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Optimal solutions of minimization problems via new best proximity point results on quasi metric spaces

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ABSTRACT. In this paper, we prove some Boyd-Wong type best proximity point results in the setting of quasi metric spaces via Q-functions. First, we modify the fundamental concepts and notations in the best proximity point theory by taking into account unsymmetrical condition of quasi metric spaces. We provide some illustrative examples to examine our notations. Then, we introduce new concepts so called proximal BW-contraction and best BW-contraction mappings. Hence, we obtain best proximity point results for such mappings. Also, we give some nontrivial and comparative examples to show the effectiveness of our results. Next, we provide some corollaries and consequences to partial metric spaces of our main results. Finally, we present an existence and uniqueness result for nonlinear Volterra integral equations.

1. INTRODUCTION

It is well known that a pseudo metric σ on a non-empty set $\mathcal{X}$ is a real valued function defined on $\mathcal{X} \times \mathcal{X}$ which satisfies the axioms of a metric except that the distance of distinct points is nonzero. In this case, many suitable famous results on metric spaces such as Baire category, Cantor intersection and Banach fixed point theorems can be easily extended to pseudo metric settings. However, when the symmetry condition is removed, many definitions of Cauchyness and completeness arise, unlike pseudo-metric spaces. Therefore, these extensions are not as easy to get as in pseudo-metric spaces. Despite this fact, studying by omitting the symmetry condition has attracted the attention of many authors due to the wide range of applications of unsymmetric distance functions in many branches as well as mathematics [8, 15, 25]. In this sense, the concept of quasi metric was first used by Wilson. [26]. Then, Kelly [14] obtained some generalizations of well known results such as Urysohn Lemma and Baire category theorem by taking into account biotopological spaces which are closely related with quasi metric spaces. Further, it was defined Cauchy sequence for a quasi pseudo metric spaces in the same paper by Kelly. However, Reilly et al. [20] observed that any convergent sequence may not be Cauchy in the sense of Kelly. To overcome this disadvantage, they proposed many kinds of definitions of Cauchy sequence in quasi metric spaces. Very recently, Altun et al. [2] proved some fixed point theorems on quasi metric spaces by classifying the definitions of Cauchyness and completeness. It can be found many nice, interesting and noteworthy results in this direction (see, for example, [4, 11] and the references therein). Now, we recall some fundamental notations and properties about quasi metric spaces: Let $\mathcal{X}$ be a nonempty set and $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ (the set of non negative real numbers) be a function such that for all $\zeta, \eta, \xi \in \mathcal{X}$,

(i) $\sigma(\zeta, \eta) = \sigma(\eta, \zeta) = 0 \iff \zeta = \eta$,
(ii) $\sigma(\zeta, \xi) \leq \sigma(\zeta, \eta) + \sigma(\eta, \xi)$.

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Then, the function σ is called quasi metric on $\mathcal{X}$. Also, the pair $(\mathcal{X},\sigma)$ is said to be a quasi metric space. In addition to (ii), if σ satisfies $\sigma(\zeta,\eta)=0 \Longleftrightarrow \zeta=\eta$, then σ is called T_1-quasi metric on $\mathcal{X}$. Let $(\mathcal{X},\sigma)$ be a quasi metric space, $\sigma^{-1}:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ and $\sigma^s:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ be mappings defined by

$$\sigma^{-1}(\zeta,\eta)=\sigma(\eta,\zeta)$$

and

$$\sigma^s(\zeta,\eta)=\max\{\sigma(\zeta,\eta),\sigma^{-1}(\zeta,\eta)\}$$

for all $\zeta,\eta\in\mathcal{X}$. Then, σ^{-1} is also a quasi metric (called conjugate of σ) and σ^s is an ordinary metric on $\mathcal{X}$. The subset M of $\mathcal{X}$ is said to be σ-open if for all $\zeta\in M$, there exists $r>0$ such that

$$B_\sigma(\zeta,r)=\{\eta\in\mathcal{X}:\sigma(\zeta,\eta)<r\}\subseteq M,$$

the subset M of $\mathcal{X}$ is said to be σ^{-1}-open if for all $\zeta\in M$, there exists $r>0$ such that

$$B_{\sigma^{-1}}(\zeta,r)=\{\eta\in\mathcal{X}:\sigma^{-1}(\zeta,\eta)<r\}\subseteq M.$$

If τ_σ and $\tau_{\sigma^{-1}}$ denote the family of all σ-open subsets of $\mathcal{X}$ and the family of all σ^{-1}-open subsets of $\mathcal{X}$, respectively, then τ_σ and $\tau_{\sigma^{-1}}$ are T_0-topologies on $\mathcal{X}$. Hence, a sequence $\{\zeta_n\}$ converges to $\zeta\in\mathcal{X}$ with respect to τ_σ if and only if $\sigma(\zeta,\zeta_n)\to 0$ as $n\to\infty$.

As we mentioned before, there are many definitions for the Cauchyness and completeness on quasi metric spaces. Throughout of this paper, we say that a quasi metric space $(\mathcal{X},\sigma)$ is complete if every σ^s-Cauchy sequence in $\mathcal{X}$ is σ^{-1}-convergent [17]. Recall that a sequence $\{\zeta_n\}$ in $\mathcal{X}$ is called σ^s-Cauchy sequence if for $\varepsilon>0$, there exists $k\in\mathbb{N}$ such that $\sigma^s(\zeta_n,\zeta_m)<\varepsilon$ whenever $m,n\geq k$.

Recently, there is a tendency to improve results proved on metric spaces by using some functions such as w-distance or τ-function [12, 21]. Therefore, Al-Homidan et al. [1] introduced a new concept, Q-function, on quasi metric spaces to generalize such functions. Then, a number of fixed point theorems has been presented by using Q-function in terms of quasi metric spaces [3, 10, 13]. In this sense, recently, Marin et al. [16] obtained some Boyd-Wong type fixed point results via Q-functions on the complete quasi metric spaces. First, they showed that Boyd-Wong fixed point theorem [7] doesn't extend to complete quasi metric spaces via quasi metric distance. However, using Q-functions, they obtained a nice quasi metric version of Boyd-Wong's result. Now, we recall the definition of Q-function given by Al-Homidan et al. [1].

Definition 1.1 ([1]). *Let $(\mathcal{X},\sigma)$ be a quasi metric space. Then, $q:\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ is called Q-function if the following conditions hold:*

- (Q_1) *$q(\zeta,\xi)\leq q(\zeta,\eta)+q(\eta,\xi)$ for all $\zeta,\eta,\xi\in\mathcal{X}$.*
- (Q_2) *If $\zeta\in\mathcal{X}$, $M>0$ and the sequence $\{\eta_n\}$ converges to a point $\eta\in\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$ and satisfies $q(\zeta,\eta_n)\leq M$ for all $n\in\mathbb{N}$, then $q(\zeta,\eta)\leq M$*
- (Q_3) *For each $\varepsilon>0$, there exists $\delta>0$ such that*

$$q(\zeta,\eta)\leq\delta \text{ and } q(\zeta,\xi)\leq\delta \Longrightarrow \sigma(\eta,\xi)\leq\varepsilon.$$

The following lemmas play important roles in our main results.

Lemma 1.1 ([16]). *Let q be a Q-function on a quasi metric space $(\mathcal{X},\sigma)$. Then, for each $\varepsilon>0$, there exists $\delta>0$ such that $q(\zeta,\eta)\leq\delta$ and $q(\zeta,\xi)\leq\delta$ imply $\sigma^s(\eta,\xi)\leq\varepsilon$.*

Lemma 1.2 ([1]). *Let $(\mathcal{X},\sigma)$ be a quasi metric space, $\{\zeta_n\}$,$\{\eta_n\}$ be sequences in $\mathcal{X}$ and $q:\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ be a Q-function. Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $\mathbb{R}^+$ such that $\alpha_n\to 0$ and $\beta_n\to 0$ as $n\to\infty$. Then, the following ones hold for all $\zeta,\eta,\xi\in\mathcal{X}$,*

- (i) *If $q(\zeta_n,\eta)\leq\alpha_n$ and $q(\zeta_n,\xi)\leq\beta_n$ for all $n\in\mathbb{N}$, then $\eta=\xi$.*

(ii) *If* $q(\zeta_n, \eta_n) \leq \alpha_n$ *and* $q(\zeta_n, \eta) \leq \beta_n$ *for all* $n \in \mathbb{N}$, *then* $\sigma^s(\eta_n, \eta) \to 0$.
(iii) *If* $q(\zeta_n, \zeta_m) \leq \alpha_n$ *for all* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{\zeta_n\}$ *is a* σ^s*-Cauchy sequence.*

On the other hand, the metric fixed point theory has been extended in a different way from the literature by using nonself mappings $F : M \to N$ where M, N are nonempty subsets of a metric space $(\mathcal{X}, \sigma)$. In case of $M \cap N = \emptyset$, there is no fixed point of the mapping F. In this case, it is reasonable to investigate whether the mapping F has a point ζ in M such that $\sigma(\zeta, F\zeta) = \sigma(M, N)$ which is called best proximity point of F. Hence, Basha and Veeramani [6] obtained an optimal solution for the minimization problem $\min_{\zeta \in M} \sigma(\zeta, F\zeta)$ and proved some best proximity point results. Recently, since the best proximity point theory includes the fixed point theory in special case $M = N = \mathcal{X}$, this topic has been studied by many authors [9, 22, 23, 24]. Now, we state some notations and definitions of best proximity point theory: Let $(\mathcal{X}, \sigma)$ be a metric space, M and N be nonempty subsets of $\mathcal{X}$. Then, N is said to be approximately compact with respect to M, if all sequence $\{\eta_n\}$ in N such that $\sigma(\zeta, \eta_n) \to \sigma(\zeta, N)$ for some $\zeta \in M$ has a convergent subsequence in N.

Basha [5] extended the Banach contraction principle to the best proximity point theory via the following definition.

Definition 1.2. *Let* $(\mathcal{X}, \sigma)$ *be a metric space,* M, N *be nonempty subsets of* $\mathcal{X}$ *and* $F : M \to N$ *be a mapping. If there exists* $k \in (0, 1)$ *such that*

$$\left.\begin{array}{l}\sigma(\eta_1, F\zeta_1) = \sigma(M, N)\\ \sigma(\eta_2, F\zeta_2) = \sigma(M, N)\end{array}\right\} \Longrightarrow \sigma(\eta_1, \eta_2) \leq k\sigma(\zeta_1, \zeta_2)$$

for all $\eta_1, \eta_2, \zeta_1, \zeta_2 \in M$, *then* F *is called proximal contraction mapping.*

Raj [19] presented a different aspect in the best proximity point theory by introducing the concept of P-property.

Definition 1.3. *Let* $(\mathcal{X}, \sigma)$ *be a metric space and* M, N *be nonempty subsets of* $\mathcal{X}$. *The pair* (M, N) *is said to have* P*-property if and only if*

$$\left.\begin{array}{l}\sigma(\zeta_1, \eta_1) = \sigma(M, N)\\ \sigma(\zeta_2, \eta_2) = \sigma(M, N)\end{array}\right\} \Longrightarrow \sigma(\zeta_1, \zeta_2) = \sigma(\eta_1, \eta_2)$$

for all $\zeta_1, \zeta_2 \in M$ *and* $\eta_1, \eta_2 \in N$.

In this paper, we prove some Boyd-Wong type best proximity point results in the setting of quasi metric spaces via Q-functions. First, we modify the fundamental concepts and notations in the best proximity point theory by taking into account unsymmetrical condition of quasi metric spaces. We provide some illustrative examples to examine our notations. Then, we introduce new concepts so called proximal BW-contraction and best BW-contraction mappings. Hence, we obtain best proximity point results for such mappings. Also, we give some nontrivial and comparative examples to show the effectiveness of our results. Next, we provide some corollaries and consequences to partial metric spaces of our main results. Finally, we present an existence and uniqueness result for nonlinear Volterra integral equations.

2. The results for proximal contractions

In quasi metric spaces, since the symmetry property is no longer available, we revised some definitions and notations of best proximity point as follows:

$$M_0^L = \{\zeta \in M : \sigma(\zeta, \eta) = \sigma(M, N) \text{ for some } \eta \in N\},$$
$$M_0^R = \{\zeta \in M : \sigma(\eta, \zeta) = \sigma(N, M) \text{ for some } \eta \in N\},$$

and

$$N_0^L = \{\eta \in N : \sigma(\eta, \zeta) = \sigma(N, M) \text{ for some } \zeta \in M\},$$
$$N_0^R = \{\eta \in N : \sigma(\zeta, \eta) = \sigma(M, N) \text{ for some } \zeta \in M\}.$$

Definition 2.4. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, M, N be nonempty subsets of $\mathcal{X}$ and $F : M \to N$ be a mapping. A point ζ is called left (right) best proximity point of F if $\sigma(\zeta, F\zeta) = \sigma(M, N)$ ($\sigma(F\zeta, \zeta) = \sigma(N, M)$). Also, we say that a point ζ is best proximity point of F if it is both left and right best proximity point of F.*

Definition 2.5. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, M and N be nonempty subsets of $\mathcal{X}$. Then, N is said to be σ-approximately compact with respect to M if all sequence $\{\eta_n\}$ in N such that $\sigma(\eta_n, \zeta) \to \sigma(N, \zeta)$ for some $\zeta \in M$ has a convergent subsequence with respect to τ_σ in N.*

Now, we provide an example to show that the set N is a σ^{-1}-approximately compact with respect to M, but it is not σ-approximately compact. Also, this example is important to show that the sequence $\{\eta_n\}$ mentioned in Definition 2.5 has a subsequence which is $\tau_{\sigma^{-1}}$-convergent, but it is not τ_{σ^s}-convergent.

Example 2.1. *Let $\mathcal{X} = \mathbb{N}$ be the set of all natural numbers with the quasi metric σ defined by $\sigma(n, n) = 0$ and $\sigma(n, m) = \frac{1}{n}$ for all $n, m \in \mathbb{N}$. Consider the subsets M and N as*

$$M = \{2n : n \in \mathbb{N}\}$$

and

$$N = \{2n + 1 : n \in \mathbb{N}\}.$$

It is clear that for every sequence $\{\eta_n\}$ in N and for every ζ in M we have $\sigma(\zeta, \eta_n) \to \sigma(\zeta, N)$. In this case, if the set $\{\eta_n : n \in \mathbb{N}\}$ is a finite set, then it is well known that $\{\eta_n\}$ has a convergent subsequence in N. Now, assume $\{\eta_n : n \in \mathbb{N}\}$ is infinite set. Then, there exists a subsequence $\{\eta_{n_k}\}$ such that $\eta_{n_k} > k$ for all $k \in \mathbb{N}$. Therefore, for every $\eta \in N$ we have $\sigma(\eta_{n_k}, \eta) \to 0$ as $k \to \infty$. Hence, for all sequence $\{\eta_n\}$ in N such that $\sigma(\zeta, \eta_n) \to \sigma(\zeta, N)$ for some $\zeta \in M$ has a convergent subsequence with respect to $\tau_{\sigma^{-1}}$ in N, that is, N is σ^{-1}-approximately compact with respect to M. Now consider the sequence $\{\eta_n\}$ in N defined by $\eta_n = 2n + 1$ for all $n \in \mathbb{N}$, then $\{\eta_n\}$ does not have a τ_σ-convergent (and so τ_{σ^s}-convergent) subsequence in N. Note that, for all $\zeta \in M$, $\sigma(2n + 1, \zeta) \to \sigma(N, \zeta) = 0$ as $n \to \infty$. Therefore, N is not σ-approximately compact with respect to M.

Now, we recall the concept of Boyd-Wong function which will be considered in our contractions: Let $\varphi : [0, \infty) \to [0, \infty)$ be a function. If φ satisfies $\varphi(0) = 0$, $\varphi(t) < t$ for all $t > 0$ and $\limsup_{r \to t^+} \varphi(r) < t$ for all $t > 0$, then it is said to be a Boyd-Wong function. We will denote the family of all Boyd-Wong functions φ by Ω.

Now, we introduce the following definition which includes the definition of proximal contraction.

Definition 2.6. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, M and N be nonempty subsets of $\mathcal{X}$. Let $F : M \to N$ be a mapping and $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Q-function. If there exists $\varphi \in \Omega$ such that*

$$\left.\begin{array}{l} \sigma(\eta_1, F\zeta_1) = \sigma(M, N) \\ \sigma(\eta_2, F\zeta_2) = \sigma(M, N) \end{array}\right\} \Longrightarrow q(\eta_1, \eta_2) \leq \varphi(q(\zeta_1, \zeta_2)) \tag{2.1}$$

for all $\eta_1, \eta_2, \zeta_1, \zeta_2 \in M$, then F is called left proximal BW-contraction mapping. If the implication

$$\left.\begin{array}{l} \sigma(F\zeta_1, \eta_1) = \sigma(N, M) \\ \sigma(F\zeta_2, \eta_2) = \sigma(N, M) \end{array}\right\} \Longrightarrow q(\eta_1, \eta_2) \leq \varphi(q(\zeta_1, \zeta_2)) \tag{2.2}$$

holds for all $\eta_1, \eta_2, \zeta_1, \zeta_2 \in M$, then F is called right proximal BW-contraction mapping.

Remark 2.1. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, M and N be nonempty subsets of $\mathcal{X}$. Let $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Q-function and $F : M \to N$ be a left (right) proximal BW-contraction mapping. In this case, the left (right) best proximity point of F in M is unique if it exists. Indeed, assume ζ^* and ζ^{**} are two left best proximity point of F in M. Then, from (2.1) we have*

$$q(\zeta^*, \zeta^{**}) \leq \varphi(q(\zeta^*, \zeta^{**})),$$

and so $q(\zeta^*, \zeta^{**}) = 0$ (otherwise we have a contradiction). Similarly, we get $q(\zeta^{**}, \zeta^*) = 0$. Therefore, we have

$$q(\zeta^*, \zeta^*) \leq q(\zeta^*, \zeta^{**}) + q(\zeta^{**}, \zeta^*) = 0,$$

hence, from Lemma 1.1, we have $\zeta^* = \zeta^{**}$.

Now, we give our main result for the left proximal BW-contraction mappings as follows:

Theorem 2.1. *Let $(\mathcal{X}, \sigma)$ be a complete quasi metric space, M, N be nonempty subsets of $\mathcal{X}$ where M is a closed subset with respect to $\tau_{\sigma^{-1}}$, $F : M \to N$ be a mapping and $M_0^L \neq \emptyset$. Let $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Q-function, N be an σ^{-1}-approximately compact subset with respect to M and $F(M_0^L) \subseteq N_0^R$. If F is a left proximal BW-contraction mapping, then F has a unique left best proximity point ζ^* in M. Moreover, $q(\zeta^*, \zeta^*) = 0$.*

Proof. Let $\zeta_0 \in M_0^L$ be an arbitrary point. Since $F\zeta_0 \in F(M_0^L) \subseteq N_0^R$, there exists $\zeta_1 \in M_0^L$ such that

$$\sigma(\zeta_1, F\zeta_0) = \sigma(M, N).$$

Similarly, since $F\zeta_1 \in F(M_0^L) \subseteq N_0^R$, there exists $\zeta_2 \in M_0^L$ such that

$$\sigma(\zeta_2, F\zeta_1) = \sigma(M, N).$$

Since F is a left proximal BW-contraction mapping, we have

$$q(\zeta_1, \zeta_2) \leq \varphi(q(\zeta_0, \zeta_1)).$$

Continuing this process, we can construct a sequence $\{\zeta_n\}$ in M such that

$$\sigma(\zeta_{n+1}, F\zeta_n) = \sigma(M, N) \tag{2.3}$$

and

$$q(\zeta_n, \zeta_{n+1}) \leq \varphi(q(\zeta_{n-1}, \zeta_n)) \tag{2.4}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $q(\zeta_{n_0}, \zeta_{n_0+1}) = 0$, then $\varphi(q(\zeta_{n_0}, \zeta_{n_0+1})) = 0$. Therefore, we have $q(\zeta_n, \zeta_{n+1}) = 0$ for all $n \geq n_0$ and so $q(\zeta_n, \zeta_{n+1}) \to 0$ as $n \to \infty$. Now, we assume that $q(\zeta_n, \zeta_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then, from (2.4) we obtain

$$\begin{aligned} q(\zeta_n, \zeta_{n+1}) &\leq \varphi(q(\zeta_{n-1}, \zeta_n)) \\ &< q(\zeta_{n-1}, \zeta_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\{q(\zeta_n, \zeta_{n+1})\}$ is a decreasing sequence in $\mathbb{R}$. Then, there exists $\eta \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} q(\zeta_n, \zeta_{n+1}) = \eta.$$

We claim that $\eta = 0$. Assume $\eta > 0$. In this case, we have

$$\begin{aligned} \eta &= \lim_{n\to\infty} q(\zeta_n, \zeta_{n+1}) \\ &\leq \lim_{n\to\infty} \varphi(q(\zeta_{n-1}, \zeta_n)) \\ &\leq \limsup_{n\to\infty} \varphi(q(\zeta_{n-1}, \zeta_n)) \\ &< \eta. \end{aligned}$$

This is a contradiction. Therefore, $\lim_{n\to\infty} q(\zeta_n, \zeta_{n+1}) = 0$. Similarly, we have $\lim_{n\to\infty} q(\zeta_{n+1}, \zeta_n) = 0$.

Now, we shall show that for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(\zeta_n, \zeta_m) < \varepsilon$ for all $m > n \geq n_0$. Assume the contrary. Then, there exist $\varepsilon_0 > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ with $m_k > n_k \geq k$ such that

$$q(\zeta_{n_k}, \zeta_{m_k}) \geq \varepsilon_0 \tag{2.5}$$

for all $k \in \mathbb{N}$ where m_k is the smallest integer satisfying (2.5) corresponding to n_k. Hence, we have

$$q(\zeta_{n_k}, \zeta_{m_k-1}) < \varepsilon_0.$$

Then, we get

$$\begin{aligned} \varepsilon_0 &\leq q(\zeta_{n_k}, \zeta_{m_k}) \\ &\leq q(\zeta_{n_k}, \zeta_{m_k-1}) + q(\zeta_{m_k-1}, \zeta_{m_k}) \\ &< \varepsilon_0 + q(\zeta_{m_k-1}, \zeta_{m_k}). \end{aligned}$$

Since $\lim_{n\to\infty} q(\zeta_n, \zeta_{n+1}) = 0$, we get

$$q(\zeta_{n_k}, \zeta_{m_k}) \to \varepsilon_0^+ \text{ as } k \to \infty.$$

Also, we obtain

$$\begin{aligned} \varepsilon_0 &\leq q(\zeta_{n_k}, \zeta_{m_k}) \\ &\leq q(\zeta_{n_k}, \zeta_{n_k+1}) + q(\zeta_{n_k+1}, \zeta_{m_k+1}) + q(\zeta_{m_k+1}, \zeta_{m_k}) \\ &\leq q(\zeta_{n_k}, \zeta_{n_k+1}) + \varphi(q(\zeta_{n_k}, \zeta_{m_k})) + q(\zeta_{m_k+1}, \zeta_{m_k}), \end{aligned}$$

and so

$$\varepsilon_0 \leq \lim_{k\to\infty} \sup \varphi(q(\zeta_{n_k}, \zeta_{m_k})) < \varepsilon_0,$$

which is a contradiction. Let $\varepsilon > 0$ and $\delta = \delta(\varepsilon)$ satisfying (Q_3). Then, there exists $n_0 \in \mathbb{N}$ such that $q(\zeta_n, \zeta_m) < \delta$ for all $m > n \geq n_0$. Since $q(\zeta_{n_0}, \zeta_n) < \delta$ and $q(\zeta_{n_0}, \zeta_m) < \delta$ for all $n, m \geq n_0$, we have $\sigma^s(\zeta_n, \zeta_m) \leq \varepsilon$. Thus, $\{\zeta_n\}$ is a σ^s-Cauchy sequence. Since $(\mathcal{X}, \sigma)$ is a complete quasi metric space and M is a closed subset of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, there exists $\zeta^* \in M$ such that

$$\sigma(\zeta_n, \zeta^*) \to 0 \text{ as } n \to \infty.$$

Let $\varepsilon > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $q(\zeta_n, \zeta_m) < \varepsilon$ for all $m > n \geq n_0$. Now, fix $n \geq n_0$. Using (Q_2), we have $q(\zeta_n, \zeta^*) \leq \varepsilon$ for all $n \geq n_0$. Hence, $q(\zeta_n, \zeta^*) \to 0$ as $n \to \infty$. Since $q(\zeta_n, \zeta_m) \to 0$ and $q(\zeta_n, \zeta^*) \to 0$ as $n, m \to \infty$, then from Lemma 1.2 (ii), we also have $\sigma^s(\zeta_n, \zeta^*) \to 0$ as $n \to \infty$. Hence, we get

$$\begin{aligned} \sigma(\zeta^*, N) &\leq \sigma(\zeta^*, F\zeta_n) \\ &\leq \sigma(\zeta^*, \zeta_{n+1}) + \sigma(\zeta_{n+1}, F\zeta_n) \\ &= \sigma(\zeta^*, \zeta_{n+1}) + \sigma(M, N) \\ &\leq \sigma^s(\zeta_{n+1}, \zeta^*) + \sigma(\zeta^*, N), \end{aligned}$$

and so $\sigma(\zeta^*, F\zeta_n) \to \sigma(\zeta^*, N)$ as $n \to \infty$. Since N is a σ^{-1}-approximately compact with respect to. M, there exists a subsequence $\{F\zeta_{n_k}\}$ of $\{F\zeta_n\}$ such that $\{F\zeta_{n_k}\}$ converges to a point η^* in N with respect to $\tau_{\sigma^{-1}}$, that is,

$$\sigma(F\zeta_{n_k}, \eta^*) \to 0 \text{ as } k \to \infty$$

for some $\eta^* \in N$. Hence, we have

$$\begin{aligned} \sigma(M,N) &\leq \sigma(\zeta^*,\eta^*) \\ &\leq \sigma(\zeta^*,\zeta_{n_k+1}) + \sigma(\zeta_{n_k+1},F\zeta_{n_k}) + \sigma(F\zeta_{n_k},\eta^*) \\ &\leq \sigma^s(\zeta_{n_k+1},\zeta^*) + \sigma(M,N) + \sigma(F\zeta_{n_k},\eta^*). \end{aligned}$$

Taking the limit $k \to \infty$ in last inequality, we get

$$\sigma(\zeta^*,\eta^*) = \sigma(M,N).$$

Since $\zeta^* \in M_0^L$, we have $F\zeta^* \in N_0^R$ and so there exists $\xi \in M_0^L$ such that

$$\sigma(\xi, F\zeta^*) = \sigma(M,N). \tag{2.6}$$

Using inequalities (2.1), (2.3) and (2.6), we get $q(\zeta_n,\xi) \to 0$ as $n \to \infty$. Because of $q(\zeta_n,\zeta^*) \to 0$ as $n \to \infty$ and from Lemma 1.2 (i), we have $\zeta^* = \xi$. Hence, we obtain $\sigma(\zeta^*,F\zeta^*) = \sigma(M,N)$, and so ζ^* is a left best proximity point of F. From Remark 2.1, the left best proximity point of F is unique. Now, we want to show that $q(\zeta^*,\zeta^*) = 0$. Assume the contrary, that is, $q(\zeta^*,\zeta^*) > 0$. Since $\sigma(\zeta^*,F\zeta^*) = \sigma(M,N)$ and F is a left proximal BW-contraction mapping, we get

$$q(\zeta^*,\zeta^*) \leq \varphi(q(\zeta^*,\zeta^*)) < q(\zeta^*,\zeta^*),$$

which is a contradiction. □

Now, we present an example to illustrate the effectiveness of our main theorem in this section. Note that, the mapping F is not a proximal BW-contraction in the sense of quasi metric σ.

Example 2.2. *Let $\mathcal{X} = [0,\infty) \times [0,\infty)$, $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ and $q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$be functions defined by*

$$\sigma(t,s) = \sigma((t_1,t_2),(s_1,s_2)) = \max\{s_1 - t_1, 0\} + |t_2 - s_2|$$

and

$$q(t,s) = q((t_1,t_2),(s_1,s_2)) = s_1 + s_2$$

for all $t,s \in \mathcal{X}$, respectively. Then, $(\mathcal{X},\sigma)$ is a complete quasi metric space and q is a Q-function. Let

$$M = [0,2] \times \{4,5\} \cup \{(0,0)\}$$

and

$$N = [2,3] \times \{0,2,7\}$$

be subsets of $\mathcal{X}$. It can be easily seen that M is a closed subsets of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, N is a σ^{-1}-approximately compact subset with respect to M. Also, we have $M_0^L = \{(0,0),(2,4),(2,5)\}$, $N_0^R = \{(2,0),(2,2),(2,7)\}$ and $\sigma(M,N) = 2$. Define the mappings $F : M \to N$ and $\varphi : [0,\infty) \to [0,\infty)$ as follows:

$$F\zeta = \begin{cases} \left(3 - \frac{a}{4}, 0\right) & \zeta = (a,b) \in M - M_0^L \\ (2,0) & \zeta = (0,0) \text{ and } \zeta = (2,4) \\ (2,2) & \zeta = (2,5) \end{cases}$$

and

$$\varphi(t) = \frac{6}{7}t.$$

In this case, $F(M_0^L) \subseteq N_0^R$. Now, we claim that F is a left proximal BW-contraction. Since there is no $\eta_1 \in M$ satisfying

$$\sigma(\eta_1, F\zeta_1) = \sigma(M,N)$$

whenever $\zeta_1 \in M - M_0^L$, it is enough to consider the following cases:

Case 1. Let $\zeta_1 = (0,0)$ and $\zeta_2 = (2,4)$. Then $\eta_1 = (0,0) = \eta_2$, and so we have

$$q(\eta_1, \eta_2) = 0 \leq \varphi(q(\zeta_1, \zeta_2))$$

and

$$q(\eta_2, \eta_1) = 0 \leq \varphi(q(\zeta_2, \zeta_1)).$$

Case 2. Let $\zeta_1 = (0,0)$ and $\zeta_2 = (2,5)$. Then $\eta_1 = (0,0)$ and $\eta_2 = (2,4)$, and so we have

$$q(\eta_1, \eta_2) = 6 \leq \varphi(q(\zeta_1, \zeta_2))$$

and

$$q(\eta_2, \eta_1) = 0 \leq \varphi(q(\zeta_2, \zeta_1)).$$

Case 3. Let $\zeta_1 = (2,4)$ and $\zeta_2 = (2,5)$. Then $\eta_1 = (0,0)$ and $\eta_2 = (2,4)$, and so we have

$$q(\eta_1, \eta_2) = 6 \leq \varphi(q(\zeta_1, \zeta_2))$$

and

$$q(\eta_2, \eta_1) = 0 \leq \varphi(q(\zeta_2, \zeta_1)).$$

Then, all hypotheses of Theorem 2.1 are satisfied. Therefore, F has a unique left best proximity point in M.

Note that, for $\zeta_1 = (2,4), \zeta_2 = (2,5), \eta_1 = (0,0)$ and $\eta_2 = (2,4)$ in M, we have

$$\left.\begin{array}{l} \sigma(\eta_1, F\zeta_1) = \sigma(M,N) \\ \sigma(\eta_2, F\zeta_2) = \sigma(M,N) \end{array}\right\}.$$

However,

$$\sigma(\eta_1, \eta_2) = 6 > \frac{6}{7} = \varphi(\sigma(\zeta_1, \zeta_2)).$$

Hence, the mapping F is not a left proximal BW-contraction in the sense of quasi metric σ.

Taking into account the subsets M_0^R and N_0^L in Theorem 2.1, we can obtain the following theorem.

Theorem 2.2. *Let $(\mathcal{X}, \sigma)$ be a complete quasi metric space, M, N be nonempty subsets of $\mathcal{X}$ where M is a closed subset with respect to $\tau_{\sigma^{-1}}$, $F : M \to N$ be a mapping and $M_0^R \neq \emptyset$. Let $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Q-function, N be a σ-approximately compact subset with respect to M and $F(M_0^R) \subseteq N_0^L$. If F is a right proximal BW-contraction mapping, then F has a unique right best proximity point ζ^* in M. Moreover $q(\zeta^*, \zeta^*) = 0$.*

Theorem 2.1 and Theorem 2.2 are independent of each other. A mapping has a left best proximity point, but it has not a right best proximitiy point and vice versa. The following example shows this fact.

Example 2.3. *Let $\mathcal{X}, \sigma, q$ be as in Example 2.2. Consider the following subsets of $\mathcal{X}$:*

$$M = [0,2] \times \{4,5\} \cup \{(0,0)\}$$

and

$$N = \left[\frac{5}{2}, 3\right] \times \{0, 2, 7\}.$$

It can be easily seen that M is a closed subsets of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, N is a σ-approximately compact subset with respect to M. Also, we have $M_0^R = \{(0,0)\}$, $N_0^L = \left[\frac{5}{2}, 3\right] \times \{0\}$ and $\sigma(N, M) = 0$. Define the mappings $F : M \to N$ and $\varphi : [0,\infty) \to [0,\infty)$ as follows:

$$F\zeta = \begin{cases} \left(3 - \frac{a}{4}, 2\right) & , \quad \zeta = (a,b) \in M - \{(0,0), (2,4), (2,5)\} \\ (3,0) & , \quad \zeta = (0,0) \text{ and } \zeta = (2,4) \\ (3,2) & , \quad \zeta = (2,5) \end{cases}$$

and

$$\varphi(t) = \frac{t}{2}.$$

In this case, although $F(M_0^R) \subseteq N_0^L$, it is obvious that $F(M_0^L) \nsubseteq N_0^R$. Also, F is a right proximal BW-contraction mapping. Then, all hypotheses of Theorem 2.2 are satisfied, and so F has a unique right best proximity point in M. However, F does not have a left proximity point in M.

3. The results with P-property

We begin this section by introducing the concepts of generalized P_q^L-Property and generalized P_q^R-Property for a pair subsets of a quasi metric space and best BW-contraction for a nonself mapping.

Definition 3.7. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, q be a Q-function on $\mathcal{X}$ and $M, N \subseteq \mathcal{X}$. The pair (M, N) is said to have generalized P_q^L-Property if and only if*

$$\left.\begin{array}{l}\sigma(\zeta_1, \eta_1) = \sigma(M, N)\\ \sigma(\zeta_2, \eta_2) = \sigma(M, N)\end{array}\right\} \Longrightarrow q(\zeta_1, \zeta_2) = q(\eta_1, \eta_2)$$

for all $\zeta_1, \zeta_2 \in M$ with $\zeta_1 \neq \zeta_2$ and $\eta_1, \eta_2 \in N$.

Definition 3.8. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, q be a Q-function on $\mathcal{X}$ and $M, N \subseteq \mathcal{X}$. The pair (M, N) is said to have generalized P_q^R-Property if and only if*

$$\left.\begin{array}{l}\sigma(\eta_1, \zeta_1) = \sigma(N, M)\\ \sigma(\eta_2, \zeta_2) = \sigma(N, M)\end{array}\right\} \Longrightarrow q(\zeta_1, \zeta_2) = q(\eta_1, \eta_2)$$

for all $\zeta_1, \zeta_2 \in M$ with $\zeta_1 \neq \zeta_2$ and $\eta_1, \eta_2 \in N$.

Definition 3.7 and Definition 3.8 are independent of each other. Indeed, let $\mathcal{X} = \mathbb{R}$, $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ and $q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be mappings defined by

$$\sigma(\zeta, \eta) = \begin{cases} \eta - \zeta, & \zeta \leq \eta \\ \\ 2(\zeta - \eta), & \text{otherwise} \end{cases}.$$

and $q(\zeta, \eta) = |\zeta - \eta|$ for all $\zeta, \eta \in \mathcal{X}$. Then, $(\mathcal{X}, \sigma)$ is a quasi metric space and q is a Q-function. Consider the subsets $M = [1, 2]$ and $N = \{\frac{1}{2}\} \cup [3, 4]$. Then, although the pair (M, N) does not have the generalized P_q^L-property, it has the generalized P_q^R-property. Indeed, for $\zeta_1 = 1$, $\zeta_2 = 2$, $\eta_1 = \frac{1}{2}$ and $\eta_2 = 3$ we have

$$\left.\begin{array}{l}\sigma(1, \frac{1}{2}) = 1 = \sigma(M, N)\\ \sigma(2, 3) = 1 = \sigma(M, N)\end{array}\right\},$$

but

$$q(\zeta_1, \zeta_2) = 1 \neq \frac{5}{2} = q(\eta_1, \eta_2).$$

Therefore, (M, N) does not have the generalized P_q^L-property. On the other hand, we have $\sigma(N, M) = \frac{1}{2}$. Note that there is no pair (ζ, η) in $M \times N$ except for $(1, \frac{1}{2})$ satisfying $\sigma(\eta, \zeta) = \sigma(N, M)$. Hence, (M, N) has the generalized P_q^R-property.

Definition 3.9. *Let $(\mathcal{X}, \sigma)$ be a quasi metric space, M, N be nonempty subsets of $\mathcal{X}$. Let $F : M \to N$ be a mapping and $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Q-function. If there exists $\varphi \in \Omega$ such that*

$$q(F\zeta, F\eta) \leq \varphi\left(q(\zeta, \eta)\right) \tag{3.7}$$

for all $\zeta, \eta \in M$, then F is called best BW-contraction mapping.

Theorem 3.3. *Let $(\mathcal{X},\sigma)$ be a complete quasi metric space, M,N be closed nonempty subsets of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, $F:M\to N$ be a mapping and $M_0^L\neq\emptyset$. Let $q:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ be a Q-function, (M,N) has the generalized P_q^L-Property and $F(M_0^L)\subseteq N_0^R$. If F is a best BW-contraction mapping, then F has a unique left best proximity point ζ^* in M.*

Proof. Let $\zeta_0\in M_0^L$ be an arbitrary point. Since $F\zeta_0\in F(M_0^L)\subseteq N_0^R$, there exists $\zeta_1\in M_0^L$ such that

$$\sigma(\zeta_1,F\zeta_0)=\sigma(M,N). \tag{3.8}$$

Similarly, there exists $\zeta_2\in M_0^L$ such that

$$\sigma(\zeta_2,F\zeta_1)=\sigma(M,N). \tag{3.9}$$

By the similar way, we can construct a sequence $\{\zeta_n\}$ in M such that

$$\sigma(\zeta_{n+1},F\zeta_n)=\sigma(M,N) \tag{3.10}$$

for all $n\in\mathbb{N}$. If there exists $k\in\mathbb{N}$ such that $\zeta_k=\zeta_{k+1}$, then, from (3.10), ζ_k is a left best proximity point. Hence, we assume that $\zeta_n\neq\zeta_{n+1}$ for all $n\in\mathbb{N}$. Since (M,N) has the generalized P_q^L-Property, from (3.10), we get

$$q(\zeta_n,\zeta_{n+1})=q(F\zeta_{n-1},F\zeta_n) \tag{3.11}$$

for all $n\in\mathbb{N}$. From (3.7), we have

$$q(F\zeta_n,F\zeta_{n+1})\leq\varphi(q(\zeta_n,\zeta_{n+1})) \tag{3.12}$$

for all $n\in\mathbb{N}$. Then, from (3.11) and (3.12), we have

$$\begin{aligned} q(\zeta_n,\zeta_{n+1}) &= q\left(F\zeta_{n-1},F\zeta_n\right)\\ &\leq \varphi(q(\zeta_{n-1},\zeta_n)) \end{aligned} \tag{3.13}$$

for all $n\in\mathbb{N}$. As in the proof of Theorem 2.1, we can obtain

$$\lim_{n\to\infty}q(\zeta_n,\zeta_{n+1})=0 \text{ and } \lim_{n\to\infty}q(\zeta_{n+1},\zeta_n)=0.$$

Now, we want to show that for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $q(\zeta_n,\zeta_m)<\varepsilon$ for all $m>n\geq n_0$. Assume the contrary. In this case, there exist $\varepsilon_0>0$ and two sequences $\{n_k\}$ and $\{m_k\}$ with $m_k>n_k>k$ such that

$$q(\zeta_{n_k},\zeta_{m_k})\geq\varepsilon_0 \tag{3.14}$$

for all $k\in\mathbb{N}$ where m_k is the smallest integer satisfying the inequality (3.14) corresponding to n_k. Hence, we have

$$q(\zeta_{n_k},\zeta_{m_k-1})<\varepsilon_0.$$

Then, we get

$$\begin{aligned} \varepsilon_0 &\leq q(\zeta_{n_k},\zeta_{m_k})\\ &\leq q(\zeta_{n_k},\zeta_{m_k-1})+q(\zeta_{m_k-1},\zeta_{m_k})\\ &< \varepsilon_0+q(\zeta_{m_k-1},\zeta_{m_k}). \end{aligned}$$

Since $\lim_{n\to\infty}q(\zeta_n,\zeta_{n+1})=0$, we get

$$q(\zeta_{n_k},\zeta_{m_k})\to\varepsilon_0^+ \text{ as } k\to\infty.$$

Now, we have two cases.
Case 1: Assume that there exists $k_0\in\mathbb{N}$ such that $\zeta_{n_k+1}\neq\zeta_{m_k+1}$ for all $k\geq k_0$. Then, we get

$$\begin{aligned}\varepsilon_0 &\leq q(\zeta_{n_k},\zeta_{m_k})\\ &\leq q(\zeta_{n_k},\zeta_{n_k+1})+q(\zeta_{n_k+1},\zeta_{m_k+1})+q(\zeta_{m_k+1},\zeta_{m_k})\\ &= q(\zeta_{n_k},\zeta_{n_k+1})+q(F\zeta_{n_k},F\zeta_{m_k})+q(\zeta_{m_k+1},\zeta_{m_k})\\ &\leq q(\zeta_{n_k},\zeta_{n_k+1})+\varphi(q(\zeta_{n_k},\zeta_{m_k}))+q(\zeta_{m_k+1},\zeta_{m_k})\end{aligned}$$

for all $k \geq k_0$.

Case 2: Assume the contrary of Case 1. Then, we have $\zeta_{n_k+1}=\zeta_{m_k+1}$ for infinitely many $k\in\mathbb{N}$. In this case, because of $\zeta_{n_k+2}\neq\zeta_{m_k+1}$ and generalized P_q^L-Property, we obtain

$$\begin{aligned}\varepsilon_0 &\leq q(\zeta_{n_k},\zeta_{m_k})\\ &\leq q(\zeta_{n_k},\zeta_{n_k+1})+q(\zeta_{n_k+1},\zeta_{n_k+2})+q(\zeta_{n_k+2},\zeta_{m_k+1})+q(\zeta_{m_k+1},\zeta_{m_k})\\ &= q(\zeta_{n_k},\zeta_{n_k+1})+q(\zeta_{n_k+1},\zeta_{n_k+2})+q(F\zeta_{n_k+1},F\zeta_{m_k})+q(\zeta_{m_k+1},\zeta_{m_k})\\ &\leq q(\zeta_{n_k},\zeta_{n_k+1})+q(\zeta_{n_k+1},\zeta_{n_k+2})+q(F\zeta_{n_k+1},F\zeta_{n_k})\\ &\quad +q(F\zeta_{n_k},F\zeta_{m_k})+q(\zeta_{m_k+1},\zeta_{m_k})\\ &\leq q(\zeta_{n_k},\zeta_{n_k+1})+q(\zeta_{n_k+1},\zeta_{n_k+2})+q(\zeta_{n_k+2},\zeta_{n_k+1})\\ &\quad +\varphi(q(\zeta_{n_k},\zeta_{m_k}))+q(\zeta_{m_k+1},\zeta_{m_k}).\end{aligned}$$

for all $k\in\mathbb{N}$. In both cases, we have

$$\varepsilon_0 \leq \lim_{k\to\infty}\sup\varphi(q(\zeta_{n_k},\zeta_{m_k})) < \varepsilon_0, \tag{3.15}$$

which is a contradiction. Let $\varepsilon>0$. From (Q_3), there exists $\delta>0$ such that $q(\zeta,\eta)\leq\delta$ and $q(\zeta,\xi)\leq\delta$ implies $\sigma^s(\eta,\xi)\leq\varepsilon$. Hence, there exists $n_0\in\mathbb{N}$ such that $q(\zeta_n,\zeta_m)<\delta$ for all $m>n\geq n_0$. Then, since

$$q(\zeta_{n_0},\zeta_n)<\delta \text{ and } q(\zeta_{n_0},\zeta_m)<\delta,$$

from Lemma 1.1, we have $\sigma^s(\zeta_n,\zeta_m)\leq\varepsilon$ and so $\{\zeta_n\}$ is a σ^s-Cauchy sequence in M. Since $(\mathcal{X},\sigma)$ is a complete quasi metric space and M is a closed subset of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, there exists $\zeta^*\in M$ such that

$$\sigma(\zeta_n,\zeta^*)\to 0 \text{ as } n\to\infty.$$

Let $\varepsilon>0$. Then, there exists $n_0\in\mathbb{N}$ such that $q(\zeta_n,\zeta_m)<\varepsilon$ for all $m>n\geq n_0$. Now, fix $n\geq n_0$. Using (Q_2), we have $q(\zeta_n,\zeta^*)\leq\varepsilon$ for all $n\geq n_0$. Hence, $q(\zeta_n,\zeta^*)\to 0$ as $n\to\infty$. Since $q(\zeta_n,\zeta_m)\to 0$ and $q(\zeta_n,\zeta^*)\to 0$ as $n,m\to\infty$, we also have $\sigma^s(\zeta_n,\zeta^*)\to 0$ as $n\to\infty$. From (3.11), since $\{F\zeta_n\}$ is a σ^s-Cauchy sequence in N and N is a closed subset of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, there exists $\eta^*\in N$ such that

$$\sigma(F\zeta_n,\eta^*)\to 0 \text{ as } n\to\infty.$$

Similarly, it can be easily shown that $q(F\zeta_n,\eta^*)\to 0$. Also, we have

$$\begin{aligned}\sigma(M,N) &\leq \sigma(\zeta^*,\eta^*)\\ &= \sigma(\zeta^*,\zeta_{n+1})+\sigma(\zeta_{n+1},F\zeta_n)+\sigma(F\zeta_n,\eta^*)\\ &\leq \sigma^s(\zeta_{n+1},\zeta^*)+\sigma(M,N)+\sigma(F\zeta_n,\eta^*)\end{aligned}$$

Taking the limit $n\to\infty$ in last inequality, we get

$$\sigma(\zeta^*,\eta^*)=\sigma(M,N). \tag{3.16}$$

Then, because of $q(\zeta_n,\zeta^*)\to 0$ as $n\to\infty$ and the inequality (3.7), we have $q(F\zeta_n,F\zeta^*)\to 0$ as $n\to\infty$. Hence, using Lemma 1.2 (i), we obtain $\sigma^s(\eta^*,F\zeta^*)=0$, and so $\eta^*=F\zeta^*$. Therefore, from (3.16), ζ^* is a left best proximity point of F in M. Now, assume ζ^* and

ζ^{**} are different two left best proximity points of F in M. Then, from the generalized P_q^L-property we have

$$q(\zeta^*,\zeta^{**}) = q(F\zeta^*, F\zeta^{**}),$$

and then, from (3.7), we get

$$q(\zeta^*,\zeta^{**}) = q(F\zeta^*, F\zeta^{**}) \leq \varphi(q(\zeta^*,\zeta^{**})).$$

Therefore, we have $q(\zeta^*,\zeta^{**}) = 0$ (otherwise we have a contradiction). Similarly we get $q(\zeta^{**},\zeta^*) = 0$. Hence, we have

$$q(\zeta^*,\zeta^*) \leq q(\zeta^*,\zeta^{**}) + q(\zeta^{**},\zeta^*) = 0,$$

and so from Lemma 1.1, we have $\zeta^* = \zeta^{**}$. This is a contradiction. □

We will now provide two examples to illustrate the effectiveness of our asymmetric approach.

Example 3.4. *Let $\mathcal{X} = [0,\infty)$, $\sigma : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ and $q : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ be mappings defined by*

$$\sigma(\zeta,\eta) = \begin{cases} 0 & , \quad \zeta = \eta \\ \frac{\zeta}{2}+\eta & , \quad \zeta \neq \eta \end{cases}$$

and

$$q(\zeta,\eta) = \begin{cases} \frac{\zeta}{2} & , \quad \zeta = \eta \\ \frac{\zeta}{2}+\eta & , \quad \zeta \neq \eta \end{cases}$$

for all $\zeta,\eta \in \mathcal{X}$, respectively. Then, $(\mathcal{X},\sigma)$ is a quasi metric space and q is a Q-function. Also $(\mathcal{X},\sigma)$ is complete. Indeed, let $\{\zeta_n\}$ be a σ^s-Cauchy sequence in $\mathcal{X}$, then $\{\zeta_n\}$ converges to 0 in the usual sense or it is eventually constant. On the other hand, since

$$B_{\sigma^{-1}}\left(\zeta,\frac{\zeta}{2}\right) = \{\zeta\} \text{ for all } \zeta > 0$$

and

$$B_{\sigma^{-1}}\left(0,\frac{\varepsilon}{2}\right) = [0,\varepsilon),$$

then every σ^s-Cauchy sequence in $\mathcal{X}$ is σ^{-1}-convergent. Consider the sets $M = [e^2,10)$, $N = [2,3]$ are closed subsets of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$. Thus, we have $\sigma(M,N) = \frac{e^2+2}{2}$. Define a mapping $F : M \to N$ as $F\zeta = \ln\zeta$ for all $\zeta \in M$. In this case, $M_0^L = \{e^2\}$, $N_0^R = \{2\}$, and so $F(M_0^L) = N_0^R$. Also, (M,N) has the generalized P_q^L-Property. Now, consider a mapping $\varphi : [0,\infty) \to [0,\infty)$ as follows.

$$\varphi(t) = \begin{cases} \frac{14t}{5e^2} & , \quad t < e^2 \\ \frac{7}{5}\ln t & , \quad t \geq e^2 \end{cases}$$

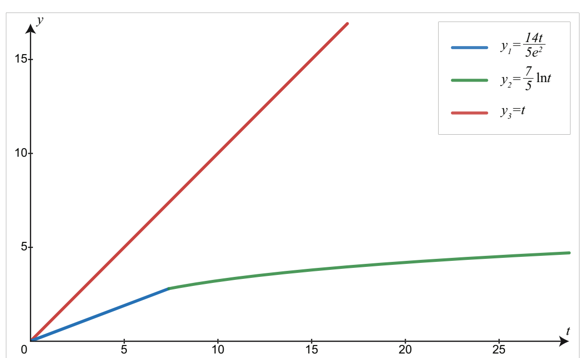

Figure 1. Figures of $\varphi(t)$ and t

Then, it can be easily seen the function $\varphi \in \Omega$ from Figure 1. We will show that the inequality (3.7) holds. We have following cases:
Case 1: Assume that $\zeta = \eta$. Then, we have $F\zeta = F\eta$, and so

$$q(F\zeta, F\eta) = \frac{F\zeta}{2} = \frac{\ln\zeta}{2} \leq \frac{7\zeta}{10e^2} = \varphi(q(\zeta, \eta)).$$

Case 2: Assume that $\zeta \neq \eta$. If $F\zeta = F\eta$, from Case 1 it is obvious. Now suppose $F\zeta \neq F\eta$. In this case, we have

$$\begin{aligned} q(F\zeta, F\eta) &= F\zeta + F\eta \\ &= \ln\zeta + \ln\eta \\ &= \ln(\zeta\eta). \end{aligned} \tag{3.17}$$

On the other hand, it is well known that $\sqrt{\sqrt{\zeta}\eta} \leq \frac{\sqrt{\zeta}+\eta}{2}$ for all $\zeta, \eta \geq 0$. Then, we have

$$(\sqrt{\zeta}\eta)^5 \leq \frac{(\sqrt{\zeta}+\eta)^{10}}{2^{10}} = \frac{(\sqrt{\zeta}+\eta)^3}{2^{10}}(\sqrt{\zeta}+\eta)^7.$$

Since $\frac{(\sqrt{\zeta}+\eta)^3}{2^{10}} \leq 1$ for all $\zeta, \eta \in M$, we get

$$(\sqrt{\zeta}\eta)^5 \leq (\sqrt{\zeta}+\eta)^7$$

for all $\zeta, \eta \in M$, and so we have, from (3.17),

$$q(F\zeta, F\eta) = \ln\left(\sqrt{\zeta}\eta\right) \leq \frac{7}{5}\ln\left(\sqrt{\zeta}+\eta\right) \leq \frac{7}{5}\ln\left(\frac{\zeta}{2}+\eta\right) = \varphi(\zeta+\eta) = \varphi(q(\zeta, \eta)).$$

Hence, all hypotheses of Theorem 3.3 are satisfied. Therefore, F has a unique left best proximity point in M.

Example 3.5. *Let $\mathcal{X}, \sigma, q$ be as in Example 3.4. Consider the sets $M = [4, 5)$, $N = [\frac{4}{5}, 2]$ are closed subsets of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$. Thus, we have $\sigma(M, N) = \frac{14}{5}$. Define two mappings $F : M \to N$ as $F\zeta = \frac{\zeta}{1+\zeta}$ for all $\zeta \in M$ and $\varphi : [0, \infty) \to [0, \infty)$ as $\varphi(t) = \frac{t^2}{1+t}$. In this case $M_0^L = \{4\}$ and $N_0^R = \{\frac{4}{5}\}$, and so $F(M_0^L) = N_0^R$. Also, the pair (M, N) has the generalized P_q^L-Property. We will show that the inequality (3.7) holds. We have following cases:*
Case 1: Assume that $\zeta = \eta$. Then, we have $F\zeta = F\eta$, and so

$$\begin{aligned} q(F\zeta, F\eta) &= \frac{F\zeta}{2} \\ &= \frac{\zeta}{2(1+\zeta)} \\ &\leq \frac{\zeta^2}{2(2+\zeta)} \\ &= \varphi(q(\zeta, \eta)). \end{aligned} \tag{3.18}$$

Case 2: Assume that $\zeta \neq \eta$. If $F\zeta = F\eta$, from Case 1 it is obvious. Now suppose $F\zeta \neq F\eta$. Then, we have

$$\begin{aligned} q(F\zeta, F\eta) &= \frac{\zeta}{2(1+\zeta)} + \frac{\eta}{1+\eta} \\ &\leq \frac{\zeta^2 + 4\eta^2 + 4\zeta\eta}{4 + 2\zeta + 4\eta} \\ &= \varphi(q(\zeta, \eta)). \end{aligned}$$

Hence, all hypotheses of Theorem 3.3 are satisfied. Therefore, F has a unique left best proximity point in M.

Similarly, we can obtain the following theorem.

Theorem 3.4. *Let $(\mathcal{X}, \sigma)$ be a complete quasi metric space, M, N be closed nonempty subsets of $\mathcal{X}$ with respect to $\tau_{\sigma^{-1}}$, $F : M \to N$ be a mapping and $M_0^R \neq \emptyset$. Let $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a Q-function, (M, N) has the generalized P_q^R-Property and $F(M_0^R) \subseteq N_0^L$. If F is a best BW-contraction mapping, then F has a unique right best proximity point ζ^* in M.*

4. Some consequences and applications

In this section, taking into account the relation between a quasi metric and a partial metric we provide some corollaries in partial metric spaces. Finally, we present an existence and uniqueness theorem for nonlinear Volterra integral equations.

In 1992, the concept of partial metric was introduced by Matthews [18] as follows:

Definition 4.10. *Let $\mathcal{X}$ be a nonempty set. Then, the mapping $p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is said to be a partial metric if the following conditions hold:*

p1) $p(\zeta, \zeta) = p(\zeta, \eta) = p(\eta, \eta)$ *if and only if* $\zeta = \eta$,
p2) $p(\zeta, \zeta) \leq p(\zeta, \eta)$,
p3) $p(\zeta, \eta) = p(\eta, \zeta)$,
p4) $p(\zeta, \xi) \leq p(\zeta, \eta) + p(\eta, \xi) - p(\eta, \eta)$

for all $\zeta, \eta, \xi \in \mathcal{X}$. The pair $(\mathcal{X}, p)$ is called partial metric space.

Each partial metric p on $\mathcal{X}$ induces a T_0 topology τ_p on $\mathcal{X}$ which has as a base the family of open balls $\{B_p(\zeta, \varepsilon) : \zeta \in \mathcal{X}, \varepsilon > 0\}$ where

$$B_p(\zeta, \varepsilon) = \{\eta \in \mathcal{X} : p(\zeta, \eta) < p(\zeta, \zeta) + \varepsilon\}.$$

Recall that a sequence $\{\zeta_n\}$ in $\mathcal{X}$ is called Cauchy sequence if the limit $\lim_{n,m\to\infty} p(\zeta_n, \zeta_m)$ exists and is finite. Also, a partial metric space $(\mathcal{X}, p)$ is complete if every Cauchy sequence in $\mathcal{X}$ is τ_p-convergent to $\zeta \in \mathcal{X}$ such that

$$\lim_{n,m\to\infty} p(\zeta_n, \zeta_m) = p(\zeta, \zeta).$$

Also, Matthews [18] showed that partial metric spaces are equivalent to a subclass of quasi metric spaces named as weightable quasi metric spaces.

Definition 4.11 ([18]). *Let $(\mathcal{X}, \sigma)$ be a quasi metric space. Then, $(\mathcal{X}, \sigma)$ is said to be a weightable if there exists a function $w : \mathcal{X} \to [0, \infty)$ such that*

$$\sigma(\zeta, \eta) + w(\zeta) = \sigma(\eta, \zeta) + w(\eta)$$

for all $\zeta, \eta \in \mathcal{X}$. In this case, w is called a weighting function.

The following example shows that there is a weightable T_1-quasi metric function but it is not ordinary metric.

Example 4.6. *Let $\mathcal{X} = [1, \infty)$ and $\sigma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a mapping defined by*

$$\sigma(\zeta, \eta) = \begin{cases} 0 & , \quad \zeta = \eta \\ \eta & , \quad \zeta \neq \eta \end{cases}$$

Then, σ is complete weightable quasi metric with the weighting function $w(\zeta) = \zeta$. Moreover, although it is a weightable T_1-quasi metric, it is not an ordinary metric. On the other hand, we get the induced partial metric p_σ as follows:

$$p_\sigma(\zeta, \eta) = \begin{cases} \zeta & , \quad \zeta = \eta \\ \zeta + \eta & , \quad \zeta \neq \eta \end{cases}.$$

It is obvious that $(\mathcal{X}, p_\sigma)$ *is complete.*

Now, we can present the equivalence of partial metric spaces and weightable quasi metric spaces given by Matthews:

Theorem 4.5 ([18])**.** *i) Let* $(\mathcal{X}, \sigma)$ *be a weightable quasi metric space with weighting function* w*. Then, the function* $p_\sigma : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *defined by* $p_\sigma(\zeta, \eta) = \sigma(\zeta, \eta) + w(\zeta)$ *for all* $\zeta, \eta \in \mathcal{X}$ *is a partial metric on* $\mathcal{X}$*. Furthermore,* $\tau_\sigma = \tau_{p_\sigma}$*.*

ii) Let $(\mathcal{X}, p)$ *be a partial metric space. Then, the function* $\sigma_p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *defined by* $\sigma_p(\zeta, \eta) = p(\zeta, \eta) - p(\zeta, \zeta)$ *for all* $\zeta, \eta \in \mathcal{X}$ *is a weightable quasi metric on* $\mathcal{X}$ *with weighting function* $w(\zeta) = p(\zeta, \zeta)$ *for all* $\zeta \in \mathcal{X}$*. Furthermore,* $\tau_{\sigma_p} = \tau_p$*.*

The following lemma which is stated and proved in [16] as Proposition 2.10 is important to find some nice examples of Q-functions on weightable quasi metric spaces.

Lemma 4.3. *Let* $(\mathcal{X}, \sigma)$ *be a weightable quasi metric space. Then, the induced partial metric* p_σ *is a* Q*-function on* $\mathcal{X}$*.*

Remark 4.2. *Note that, when Theorem 4.5 and Lemma 4.3 are considered together, it can be seen that every partial metric* p *on* $\mathcal{X}$ *is a* Q*-function on the quasi metric space* $(\mathcal{X}, \sigma_p)$*.*

Hence, considering Theorem 2.1 and Theorem 3.3 together with Remark 4.2 we can present the following results:

Corollary 4.1. *Let* $(\mathcal{X}, p)$ *be a complete partial metric space,* σ_p *be the induced weightable quasi metric,* M, N *be nonempty subsets of* $\mathcal{X}$ *where* M *is a closed subset with respect to* $\tau_{\sigma_p^{-1}}$*,* $F : M \to N$ *be a mapping and* $M_0^L \neq \emptyset$*. Let* N *be a* σ_p^{-1}*-approximately compact subset with respect to* M *and* $F(M_0^L) \subseteq N_0^R$*. If* F *is a left proximal* BW*-contraction mapping with respect to* p*, that is,* F *satisfies*

$$\left.\begin{array}{l}\sigma_p(\eta_1, F\zeta_1) = \sigma_p(M, N)\\ \sigma_p(\eta_2, F\zeta_2) = \sigma_p(M, N)\end{array}\right\} \Longrightarrow p(\eta_1, \eta_2) \leq \varphi(p(\zeta_1, \zeta_2))$$

for all $\eta_1, \eta_2, \zeta_1, \zeta_2 \in M$*, then* F *has a unique left best proximity point* ζ^* *in* M*. Moreover* $p(\zeta^*, \zeta^*) = 0$*.*

Corollary 4.2. *Let* $(\mathcal{X}, p)$ *be a complete partial metric space,* σ_p *be the induced weightable quasi metric,* M, N *be closed nonempty subsets of* $\mathcal{X}$ *with respect to* $\tau_{\sigma_p^{-1}}$*,* $F : M \to N$ *be a mapping and* $M_0^L \neq \emptyset$*. Let the pair* (M, N) *has the* P_p^L*-Property and* $F(M_0^L) \subseteq N_0^R$*. If* F *is best* BW*-contraction mapping with respect to* p*, that is,* F *satisfies*

$$p(F\zeta, F\eta) \leq \varphi\left(p(\zeta, \eta)\right)$$

for all $\zeta, \eta \in M$*, then* F *has a unique left best proximity point* ζ^* *in* M*. Moreover* $p(\zeta^*, \zeta^*) = 0$*.*

Finally, we present an existence and uniqueness theorem for nonlinear Volterra integral equations of the form

$$\eta(t) = \int_0^t K(t, s, \eta(s))ds, \tag{4.19}$$

where $K : [0, 1]^2 \times [0, \infty) \to [0, \infty)$ is a continuous function. Taking into account Theorem 3.3, we give an existence and uniqueness result for solution of equation (4.19). We will consider the space $\mathcal{X}$ as the positive cone of $C[0, 1]$, that is,

$$\mathcal{X} = \{\eta \in C[0, 1] : \eta(t) \geq 0 \text{ for all } t \in [0, 1]\}.$$

Theorem 4.6. *Suppose the following conditions hold:*
(i) there exist a BW-function $\varphi : [0,\infty) \to [0,\infty)$ *satisfying*

$$\sup_{t\in[0,1]} \varphi(w(t)) \leq \varphi(\sup_{t\in[0,1]} w(t))$$

for all $w \in \mathcal{X}$ *and a continuous function* $p : [0,1]^2 \to [0,\infty)$ *such that*

$$K(t,s,\eta) + K(t,s,\nu) \leq p(t,s)\varphi\left(\frac{\eta}{2} + \nu\right)$$

for all $t,s \in [0,1]$ *and* $\eta, \nu \in [0,\infty)$.
(ii) $\sup_{t\in[0,1]} \int_0^t p(t,s)ds \leq 1$.
Then, the integral equation (4.19) has a unique positive solution.

Proof. Define a quasi metric on $\mathcal{X}$ as

$$\sigma(\eta,\nu) = \begin{cases} 0 & , \quad \eta = \nu \\ \sup_{t\in[0,1]}\left\{\frac{\eta(t)}{2} + \nu(t)\right\} & , \quad \eta \neq \nu \end{cases}$$

and a Q-function as

$$q(\eta,\nu) = \begin{cases} \sup_{t\in[0,1]}\left\{\frac{\eta(t)}{2}\right\} & , \quad \eta = \nu \\ \sup_{t\in[0,1]}\left\{\frac{\eta(t)}{2} + \nu(t)\right\} & , \quad \eta \neq \nu \end{cases}.$$

In this case, $(\mathcal{X},\sigma)$ is a complete quasi metric space. Consider an operator $F : \mathcal{X} \to \mathcal{X}$ by

$$F\eta(t) = \int_0^t K(t,s,\eta(s))ds.$$

Now, for all $t \in [0,1]$ and $\eta,\nu \in \mathcal{X}$, we have

$$\begin{aligned} F\eta(t) + F\nu(t) &= \int_0^t K(t,s,\eta(s))ds + \int_0^t K(t,s,\nu(s))ds \\ &= \int_0^t [K(t,s,\eta(s)) + K(t,s,\nu(s))]ds \\ &\leq \int_0^t p(t,s)\varphi(\eta(s)+\nu(s))ds \\ &\leq \int_0^t p(t,s) \sup_{s\in[0,1]} \varphi\left(\frac{\eta(s)}{2} + \nu(s)\right) ds \\ &\leq \int_0^t p(t,s)\varphi\left(\sup_{s\in[0,1]}\left\{\frac{\eta(s)}{2} + \nu(s)\right\}\right) ds \\ &\leq \varphi\left(\sup_{s\in[0,1]}\left\{\frac{\eta(s)}{2} + \nu(s)\right\}\right)\int_0^t p(t,s)ds \\ &= \varphi(q(\eta,\nu))\int_0^t p(t,s)ds \end{aligned}$$

hence, from (ii), we have

$$q(F\eta, F\nu) \leq \varphi(q(\eta,\nu)).$$

Then, by Theorem 3.3 F has a unique left best proximity point. Therefore, the equation (4.19) has a unique positive solution because of the definition of the quasi metric σ. □

Example 4.7. *The following nonlinear Volterra integral equation*

$$\eta(t) = \frac{\pi}{2}\int_0^t \frac{t\sin(\pi ts)\,|\eta(s)|}{1+|\eta(s)|}ds \tag{4.20}$$

has a unique positive solution in $C\,[0,1]$.

Proof. Define a function $K : [0,1]^2 \times [0,\infty) \to [0,\infty)$ by

$$K(t,s,\eta) = \frac{\pi t\sin(\pi ts)\eta}{4(1+\eta)},$$

then it satisfies the condition (i) of Theorem 4.6 with $p(t,s) = \dfrac{\pi t\sin(\pi ts)}{2}$ and $\varphi(t) = \dfrac{t^2}{1+t}$. Also, the condition (ii) holds, since

$$\sup_{t\in[0,1]}\int_0^t p(t,s)ds = \sup_{t\in[0,1]}\int_0^t \frac{\pi t\sin(\pi ts)}{2}ds = 1.$$

Hence from Theorem 4.6, the equation (4.20) has a unique positive solution. □

5. Conclusions

In this paper, we first modify the fundamental concepts and notations in the best proximity point theory by taking into account unsymmetrical condition of quasi metric spaces. Then, we introduce new concepts so called proximal BW-contraction and best BW-contraction mappings. Thus, we obtain some Boyd-Wong type best proximity point results in the setting of quasi metric spaces via Q-functions. Next, we provide some corollaries and consequences to partial metric spaces of our main results. Finally, we present an existence and uniqueness result for nonlinear Volterra integral equations.

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