamma)$ is in the form (1). In the case where γ is not injective, two cases arise.

Case 1: $|Y\gamma| = 1$. Then $Y \in \pi(\gamma)$ and $\pi(\gamma)$ is in the form (2).

Case 2: $|Y\gamma| > 1$. From γ is not injection and $\{z\} \in \pi(\gamma)$ for all $z \in X \setminus Y$. There exists two distinct $a, b \in Y$ such that $a\gamma = b\gamma$. This implies that $a \in A$ for some $A \in X/E$. To demonstrate that $b \in A$, we assume that $b \in B$ for some $B \in X/E$ and $B \neq A$. Since $|Y\gamma| \neq 1$, there exists $y' \in Y$ such that $y'\gamma \neq a\gamma$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A \cap Y, \\ y' & \text{if } x \in B \cap Y, \\ x & \text{otherwise.} \end{cases}$$

Thus, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq \mathrm{id}_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \gamma$.

By Theorem 2.1, we obtain $\pi(\gamma)$ refines $\pi(\alpha\gamma)$. However, $a\gamma = b\gamma$ while $a\alpha\gamma = a\gamma \neq y'\gamma = b\alpha\gamma$, leading to a contradiction. Hence $b \in A$ and so $\gamma|_{A\cap Y}$ is constant. To show that $\gamma|_{B\cap Y}$ is a constant for every $B \in X/E$. Assume that there exists $B \in X/E$ such that $B \neq A$ and $\gamma|_{B\cap Y}$ is a injection. Let b_1 and b_2 be two distinct elements in $B \cap Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b_1 & \text{if } x = a, \\ b_2 & \text{otherwise.} \end{cases}$$

$\square$

From $X\alpha \subseteq B\cap Y$, we get $\alpha \in \overline{S}_E(X, Y)$. Similarly, $a\gamma = b\gamma$ and $a\alpha\gamma = b_1\gamma \neq b_2\gamma = b\alpha\gamma$ which is a contradiction. Thus $A \cap Y \in E_Y(\gamma)$ for all $A \in X/E$. Then, we conclude that $\pi(\gamma)$ is in form (3).

Lemma 4.8. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible such that $|X\gamma| > 1$. Then $E_Y(\gamma)$ is one of the following forms.*

(1) $E_Y(\gamma) = \{Y\}$.
(2) $E_Y(\gamma) = \{A \cap Y : A \in X/E\}$.
(3) $E_Y(\gamma) = \{X\}$.
(4) $E_Y(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$.
(5) $E_Y(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$.

Proof. By Lemma 4.2, we have $(X \setminus Y)\gamma \subseteq Y$ or $(X \setminus Y)\gamma \subseteq X \setminus Y$.

Case 1: $(X \setminus Y)\gamma \subseteq X \setminus Y$. We consider two subcases.

Subcase 1.1: $Y\gamma \subseteq A \cap Y$ for some $A \in X/E$. In this case $Y = (A \cap Y)\gamma^{-1}$, and $E_Y(\gamma)$ is in the form (1).

Subcase 1.2: $Y\gamma \nsubseteq A\cap Y$ for all $A \in X/E$. Hence, $|Y\gamma| > 1$. Considering any $A \in X/E$, there exists $A' \in X/E$ such that $(A \cap Y)\gamma \subseteq A' \cap Y$. Then $A \cap Y \subseteq (A' \cap Y)\gamma^{-1} \in E_Y(\gamma)$. To demonstrate that $A \cap Y = (A' \cap Y)\gamma^{-1}$, we assume, to the contrary, that there exists $a' \in (A' \cap Y)\gamma^{-1} \setminus (A \cap Y)$. Then $a' \in B \cap Y$ for some $B \in X/E$, and $a'\gamma \in A' \cap Y$. By Lemma 4.4, we conclude that $Y\gamma \subseteq A' \cap Y$, leading to a contradiction. Thus, $A \cap Y = (A' \cap Y)\gamma^{-1} \in E_Y(\gamma)$. Since A is arbitrary, we obtain $\{A\cap Y : A \in X/E\} \subseteq E_Y(\gamma)$ and the equality holds due to $(X \setminus Y)\gamma \subseteq X \setminus Y$. Therefore, $E_Y(\gamma)$ takes the form (2).

Case 2: $(X \setminus Y)\gamma \subseteq Y$. Two cases arise.

Subcase 2.1: $(X \setminus Y)\gamma \subseteq A \cap Y$ for some $A \in X/E$. If $|X \setminus Y| > 1$, then Lemma 4.5, $E_Y(\gamma) = \{X\}$. Thus, we can conclude that $E_Y(\gamma)$ is in the form (3). If $|X \setminus Y| = 1$, then similar Subcase 1.1, we get $E_Y(\gamma) = \{Y, X \setminus Y\}$ or $E_Y(\gamma) = \{X\}$ and similar Subcase 1.2, we have $E_Y(\gamma) = \{A \cap Y : A \in X/E\} \cup \{X \setminus Y\}$. Hence, we can conclude that $E_Y(\gamma)$ is in the form (4) and (5).

Subcase 2.2: $(X \setminus Y)\gamma \nsubseteq A\cap Y$ for all $A \in X/E$. By Lemma 4.5, we have $|A \cap Y \cap (X \setminus Y)\gamma| \leq 1$. Furthermore, according to Lemma 4.3, $(w\gamma, z\gamma) \notin E$ for all $w \in X$, $z \in X \setminus Y$ and $z \neq w$, implying that $\{\{x\} : x \in X \setminus Y\} \subseteq E_Y(\gamma)$. Consider $Y\gamma$. If $Y\gamma \subseteq A \cap Y$ for some $A \in X/E$, then $Y = (A \cap Y)\gamma^{-1} \in E_Y(\gamma)$, and hence $E_Y(\gamma)$ is in the form (4). If $Y\gamma \nsubseteq A\cap Y$ for all $A \in X/E$, then, by the same proof as given in Subcase 1.2, we conclude that $\{A \cap Y : A \in X/E\} \subseteq E_Y(\gamma)$. This implies $E_Y(\gamma)$ is in the form (5). $\square$

Theorem 4.9. *Let $\gamma \in \overline{S}_E(X,Y)$. Then γ is right-compatible if and only if γ is constant or $\pi(\gamma)$ and $E_Y(\gamma)$ satisfy Lemma 4.7 and 4.8, respectively.*

Proof. By Lemmas 4.7 and 4.8, the sufficient part remains to be proven. Let $\alpha, \beta \in \overline{S}_E(X,Y)$ such that $\alpha \leq \beta$ and $|X\gamma| > 1$. It is evident that $X\alpha\gamma \subseteq X\beta\gamma$, and for each $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha\gamma \subseteq (B \cap Y)\alpha\gamma$. To demonstrate that $\alpha\gamma$ and $\beta\gamma$ satisfy the remaining conditions in Theorem 2.1, we first establish that $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$. Let $P \in \pi(\beta\gamma)$. Then $P = x(\beta\gamma)^{-1}$ for some $x \in X\beta\gamma$. Three cases are considered.

Case 1: $\pi(\gamma) = \{\{x\} : x \in X\}$. Then $P = z\beta^{-1}$, where $z \in X\beta$ and $z\gamma = x$. Since $\pi(\beta)$ refines $\pi(\alpha)$, we have $P = z\beta^{-1} \subseteq z'\alpha^{-1}$ for some $z' \in X\alpha$. Let $z'\gamma = x'$. Then $x' \in X\alpha\gamma$ and $P = z\beta^{-1} \subseteq z'\alpha^{-1} = x'(\alpha\gamma)^{-1}$.

Case 2: $\pi(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$. In this case $P = Y\beta^{-1}$ or $P = z\beta^{-1}$, where $x\beta = z \in X \setminus Y$. If $P = Y\beta^{-1} = \bigcup_{A\in X/E}(A \cap Y)\beta^{-1}$, then since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have

$$\bigcup_{A\in X/E} (A \cap Y)\beta^{-1} \subseteq \bigcup_{B\in X/E} (B \cap Y)\alpha^{-1} = Y\alpha^{-1} = y\gamma^{-1}\alpha^{-1} = y(\alpha\gamma)^{-1},$$

where $y \in Y \cap X\alpha\gamma$. For the case $P = z\beta^{-1}$, where $x\beta = z \in X \setminus Y$, we have $P \in \pi(\beta)$. Since $\pi(\beta)$ refines $\pi(\alpha)$, and $\pi(\alpha)$ also refines $\pi(\alpha\gamma)$, there exists $P' \in \pi(\alpha\gamma)$ such that $P \subseteq P'$.

Case 3: $\pi(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$. Then, $P = (A \cap Y)\beta^{-1}$ with $x\beta \in A \cap Y$, or $P = z\beta^{-1}$ with $x\beta = z \in X \setminus Y$. If $P = (A \cap Y)\beta^{-1}$, due to the fact that $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $(A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ with $B \cap Y \cap X\alpha \neq \emptyset$. Since $\pi(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$, it implies that $B \cap Y = x'\gamma^{-1}$ for some $x' \in X\gamma$. Since $B \cap Y \cap X\alpha \neq \emptyset$, we have $x' \in X\alpha\gamma$, and thus, $P \subseteq (B \cap Y)\alpha^{-1} = x'\gamma^{-1}\alpha^{-1} = x'(\alpha\gamma)^{-1}$. The remaining case can be proven similarly to Case 2.

Considering the above three cases, we can conclude that $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$.

To demonstrate that $E_Y(\beta\gamma)$ refines $E_Y(\alpha\gamma)$, consider $U \in E_Y(\beta\gamma)$. Then, $U = (A \cap Y)(\beta\gamma)^{-1}$ for some $A \in X/E$. There are five cases to consider.

Case 1: $E_Y(\gamma) = \{Y\}$. In this case, as $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $U = Y\beta^{-1} = \bigcup_{A\in X/E}(A \cap Y)\beta^{-1} \subseteq \bigcup_{B\in X/E}(B \cap Y)\alpha^{-1}$. Therefore, $U \subseteq \bigcup_{B\in X/E}(B \cap Y)\alpha^{-1} = Y\alpha^{-1} = (A \cap Y)\gamma^{-1}\alpha^{-1} = (A \cap Y)(\alpha\gamma)^{-1}$.

Case 2: $E_Y(\gamma) = \{A\cap Y : A \in X/E\}$. In this case $U = (A'\cap Y)\beta^{-1}$, where $(A\cap Y)\gamma^{-1} = A'\cap Y$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $(A'\cap Y)\beta^{-1} \subseteq (B\cap Y)\alpha^{-1}$ for some $B \in X/E$ with $B\cap Y\cap X\alpha \neq \emptyset$. Furthermore, $B\cap Y = (C\cap Y)\gamma^{-1}$ for some $C \in X/E$. This implies that $U \subseteq (B\cap Y)\alpha^{-1} = (C\cap Y)\gamma^{-1}\alpha^{-1} = (C\cap Y)(\alpha\gamma)^{-1}$.

Case 3: $E_Y(\gamma) = \{X\}$. In this case $U = (A\cap Y)(\beta\gamma)^{-1} = X = (A\cap Y)(\alpha\gamma)^{-1}$.

Case 4: $E_Y(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$. In this case $U = Y\beta^{-1}$ or $U = z\beta^{-1}$, where $z \in X\beta \setminus Y$. The case $U = Y\beta^{-1}$ can be proven similar to Case 1. For the case $U = z\beta^{-1}$, as $\pi(\beta)$ refines $\pi(\alpha)$, we have $z\beta^{-1} \subseteq w\alpha^{-1}$ for some $w \in X\alpha$. If $w \in Y$, then $w\alpha^{-1} \subseteq Y\alpha^{-1} = (B \cap Y)\gamma^{-1}\alpha^{-1}$ for some $B \in X/E$. This implies that $U = z\beta^{-1} \subseteq (B\cap Y)(\alpha\gamma)^{-1}$. If $w \in X \setminus Y$, then $\{w\} = (B\cap Y)\gamma^{-1}$ for some $B \in X/E$, implying that $U \subseteq (B\cap Y)\gamma^{-1}\alpha^{-1} = (B\cap Y)(\alpha\gamma)^{-1}$.

Case 5: $E_Y(\gamma) = \{A\cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$. In this case $U = (A'\cap Y)\beta^{-1}$, where $(A\cap Y)\gamma^{-1} = A'\cap Y$, or $U = z\beta$, where $z \in X\beta \setminus Y$. Both instances can be demonstrated similarly to Case 2 and Case 4, respectively.
From the five cases outlined above, we can infer that $E_Y(\beta\gamma)$ refines $E_Y(\alpha\gamma)$.

Finally, let $x \in X$ be such that $x\beta\gamma \in X\alpha\gamma$. This implies $x\beta\gamma = z\alpha\gamma$ for some $z \in X$. Hence, $x\beta$ and $z\alpha$ belong to $w\gamma^{-1}$ for some $w \in X\gamma$. If $|w\gamma^{-1}| = 1$, then $x\beta = z\alpha \in X\alpha$, indicating $x\beta\gamma = x\alpha\gamma$. In the case where $w\gamma^{-1} = Y$, then $x\beta \in A\cap Y$ for some $A \in X/E$. This leads to $x \in (A\cap Y)\beta^{-1} \subseteq (B\cap Y)\alpha^{-1}$ for some $B \in X/E$. Thus, $x\alpha \in Y$, and so $x\alpha\gamma = w = x\beta\gamma$. If $w\gamma^{-1} = A\cap Y$ for some $A \in X/E$, then $x\beta, z\alpha \in A\cap Y$. Considering $X\alpha \subseteq X\beta$, we have $z\alpha = z'\beta = z'\alpha$ for some $z' \in X$. Therefore, $x\beta, z'\beta \in A\cap Y$, and so $x, z' \in (A\cap Y)\beta^{-1} \subseteq (B\cap Y)\alpha^{-1}$ for some $B \in X/E$ with $B\cap Y\cap X\alpha \neq \emptyset$. Consequently, $x\alpha, z'\alpha \in B\cap Y$. Since $z'\alpha = z\alpha \in A\cap Y$, we obtain $B = A$. Hence, $x\alpha, x\beta \in A\cap Y$, and then $x\alpha\gamma = w = x\beta\gamma$. □

We conclude this article by deriving the necessary and sufficient conditions for elements of $\overline{S}_E(X,Y)$ to be left-compatible in the case where the partition X/E of X is finite.

Lemma 4.10. *Let X/E be finite and $\gamma \in \overline{S}_E(X,Y)$. Then, $Y\gamma = Y$ if and only if, for each $A \in X/E$, there exists $B \in X/E$ such that $A\cap Y = (B\cap Y)\gamma$.*

Proof. Suppose $Y\gamma = Y$. Let $A \in X/E$ and $a \in A\cap Y$. As $Y\gamma = Y$, there exists $B \in X/E$ and $b \in B\cap Y$ such that $b\gamma = a$. Clearly, $(B\cap Y)\gamma \subseteq A\cap Y$. To establish the equality, assume, to the contrary, that there exists $c \in (A\cap Y) \setminus (B\cap Y)\gamma$. Again, since $Y\gamma = Y$, there exists $C \in X/E$ and $c' \in C\cap Y$ such that $c'\gamma = c$. This implies that $C \neq B$, and both $(B\cap Y)\gamma$ and $(C\cap Y)\gamma$ are subsets of $A\cap Y$. As X/E is finite, there exists $D \in X/E$ such that $D\cap Y \neq \emptyset$, but $D\cap Y\cap Y\gamma = \emptyset$. Consequently, $Y\gamma \neq Y$, leading to a contradiction.

Conversely, assume the condition holds. Then, for each $A \in X/E$, there exists $B_A \in X/E$ such that $A\cap Y = (B_A\cap Y)\gamma$. This implies that

$$Y = \bigcup_{A\in X/E}(A\cap Y) = \bigcup_{A\in X/E}(B_A\cap Y)\gamma \subseteq Y\gamma \subseteq Y.$$

This completes the proof. □

Theorem 4.11. *Let X/E be finite and $\gamma \in \overline{S}_E(X,Y)$. Then γ is left-compatible if and only if $Y\gamma = Y$ and either one of the following conditions hold:*

(1) $(X \setminus Y)\gamma \subseteq Y$.

(2) $X \setminus Y \subseteq X\gamma$.

Proof. Suppose that $Y\gamma \neq Y$ or that neither $(X \setminus Y)\gamma \subseteq Y$ nor $X \setminus Y \subseteq X\gamma$.

Case 1: $Y\gamma \neq Y$. Then there exists $a \in Y \setminus Y\gamma$ such that $a \in A \cap Y$ for some $A \in X/E$.

Subcase 1.1: $(A \cap Y) \cap Y\gamma \neq \emptyset$. In this instance, there exists $B \in X/E$ such that $B \cap Y \neq \emptyset$ and $(B \cap Y)\gamma \subseteq A \cap Y$. Choose $b \in (B \cap Y)\gamma$ and define $\alpha \in \overline{S}_E(X,Y)$ as follows:

$$x\alpha = \begin{cases} b & \text{if } x = a, \\ a & \text{otherwise.} \end{cases}$$

Clearly, $\chi_b \leq \alpha$ and $(A \cap Y)\gamma\chi_b = \{b\}$. However, for any $C \in X/E$, we have $(C \cap Y)\gamma\alpha \subseteq (Y \setminus \{a\})\alpha = \{a\}$. Therefore, $\gamma\chi_b \nleq \gamma\alpha$.

Subcase 1.2: $(A \cap Y) \cap Y\gamma = \emptyset$. Then $(A \cap Y)\gamma \subseteq B \cap Y$ for some $B \in X/E$. Choose $b \in (A \cap Y)\gamma \subseteq Y\gamma$ and define $\alpha \in \overline{S}_E(X,Y)$ as follows:

$$x\alpha = \begin{cases} b & \text{if } x \in A \cap Y, \\ a & \text{otherwise.} \end{cases}$$

Clearly $\chi_b \leq \alpha$ and $(A \cap Y)\gamma\chi_b = \{b\}$. Again, for any $C \in X/E$, we have $(C \cap Y)\gamma\alpha \subseteq (Y \setminus (A \cap Y))\alpha = \{a\}$, implying that $\gamma\chi_b \nleq \gamma\alpha$.

Case 2: $(X \setminus Y)\gamma \nsubseteq Y$ and $X \setminus Y \nsubseteq X\gamma$. Consequently, there exist $w \in (X \setminus Y)\gamma \setminus Y$ and $z \in (X \setminus Y) \setminus X\gamma$. Choose $y \in Y$ and let $\alpha, \beta \in \overline{S}_E(X,Y)$ be defined as follow:

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ z & \text{otherwise,} \end{cases} \quad \text{and } x\beta = \begin{cases} y & \text{if } x \in Y, \\ z & \text{if } x = z, \\ w & \text{otherwise.} \end{cases}$$

It is evident that $\alpha \leq \beta$. However, $z \in X\gamma\alpha$ while $z \notin X\gamma\beta$, resulting in $X\gamma\alpha \nsubseteq X\gamma\beta$. Therefore, $\gamma\alpha \nleq \gamma\beta$.
Considering the above two cases, we can conclude that γ is not left-compatible.

Conversely, assume that $Y\gamma = Y$ and $(X \setminus Y)\gamma \subseteq Y$ or $X \setminus Y \subseteq (X \setminus Y)\gamma$. Let $\alpha, \beta \in \overline{S}_E(X,Y)$ be such that $\alpha \leq \beta$. To demonstrate $\gamma\alpha \leq \gamma\beta$, we first verify that $\pi(\gamma\beta)$ refines $\pi(\gamma\alpha)$. Let $P \in \pi(\gamma\beta)$. Then $P = z(\gamma\beta)^{-1} = (z\beta^{-1})\gamma^{-1}$ for some $z \in X\gamma\beta$. Since $z = z'\gamma\beta$ for some $z' \in X$, we have $z'\gamma \in z\beta^{-1} \subseteq w\alpha^{-1}$ because $\pi(\beta)$ refines $\pi(\alpha)$. Hence, $w = z'\gamma\alpha \in X\gamma\alpha$ and $P = z(\gamma\beta)^{-1} = (z\beta^{-1})\gamma^{-1} \subseteq (w\alpha^{-1})\gamma^{-1} = w(\gamma\alpha)^{-1}$. To show that $E_Y(\gamma\beta)$ refines $E_Y(\gamma\alpha)$, consider $U \in E_Y(\gamma\beta)$. This implies $U = (A \cap Y)(\gamma\beta)^{-1} = (A \cap Y)\beta^{-1}\gamma^{-1}$ for some $A \in X/E$ and $A \cap Y \cap X\gamma\beta \neq \emptyset$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, it follows that $(A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ and $B \cap Y \cap X\alpha \neq \emptyset$. Consequently, $U = (A \cap Y)\beta^{-1}\gamma^{-1} \subseteq (B \cap Y)\alpha^{-1}\gamma^{-1} = (B \cap Y)(\gamma\alpha)^{-1}$. To confirm that $X\gamma\alpha$ is a subset of $X\gamma\beta$, we first consider the case where $(X \setminus Y)\gamma \subseteq Y$. In this situation, we observe that $X\gamma\alpha = Y\alpha \subseteq Y\beta = X\gamma\beta$. For the case where $X \setminus Y \subseteq (X \setminus Y)\gamma$, we have $X\gamma = X$, so $X\gamma\alpha = X\alpha \subseteq X\beta = X\gamma\beta$. Now, considering an element $x \in X$ such that $x\beta \in X\alpha$, we find that $(x\gamma)\beta \in X\gamma\alpha \subseteq X\alpha$, implying that $x\gamma\beta = x\gamma\alpha$. Finally, consider $A \in X/E$. Then, $(A \cap Y)\gamma \subseteq B \cap Y$ for some $B \in X/E$. Since $\alpha \leq \beta$, there exists $C \in X/E$ such that $(B \cap Y)\alpha \subseteq (C \cap Y)\beta$. According to 4.10, there exists $D \in X/E$ such that $C \cap Y = (D \cap Y)\gamma$. This implies that $(A \cap Y)\gamma\alpha \subseteq (B \cap Y)\alpha \subseteq (C \cap Y)\beta = (D \cap Y)\gamma\beta$. Therefore, $\gamma\alpha \leq \gamma\beta$, indicating that γ is left-compatible. $\square$

5. CONCLUSIONS

In this paper, we study the semigroup $\overline{S}_E(X, Y)$, which generalizes several well-known semigroups. We provide a characterization of the natural partial order on the semigroup $\overline{S}_E(X, Y)$ and classify its elements that are minimal, maximal, left-compatible, and right-compatible with respect to this order. Noting that $\overline{S}_E(X, X) = T_E(X)$, by setting $Y = X$ in Theorem 2.1, we derive the characterization of the natural partial order on $T_E(X)$, originally presented in [10, Theorem 2.1]. Moreover, if we set E as the identity relation on X, we obtain $\overline{S}_E(X, Y) = S(X, Y)$ and $E_Y(\alpha) = \pi_Y(\alpha) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}$. Using these results, Theorem 2.1 enables us to derive the characterization of this order on $S(X, Y)$, which was originally presented in [9, Theorem 2.1]. Finally, If E is the identity relation on X and $X = Y$, then $\overline{S}_E(X, Y) = T(X)$, $X/E = \{\{x\} : x \in X\}$, and $E_Y(\alpha) = E(\alpha) = \pi(\alpha)$. These results, combined with Theorem 2.1, provide the characterization of this order on $T(X)$, originally given in [4, Proposition 2.3].

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