

Duality of Natural and Euclidean Gradients on the Multinomial Statistical Manifold

IOANA RĂDULESCU (LĂZĂRESCU)

ABSTRACT. This paper explores the duality of the Natural Gradient and Euclidean Gradient in the statistical manifold of multinomial distributions by examining the relationship between these gradients in dual coordinate spaces. We derive the canonical exponential form of the multinomial distribution and compute the Fisher metric using a change of basis method. The duality between the Natural and Euclidean Gradients in these spaces is demonstrated through both computational derivations and experimental validation. In a small-scale experiment, we compare the convergence rates of these gradients, confirming their duality and highlighting the practical advantages of the Natural Gradient in optimization using gradient descent methods.

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¹ INTERDISCIPLINARY DOCTORAL SCHOOL, TRANSILVANIA UNIVERSITY OF BRAȘOV, BD. EROILOR 29, BRAȘOV, 500036, ROMANIA,

Email address: ioana.radulescu@unitbv.ro