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A Study of a Plate Equation with Nonlocal Weak Damping and Logarithmic Sources: Existence and Decay

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ABSTRACT. In this paper, we investigate a plate equation incorporating a nonlocal damping term of the form $\|u_t\|_2^q u_t$ and a logarithmic source term $u \ln |u|$. Employing the Galerkin method, we rigorously establish the existence of solutions for the proposed problem. Furthermore, we derive an explicit and generalized decay rate result by utilizing the multiplier method alongside key properties of convex functions, providing a comprehensive analysis of the system asymptotic behavior.

1. INTRODUCTION

The study of nonlinear plate equations has received significant attention due to their broad applications in engineering, physics, and mathematical modeling of elastic structures. In particular, understanding the dynamic behavior of plates subject to various nonlinear effects is crucial in ensuring stability and long-term performance in practical applications, such as aerospace engineering, mechanical vibrations, and material science. One of the key challenges in this field is analyzing the influence of nonlinear source terms and damping mechanisms on the qualitative properties of solutions.

Among the various types of nonlinearities, the logarithmic source term has been of increasing interest due to its connection to mathematical models arising in elasticity, viscoelasticity, and thermomechanics. The logarithmic function is known for its slow growth, making it fundamentally different from polynomial-type nonlinearities, yet still capable of inducing complex dynamical behaviors such as finite-time blow-up or global existence under suitable conditions. The presence of such a term introduces significant mathematical difficulties, particularly in proving stability results and deriving precise energy estimates.

To ensure the stability of solutions to plate equations with logarithmic nonlinearities, incorporating appropriate damping mechanisms is essential. A widely studied and effective class of damping is the nonlocal damping of the form $\|u_t\|_2^q u_t$. This type of damping has shown to exhibit strong dissipative effects by controlling the global energy of the system and influencing the long-time behavior of solutions. Unlike local damping terms, which act at every point independently, nonlocal damping takes into account the overall energy distribution, making it particularly effective in preventing blow-up and ensuring uniform decay rates.

1.1. **Problems with nonlocal damping.** Recently, evolution equations with nonlocal dissipative effects have been widely studied by numerous researchers in various contexts. One notable example is the introduction of a nonlocal damping term of the form $M(\|\nabla u\|_2^2)u_t$, which was first proposed by Lange and Perla Menzala [1] for the beam equation:

$$u_{tt} + \Delta^2 u + M(\|\nabla u\|_2^2)u_t = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+.$$

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Cavalcanti et al. [2] extended the study of this equation by incorporating a memory term $-(g * \Delta u)(t)$. Using the Galerkin method and the multiplier technique, they established results on global existence and energy decay. Jorge Silva and Narciso [3] examined the following model:

$$u_{tt} + \Delta^2 u + M(\|\nabla u\|_2^2)u_t + f(u) = h(x),$$

within a bounded domain Ω, and established the existence of a global attractor with finite dimension. When considering a nonlocal damping term of the form $M(\|\nabla u\|_2^2)g(u_t)$, involving the product of two nonlinear terms, Narciso et al. [4, 5] analyzed the long-term dynamics of related models. Further studies on damping terms of the form $M(\|\nabla u\|_2^2)(-\Delta)^\theta u_t$ can be found in [6, 7, 8].

All the aforementioned damping terms involved a Kirchhoff-type coefficient $M(\|\nabla u\|_2^2)$. However, there are also studies on damped wave-type equations where the nonlocal damping coefficient depends on u_t. In a pioneering work, Balakrishnan and Taylor [9] introduced beam equations with nonlocal energy damping of the form $(\|\Delta u\|_2^2 + \|u_t\|_2^2)^q \Delta u_t$ to model damping phenomena in flight structures. More recently, Jorge Silva et al. [10] considered the following beam equation:

$$u_{tt} - \kappa\Delta u + \Delta^2 u - \gamma \left[\int_\Omega (|\Delta u|^2 + |u_t|^2)\, dx\right]^q \Delta u_t + f(u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

for $q \geq 1$, where they proved global existence, uniqueness, and polynomial decay stability. For further insights into beam equations with nonlocal energy damping and which discuss long-term behavior, we refer to [11, 12, 13].

Shifting our focus to evolution equations with nonlocal weak damping of the form $\|u_t\|_2^q u_t$, Zhao et al. [14, 15, 16] conducted a series of studies on the existence of a global attractor under suitable conditions. More specifically, Zhao et al. [14, 16] investigated the initial-boundary-value problem for the wave equation with nonlocal damping and anti-damping:

$$u_{tt} - \Delta u + k\|u_t\|_2^q u_t + f(u) = \int_\Omega K(x,y)u_t(y)\, dy + h(x),$$

under both subcritical and critical nonlinearities. Zhao et al. [15] analyzed the long-term behavior of the following extensible beam equation incorporating nonlocal damping:

$$u_{tt} + \Delta^2 u - m(\|\nabla u\|_2^2)\Delta u + k\|u_t\|_2^q u_t + f(u) = h(x).$$

Peng and Zhang [17] established the existence of a global attractor for a coupled wave-plate system featuring nonlocal weak damping and nonlocal anti-damping on Riemannian manifolds. Their results were obtained using the semi-group and multiplier methods. For further discussion on nonlocal weak damping (also referred to as averaged damping), we refer to [18] and the references therein.

Most of the aforementioned studies focused on equations where $f(u)$ acts as a damping term, ensuring that the system retains a positive energy. However, when $f(u)$ serves as a source term, making the system energy non-positive, the behavior becomes significantly different. In such cases, solutions may not exist globally, and finite-time blow up may occur. To address this phenomenon, we recall the potential well method which has been extensively used to analyze such scenarios. There exist numerous results concerning the existence and non-existence of solutions for evolution equations with damping terms (e.g., $|u_t|^{m-2}u_t$ for $m \geq 2$, $-\Delta u_t$, among others), but we omit specific references here for brevity.

Regarding evolution equations that feature both nonlocal damping (particularly nonlocal weak damping) and a source term, the possibility of global nonexistence has been partially explored. Zhang et al. [19] investigated the initial-boundary-value problem for a wave equation with degenerate nonlocal damping and source terms:

$$u_{tt} - \Delta u + \|\nabla u\|_2^l |u_t|^{m-2} u_t = |u|^{p-2}u.$$

Using the potential well theory, they established conditions for asymptotic stability and blow-up of solutions. Liu et al. [20] examined the initial-boundary-value problem for the following beam equation with energy damping:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2)\Delta u - (\|\Delta u\|_2^2 + \|u_t\|_2^2)^q \Delta u_t = |u|^{p-2}u.$$

By combining the potential well theory with Nakao's inequality, they established results on global existence and energy decay rates for solutions initiated within the stable set. Furthermore, Hu et al. [21] considered the initial-boundary-value problem for the wave equation with nonlocal damping

$$u_{tt} - \Delta u + \|u_t\|_2^m u_t = \|u\|_p^r u^{p-2} u$$

and provided sufficient conditions for finite-time blow up of weak solutions under appropriate initial data constraints.
Recently, Liu et al. [22] studied the following boundary-value-problem for the wave equation with a nonlocal damping term and a nonlinear source term

$$\begin{cases} u_{tt} - \Delta u + \|u_t\|_2^q u_t = |u|^{p-2}u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases}$$

They conducted a comprehensive investigation of the dynamic behavior of the solutions under three different energy levels. Specifically, they established results on global existence, energy decay estimates, and blow up of solutions at both subcritical ($E(0) < d$) and critical ($E(0) = d$) energy levels. Furthermore, they demonstrated the finite-time blow up of the solution for initial data at arbitrarily high energy levels, along with estimates for the lower and upper bounds of the blow-up time.

1.2. **Problems with logarithmic nonlinearity.** Logarithmic nonlinearity plays a fundamental role in various branches of physics, including nuclear physics, optics, and geophysics [23, 24, 25]. This type of nonlinearity naturally emerges in diverse physical contexts, such as inflationary cosmology, supersymmetric field theories, quantum mechanics, and nuclear physics [26, 27]. Due to its significant applications, mathematical investigations of equations involving logarithmic nonlinearities have attracted considerable attention.

A pioneering contribution to this field was made by Birula and Mycielski [24, 28], who studied the following problem:

$$\begin{cases} u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 = 0, & (x,t) \in [a,b] \times (0,T), \\ u(a,t) = u(b,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [a,b]. \end{cases} \tag{1.1}$$

This equation provides a relativistic formulation of logarithmic quantum mechanics and can also be derived as the limiting case $p \to 1$ in the p-adic string equation [29].

Further contributions in this domain include the work of Cazenave and Haraux [30], who investigated the Cauchy problem associated with

$$u_{tt} - \Delta u = u \ln |u|^k, \quad \text{in } \mathbb{R}^3 \tag{1.2}$$

and established the existence and uniqueness of solutions, providing key insights into the well-posedness of logarithmic wave equations.

A notable advancement in the global existence theory was made by Gorka [25], who applied compactness arguments to establish the global existence of weak solutions for the one-dimensional case of equation (1.2), with initial conditions $(u_0, u_1) \in H_0^1 \times L^2$. Later, Bartkowski and Gorka [23] extended this analysis to the one-dimensional Cauchy problem, proving the existence of both classical and weak solutions.

Another important study in this field was conducted by Hiramatsu et al. [31], who introduced the equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|, \tag{1.3}$$

to explore the dynamics of Q-balls in theoretical physics. While their work primarily relied on numerical simulations, it lacked a rigorous theoretical foundation.

Han [32] later addressed this gap by proving the global existence of weak solutions for the initial-boundary-value problem associated with equation (1.3) in $\mathbb{R}^3$, for initial data $(u_0, u_1) \in H_0^1 \times L^2$. Al-Gharabli and Messaoudi [33] investigated a logarithmically nonlinear plate equation incorporating a nonlinear frictional damping

$$\begin{cases} u_{tt} + \Delta^2 u + u + h(u_t) = ku \ln |u|, & x \in \Omega, \quad t > 0, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & t > 0. \end{cases} \tag{1.4}$$

Using the Galerkin method, they established the existence of solutions. Furthermore, by employing the multiplier method and properties of convex functions, they derived an explicit and general decay rate, providing a deeper understanding of the long-term behavior of solutions.

This growing body of research highlights the significance of logarithmic nonlinearities in mathematical physics, particularly in wave propagation, quantum mechanics, and nonlinear elasticity. The continued exploration of such models is essential for uncovering new stability properties, decay estimates, and solution structures in nonlinear partial differential equations.

Inspired by the aforementioned studies, we examine the following plate equation incorporating nonlocal damping and a logarithmic source term

$$\begin{cases} u_{tt} + \Delta^2 u + u + \|u_t\|_2^q u_t = ku \ln |u|, & x \in \Omega, \quad t > 0 \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad t > 0, \end{cases} \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$ and k is a positive real number. Our goal is to establish precise energy decay estimates and analyze how the interplay between these nonlinear effects influences the overall system dynamics. The results obtained contribute to the broader understanding of dissipative mechanisms in elastic structures and provide valuable insights into the long-term behavior of solutions in mathematical physics and engineering applications.

2. Preliminaries

In this section, we introduce essential mathematical tools and preliminary results that will be used in the proofs of our main theorems. We consider the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^2(\Omega)$, equipped with their usual norms and inner products. Throughout the paper, we use c to denote a generic positive constant.

To ensure the well-posedness and stability of our problem, we impose the following assumption:

(A) The constant k in equation (1.5) satisfies $0 < k < k_0$, where k_0 is uniquely determined by the condition:

$$\frac{2\pi}{k_0} = e^{-3-\frac{2}{k_0}}. \tag{2.6}$$

Remark 2.1. *The function*

$$f(s) = \sqrt{\frac{2\pi}{s}} - e^{\left(-\frac{3}{2}-\frac{1}{s}\right)}$$

is strictly decreasing on $(0,\infty)$*, with*

$$\lim_{s\to 0^+} f(s) = +\infty \quad \text{and} \quad \lim_{s\to+\infty} f(s) = -e^{-\frac{3}{2}}.$$

Thus, there exists a unique k_0 *such that* $f(k_0) = 0$.

The energy functional associated with problem (1.5) is given by

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|u\|_2^2 - \int_\Omega u^2 \ln |u|^k dx\right) + \frac{k}{4}\|u\|_2^2. \tag{2.7}$$

Differentiating (2.7) with respect to t and using (1.5), we obtain

$$E'(t) = -\|u_t\|_2^{q+2} \le 0. \tag{2.8}$$

This implies that the energy is non-increasing over time, which plays a crucial role in establishing the stability results.
We recall two fundamental inequalities which will be essential in controlling the logarithmic nonlinearity and deriving the stability estimates for the solutions.

Lemma 2.1. [34, 35] *(Logarithmic Sobolev inequality) Let* u *be any function in* $H_0^2(\Omega)$ *and let* $a > 0$ *be an arbitrary constant. Then the following inequality holds:*

$$\int_\Omega u^2 \ln |u| dx \le \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi}\|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.9}$$

Lemma 2.2. [36] *(Logarithmic Gronwall inequality) Let* $c > 0$, $\gamma \in L^1(0,T;\mathbb{R}^+)$*, and suppose that the function* $w : [0,T] \to [1,\infty)$ *satisfies*

$$w(t) \le c\left(1 + \int_0^t \gamma(s)w(s) \ln w(s) ds\right), \quad 0 \le t \le T. \tag{2.10}$$

Then, we have the following bound

$$w(t) \le c \exp\left(c\int_0^t \gamma(s) ds\right), \quad 0 \le t \le T. \tag{2.11}$$

3. WELL-POSEDNESS

In this section, we state and prove an existence result for problem (1.5).

Definition 3.1. *A function*

$$u \in L^\infty([0,T], H_0^2(\Omega)) \cap L^\infty([0,T], L^2(\Omega)) \cap L^\infty([0,T], H^{-2}(\Omega))$$

is called a weak solution of (1.5) *if*

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t(x,t)w(x)dx + \int_\Omega \Delta u(x,t)\Delta w(x)dx + \int_\Omega u(x,t)w(x)dx \\ \quad + \int_\Omega \|u_t\|_2^q u_t w(x)dx = \int_\Omega u(x,t)w(x) \ln |u(x,t)|^k dx, \ \forall w \in H_0^2(\Omega) \\ \quad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases} \tag{3.12}$$

Theorem 3.1. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. *Then problem* (1.5) *has a global weak solution*

$$u \in L^\infty([0,T], H_0^2(\Omega)) \cap L^\infty([0,T], L^2(\Omega)) \cap L^\infty([0,T], H^{-2}(\Omega)). \tag{3.13}$$

Proof. To establish the existence of problem (1.5), we use the standard Faedo-Galerkin method. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the "separable" space $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let $V_m = span\{w_1, w_2, ..., w_m\}$ and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$u_0^m(x) = \sum_{j=1}^m a_j w_j(x), \qquad u_1^m(x) = \sum_{j=1}^m b_j w_j(x),$$

such that

$$u_0^m \to u_0 \text{ in } H_0^2(\Omega) \text{ and } u_1^m \to u \text{ in } L^2(\Omega), \text{ as } m \to \infty. \tag{3.14}$$

We seek a solution of the form

$$u^m(x,t) = \sum_{j=1}^m g_j^m(t) w_j(x),$$

in V_m, of the approximate problem :

$$\begin{cases} \int_\Omega \left(u_{tt}^m w + \Delta u^m \Delta w + u^m w + \|u_t^m\|_2^q u_t^m w\right) dx = \int_\Omega w u^m \ln|u^m|^k dx, \ \forall w \in V_m \\ u^m(0) = u_0^m = \sum_{j=1}^m (u_0, w_j) w_j \\ u_t^m(0) = u_1^m = \sum_{j=1}^m (u_1, w_j) w_j. \end{cases} \tag{3.15}$$

This leads to a system of ODEs for unknown functions $g_j^m(t)$. Based on the standard existence theory for ODE, one can obtain functions

$$g_j : [0, t_m) \to \mathbb{R}, \ j = 1, 2, ..., m,$$

which satisfy (3.15) in a maximal interval $[0, t_m), t_m \in (0, T]$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independently of m and t. For this purpose, let $w = u_t^m$ in (3.15) and integrate by parts to obtain

$$\frac{d}{dt} E^m(t) = -\|u_t^m\|_2^{q+2} \le 0, \tag{3.16}$$

where

$$E^m(t) = \frac{1}{2}\left(\|u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \left(\frac{k+2}{2}\right)\|u^m\|_2^2 - \int_\Omega |u^m|^2 \ln|u^m|^k dx\right). \tag{3.17}$$

Integrating (3.16) over $(0,t)$ gives, for $t \in [0, t_m)$,

$$E^m(t) + \int_0^t \|u_t^m\|_2^{q+2} ds = E^m(0). \tag{3.18}$$

Combining (3.17) and (3.18) and applying the Logarithmic Sobolev inequality, we get

(3.19)

$$\|u_t^m\|_2^2 + \left(1 - \frac{ka^2}{2\pi}\right)\|\Delta u^m\|_2^2 + \left[\left(\frac{k+2}{2}\right) + k(1+\ln a)\right]\|u^m\|_2^2 + 2\int_0^t \|u_t^m(s)\|_2^{q+2} ds \\ \le C + \|u^m\|_2^2 \ln \|u^m\|_2^2.$$

Choosing $e^{\left(-\frac{3}{2}-\frac{1}{2k}\right)} < a < \sqrt{\frac{2\pi}{k}}$ will make

$$1-\frac{ka^2}{2\pi}>0, \quad \left(\frac{k+2}{2}\right)+k\left(1+\ln a\right)>0.$$

This choice is feasible due to Condition (A), which allows us to obtain:

$$\|u_t^m\|_2^2+\|\Delta u^m\|_2^2+\|u^m\|_2^2+2\int_0^t\|u_t^m(s)\|_2^{q+2}ds\le C(1+\|u^m\|_2^2\ln\|u^m\|_2^2). \tag{3.20}$$

Let us note that

$$u^m(.,t)=u^m(.,0)+\int_0^t\frac{\partial u^m}{\partial s}(.,s)ds.$$

Then, using Cauchy-Schwarz' inequality, we get

$$\begin{aligned}\|u^m(t)\|_2^2&\le 2\|u^m(0)\|_2^2+2\left\|\int_0^t\frac{\partial u^m}{\partial s}(s)ds\right\|_2^2\\&\le 2\|u^m(0)\|_2^2+2T\int_0^t\|u_t^m(s)\|_2^2ds,\end{aligned} \tag{3.21}$$

and, consequently, inequality (3.20) yields

$$\|u_t^m\|_2^2\le\|u_t^m\|_2^2+\|\Delta u^m\|_2^2+\|u^m\|_2^2+2\int_0^t\|u_t^m(s)\|_2^{q+2}ds\le C(1+\|u^m\|_2^2\ln\|u^m\|_2^2), \tag{3.22}$$

hence,

$$2T\int_0^t\|u_t^m(s)\|_2^2ds\le 2T\int_0^t C(1+\|u^m\|_2^2\ln\|u^m\|_2^2)ds \tag{3.23}$$

Combining (3.21) and (3.23), we obtain

$$\|u^m\|_2^2\le 2\|u^m(0)\|_2^2+2TC\left(1+\int_0^t\|u^m\|_2^2\ln\|u^m\|_2^2ds\right). \tag{3.24}$$

If we put $C_1=\max\left\{2TC,\ 2\|u^m(0)\|_2^2\right\}$, then (3.24) leads to

$$\|u^m\|_2^2\le C_1\left(1+\int_0^t\|u^m\|_2^2\ln\left(\|u^m\|_2^2\right)ds\right).$$

Without loss of generality, we assume $C_1\ge 1$, yields

$$\|u^m\|_2^2\le C_1\left(1+\int_0^t\left(C_1+\|u^m\|_2^2\right)\ln\left(C_1+\|u^m\|_2^2\right)ds\right).$$

By applying the Logarithmic Gronwall inequality to the above estimate, we derive the following bound

$$\|u^m\|_2^2\le C_1e^{C_1T}\le C_2. \tag{3.25}$$

Consequently, by combining (3.20) with (3.25), we obtain:

$$\|u_t^m\|_2^2+\|\Delta u^m\|_2^2+\|u^m\|_2^2+2\int_0^t\|u_t^m(s)\|_2^{q+2}ds\le C(1+C_2\ln C_2)=C_3, \tag{3.26}$$

where C_3 is a positive constant independent of m and t. This leads to the uniform bound:

$$\sup_{t\in(0,t_m)}\|u_t^m\|_{L^2(\Omega)}^2+\sup_{t\in(0,t_m)}\|\Delta u^m\|_{L^2(\Omega)}^2+\sup_{t\in(0,t_m)}\|u^m\|_{L^2(\Omega)}^2+2\int_0^{t_m}\|u_t^m(s)\|_2^{q+2}ds\le 4C_3. \tag{3.27}$$

Thus, we can extend t_m to T. Furthermore, from (3.27), we deduce the following uniform estimates

$$\begin{cases} u^m \text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)), \\ u_t^m \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)), \\ u_t^m \text{ is uniformly bounded in } L^{q+2}(0,T;L^2(\Omega)), \\ \|u_t^m\|_2^q u_t^m \text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{3.28}$$

These estimates ensure the existence of a subsequence of u^m (still denoted by u^m) satisfying the weak convergences:

$$\begin{cases} u^m \rightharpoonup u \quad \text{weakly * in } L^\infty(0,T;H_0^2(\Omega)), \\ u_t^m \rightharpoonup u_t \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^2(\Omega)), \\ u_t^m \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \\ u_t^m \rightharpoonup u_t \quad \text{weakly in } L^{q+2}(0,T;L^2(\Omega)), \\ \|u_t^m\|_2^q u_t^m \rightharpoonup \chi \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{3.29}$$

Applying the Aubin-Lions compactness theorem, we obtain, up to a subsequence,

$$u^m \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega))$$

and

$$u^m \to u \quad \text{almost everywhere in } \Omega \times (0,T).$$

Since the function $s \mapsto s \ln |s|^k$ is continuous, we conclude that:

$$u^m \ln |u^m|^k \to u \ln |u|^k \quad \text{almost everywhere in } \Omega \times (0,T).$$

Moreover, using the embedding of $H_0^2(\Omega)$ embeds in $L^\infty(\Omega)$ ($\Omega \subset \mathbb{R}^2$), we deduce that $u^m \ln |u^m|^k$ is bounded in $L^\infty(\Omega \times (0,T))$. Consequently, by the Bounded Convergence Theorem (as Ω is bounded), we obtain

$$u^m \ln |u^m|^k \to u \ln |u|^k \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.30}$$

Now, we integrate (3.15) over $(0,t)$ to obtain

$$\begin{aligned} &\int_\Omega u_t^m w dx - \int_\Omega u_1^m w dx + \int_0^t \int_\Omega \Delta u^m(s) \Delta w dx ds + \int_0^t \int_\Omega u^m(s) w dx ds \\ &\quad + \int_0^t \int_\Omega \|u_t^m(s)\|_2^q u_t^m w dx ds = \int_\Omega \int_0^t w u^m(s) \ln |u^m(s)|^k dx ds, \ \forall w \in V_m. \end{aligned} \tag{3.31}$$

Convergences (3.14), (3.29) and (3.30) are sufficient to pass to the limit in (3.31), as $m \to +\infty$, and arrive at

$$\begin{aligned} \int_\Omega u_t w dx = &\int_\Omega u_1 w dx - \int_0^t \int_\Omega \Delta u(s) \Delta w dx ds - \int_0^t \int_\Omega u(s) w dx ds \\ &- \int_0^t \int_\Omega \chi(s) w dx ds + \int_\Omega \int_0^t u(s) w \ln |u(s)|^k dx, \ \forall w \in V_m, \ \forall m \geq 1, \end{aligned} \tag{3.32}$$

which implies that (3.32) is valid for any $w \in H_0^2(\Omega)$. Using the fact that the terms in the right-hand side of (3.32) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence differentiable for a.e. $t \in \mathbb{R}^+$. Thus, differentiating (3.32), we obtain for a.e. $t \in (0,T)$

$$\begin{aligned}(3.33)\qquad &\frac{d}{dt}\int_\Omega u_t(x,t)w(x)dx+\int_\Omega \Delta u(x,t).\Delta w(x)dx+\int_\Omega u(x,t)w(x)dx\\ &+\int_\Omega \chi(t)w(x)dx=\int_\Omega w(x)u(x,t)\ln|u(x,t)|^k dx,\ \forall w\in H_0^2(\Omega).\end{aligned}$$

On the other hand, using Cauchy Schwarz' inequality one has, for all $v\in L^2(0,T;L^2(\Omega))$,

(3.34)

$$\begin{aligned}\int_\Omega (u_t^m-v)\left(\|u_t^m\|_2^q u_t^m-\|v\|_2^q v\right)dx &= \int_\Omega \|u_t^m\|_2^q\left(\|u_t^m\|_2^2-u_t^m v\right)dx+\int_\Omega \|v\|_2^q\left(\|v\|_2^2-u_t^m v\right)dx\\ &\geq \int_\Omega \|u_t^m\|_2^q\left(\|u_t^m\|_2^2-\frac{1}{2}\left(\|u_t^m\|_2^2+\|v\|_2^2\right)\right)dx\\ &\quad+\int_\Omega \|v\|_2^q\left(\|v\|_2^2-\frac{1}{2}\left(\|u_t^m\|_2^2+\|v\|_2^2\right)\right)dx\\ &=\frac{1}{2}\left(\|u_t^m\|_2^2-\|v\|_2^2\right)\left(\|u_t^m\|_2^q-\|v\|_2^q\right)\geq 0.\end{aligned}$$

Consequently, we have

$$X^m:=\int_0^T\int_\Omega (u_t^m-v)\left(\|u_t^m\|_2^q u_t^m-\|v\|_2^q v\right)dxdt\geq 0,\ \text{for all } v\in L^2(0,T;L^2(\Omega)).$$

Now, integrate (3.17) over $(0,t)$ to get

$$\begin{aligned}(3.35)\qquad X^m &= \|u_1^m\|_2^2+\|\Delta u_0^m\|_2^2+\left(\frac{k+2}{4}\right)\|u_0^m\|_2^2-\int_\Omega |u_0^m|\ln|u_0^m|dx\\ &-\left(\|u_t^m\|_2^2+\|\Delta u^m\|_2^2+\left(\frac{k+2}{4}\right)\|u^m\|_2^2-\int_\Omega |u^m|\ln|u^m|dx\right)\\ &-\int_0^t\int_\Omega v\|u_t^m\|_2^q u_t^m dxds-\int_0^t\int_\Omega (u_t^m-v)\|v\|_2^q v dxds.\end{aligned}$$

Taking $m\to+\infty$, we obtain

$$\begin{aligned}(3.36)\qquad &0\leq \limsup X^m=\|u_1\|_2^2+\|\Delta u_0\|_2^2+\left(\frac{k+2}{4}\right)\|u_0\|_2^2-\int_\Omega |u_0|\ln|u_0|dx\\ &-\left(\|u_t\|_2^2+\|\Delta u\|_2^2+\left(\frac{k+2}{4}\right)\|u\|_2^2-\int_\Omega |u|\ln|u|dx\right)-\int_0^t\int_\Omega \chi(t)vdxds\\ &-\int_0^t\int_\Omega (u_t-v)\|v\|_2^q vdxds.\end{aligned}$$

Replacing w by u_t in (3.33) and integrating over $(0,t)$, we obtain

$$\begin{aligned}(3.37)\qquad &0\leq \limsup X^m=\|u_1\|_2^2+\|\Delta u_0\|_2^2+\left(\frac{k+2}{4}\right)\|u_0\|_2^2-\int_\Omega |u_0|\ln|u_0|dx\\ &-\left(\|u_t\|_2^2+\|\Delta u\|_2^2+\left(\frac{k+2}{4}\right)\|u\|_2^2-\int_\Omega |u|\ln|u|dx\right)-\int_0^t\int_\Omega \chi(t)vdxds.\end{aligned}$$

Combining (3.36) with (3.37)

(3.38)
$$
\begin{aligned}
0 \leq \limsup X^m &= \int_0^t \int_\Omega \chi(t) u_t dxds - \int_0^t \int_\Omega \chi(t) v dxds - \int_0^t \int_\Omega \|v\|_2^q v(u_t - v) v dxds \\
&\leq \int_0^t \int_\Omega (\chi(t) - \|v\|_2^q v)(u_t - v) dxds.
\end{aligned}
$$

Hence,

$$\int_0^t \int_\Omega (\chi(t) - \|v\|_2^q v)(u_t - v) dxds \geq 0, \ \forall v \in L^2(0,T; L^2(\Omega)).$$

Let $v = \lambda\psi + u_t, \ \psi \in L^2(0,T; L^2(\Omega))$. So, we get, $\forall \lambda \neq 0$,

$$-\lambda \int_0^t \int_\Omega (\chi(t) - \|\lambda\psi + u_t\|_2^q (\lambda\psi + u_t)) \psi dxds \geq 0, \ \forall \psi \in L^2(0,T; L^2(\Omega)).$$

Let $\lambda > 0$, then we have

$$\int_0^t \int_\Omega (\chi(t) - \|\lambda\psi + u_t\|_2^q (\lambda\psi + u_t)) \psi dxds \leq 0, \ \forall \psi \in L^2(0,T; L^2(\Omega)).$$

As $\lambda \to 0$, then we obtain

$$\int_0^t \int_\Omega (\chi(t) - \|u_t\|_2^q u_t) \psi dxds \leq 0, \ \forall \psi \in L^2(0,T; L^2(\Omega)). \tag{3.39}$$

Similarly, for $\lambda < 0$, we get

$$\int_0^t \int_\Omega (\chi(t) - \|u_t\|_2^q u_t) \psi dxds \geq 0, \ \forall \psi \in L^2(0,T; L^2(\Omega)). \tag{3.40}$$

Thus, (3.37) and (3.39) imply that $\chi = \|u_t\|_2^q u_t$. Hence (3.33) becomes

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega u_t(x,t) w(x) dx + \int_\Omega \Delta u(x,t).\Delta w(x) dx + \int_\Omega u(x,t) w(x) dx \\
&\qquad + \int_\Omega \|u_t\|_2^q u_t w(x) dx = \int_\Omega w(x) u(x,t) \ln |u(x,t)|^k dx, \ \forall w \in H_0^2(\Omega).
\end{aligned} \tag{3.41}
$$

To handle the initial conditions, we note that

$$
\begin{aligned}
u^m &\rightharpoonup u \quad \text{weakly in } L^2(0,T; H_0^2(\Omega)) \\
u_t^m &\rightharpoonup u_t \quad \text{weakly in } L^2(0,T; L^2(\Omega)).
\end{aligned} \tag{3.42}
$$

Thus, using Lion's Lemma [37], we obtain

$$u^m \to u \text{ in } C([0,T], L^2(\Omega)). \tag{3.43}$$

Therefore, $u^m(x,0)$ makes sense and

$$u^m(x,0) \to u(x,0) \text{ in } L^2(\Omega).$$

Also, we have

$$u^m(x,0) = u_0^m(x) \to u_0(x) \text{ in } H_0^2(\Omega).$$

Hence,

$$u(x,0) = u_0(x).$$

Next, multiply (3.15) by $\phi \in C_0^\infty(0,T)$ and integrate over $(0,T)$ to obtain, for any $w \in V_m$,

(3.44)

$$-\int_0^T\int_\Omega u_t^m(t)w\phi'(t)dxdt = -\int_0^T\int_\Omega \Delta u^m(t)\Delta w\phi(t)dxdt$$
$$-\int_0^T\int_\Omega u^m w\phi(t)dxdt - \int_0^T\int_\Omega \|u_t^m\|_2^q u_t^m w\phi(t)dxdt + \int_0^T\int_\Omega wu_m \ln|u_m|^k\phi(t)dxdt.$$

As $m\to+\infty$, we have for any $w\in H_0^2(\Omega)$ and any $\phi\in C_0^\infty((0,T))$,

(3.45)

$$-\int_0^T\int_\Omega u_t(t)w\phi'(t)dxdt = -\int_0^T\int_\Omega \Delta u(t)\Delta w\phi(t)dxdt$$
$$-\int_0^T\int_\Omega uw\phi(t)dxdt - \int_0^T\int_\Omega \|u_t\|_2^q u_t w\phi(t)dxdt + \int_0^T\int_\Omega w\phi(t)u\ln|u|^k dxdt.$$

This means (see [38]),

$$u_{tt}\in L^2([0,T),H^{-2}(\Omega)).$$

Recalling that $u_t\in L^2((0,T),L^2(\Omega))$, we obtain

$$u_t\in C([0,T),H^{-2}(\Omega)).$$

So, $u_t^m(x,0)$ makes sense and

$$u_t^m(x,0)\to u_t(x,0) \text{ in } H^{-2}(\Omega).$$

But

$$u_t^m(x,0)=u_1^m(x)\to u_1(x) \text{ in } L^2(\Omega).$$

Hence

$$u_t(x,0)=u_1(x).$$

This concludes the proof of Theorem 3.1. □

4. Potential wells

In this section, we present and discuss the potential wells corresponding to the logarithmic nonlinearity. First, we define

$$J(u)=J(u(t))=\frac{1}{2}\left(\|\Delta u\|_2^2+\|u\|_2^2-\int_\Omega u^2\ln|u|^k dx\right)+\frac{k}{4}\|u\|_2^2 \tag{4.46}$$

$$I(u)=I(u(t))=\|\Delta u\|_2^2+\|u\|_2^2-\int_\Omega u^2\ln|u|^k dx \tag{4.47}$$

Remark 4.2.

1. *From the above definitions, it is clear that*

$$J(u)=\frac{1}{2}I(u)+\frac{k}{4}\|u\|_2^2, \tag{4.48}$$

and

$$E(t)=\frac{1}{2}\|u_t\|_2^2+J(u)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}I(u)+\frac{k}{4}\|u\|_2^2. \tag{4.49}$$

2. *Using the Logarithmic Sobolev inequality, $J(u)$ and $I(u)$ are well defined.*

The potential well depth is defined as

$$0 < d = \inf_u \{\sup_{\lambda \geq 0} J(\lambda u) : u \in H_0^2(\Omega), \|\Delta u\|_2 \neq 0\} \tag{4.50}$$

and the well-known Nehari manifold

$$N = \{u : u \in H_0^2(\Omega)/I(u) = 0, \|\Delta u\|_2 \neq 0\}. \tag{4.51}$$

Similar to the results in [36, 39], one has

$$0 < d = \inf_{u \in N} J(u). \tag{4.52}$$

Then, we introduce

$$W = \{u : u \in H_0^2(\Omega)/I(u) > 0, J(u) < d\} \cup \{0\}.$$

Lemma 4.3. *Let $u_0 \in H_0^2(\Omega)$ and $\ell = \frac{2\pi}{k} e^{2+\frac{2}{k}}$.*

1. *If $0 < \|u\|_2^2 < \ell$, then $I(u) > 0$.*
2. *If $I(u) = 0$ and $\|u\| \neq 0$, i.e. $u \in N$, then $\|u\|^2 > \ell$.*

Proof. Using the logarithmic Sobolev inequality (2.9), for any $a > 0$, we have

$$\begin{aligned} I(u) &= ||\Delta u||_2^2 + \|u\|_2^2 - \int_\Omega u^2 \ln |u|^k dx \\ &\geq \left(1 - \frac{ka^2}{2\pi}\right) \|\Delta u\|_2^2 + \|u\|_2^2 + k(1 + \ln a)\|u\|_2^2 - \frac{k}{2}\|u\|_2^2 \ln \|u\|_2^2. \end{aligned} \tag{4.53}$$

Taking $a = \sqrt{\frac{2\pi}{k}}$ in (4.53), we obtain

$$I(u) \geq \left(1 + k\left(1 + \ln \sqrt{\frac{2\pi}{k}}\right) - \frac{k}{2} \ln \|u\|_2^2\right) \|u\|^2. \tag{4.54}$$

If $0 < \|u\|_2^2 < \ell$, then $I(u) > 0$ from (4.54). If $I(u) = 0$ and $\|u\| \neq 0$, then by (4.54), we have

$$1 + k\left(1 + \ln \sqrt{\frac{2\pi}{k}}\right) - \frac{k}{2} \ln \|u\|_2^2 \leq 0,$$

which gives

$$\frac{k}{2} \ln \|u\|_2^2 \geq 1 + k\left(1 + \ln \sqrt{\frac{2\pi}{k}}\right),$$

that is

$$\|u\|_2^2 \geq \frac{2\pi}{k} e^{2+\frac{2}{k}} = \ell$$

□

Lemma 4.4. *The constant d in* (4.50) *satisfies*

$$d \geq \frac{\pi}{2} e^{2+\frac{2}{k}}.$$

Proof. Let $u \in N$, then $I(u) = 0$ and $\|u\| \neq 0$, then by Lemma 4.3, $\|u\|_2^2 \geq \ell$. By invoking (4.48), we arrive at

$$J(u) = \frac{1}{2} I(u) + \frac{k}{4}\|u\|_2^2 \geq \frac{k}{4}\ell = \frac{\pi}{2} e^{2+\frac{2}{k}}. \tag{4.55}$$

Recalling the definition of d, we obtain

$$d \geq \frac{\pi}{2} e^{2+\frac{2}{k}}.$$

□

Lemma 4.5. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *such that*

$$0 < E(0) < d \text{ and } I(u_0) > 0. \tag{4.56}$$

Then any solution of (1.5), $u \in W$.

Proof. Let T be the maximal existence time of the weak solution u. From (4.49) and (2.8), we obtain the estimate:

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq \frac{1}{2}\|u_1\|^2 + J(u_0) < d, \quad \text{for all } t \in [0, T). \tag{4.57}$$

We now claim that $u(t) \in W$ for all $t \in [0, T)$. Suppose, by contradiction, that there exists some $t_0 \in (0, T)$ such that $u(t_0) \in \partial W$. This implies either $I(u(t_0)) = 0$ with $\|\Delta u(t_0)\|_2 \neq 0$, or $J(u(t_0)) = d$. From (4.57), we know that $J(u(t_0)) < d$, which forces $I(u(t_0)) = 0$ and $\|\Delta u(t_0)\|_2 \neq 0$. However, from (4.52), we also have $J(u(t_0)) \geq d$, leading to a contradiction with (4.57). Thus, our assumption is incorrect, and we conclude that $u(t) \in W$ for all $t \in [0, T)$.

□

Lemma 4.6. *Under the conditions of Lemma 4.5, there exists a positive constant* c_0*, such that*

$$I(u) \geq c_0\|u\|_2^2. \tag{4.58}$$

Proof. Since $I(u_0) \geq 0$, then using Lemma 4.5, we have $I(u) > 0$, which implies that

$$J(u) = \frac{1}{2}I(u) + \frac{k}{4}\|u\|_2^2 \geq \frac{k}{4}\|u\|_2^2. \tag{4.59}$$

Combining (4.59), (2.8) and condition $E(0) < \frac{\pi}{2}e^{2+\frac{2}{k}}$, we have

$$\|u\|_2^2 \leq \frac{4}{k}J(u) \leq \frac{4}{k}E(t) \leq \frac{4}{k}E(0) < \frac{2\pi}{k}e^{2+\frac{2}{k}}. \tag{4.60}$$

Then, (4.60) and (4.54) lead to (4.58). □

5. Stability

In this section, we state and prove the main decay result. For this purpose, we present first some lemmas.

Lemma 5.7. [40] *Let* $\phi(t)$ *be a nonincreasing non-negative function on* $[0, \infty)$ *and satisfy*

$$\phi^{1+r}(t) \leq k_0(\phi(t) - \phi(t+1)), \quad t \in [0, T],$$

where $k_0 > 1$ *and* r *is a non-negative constant. Then, we have the following:*

(1) if $r > 0$*, then* $\phi(t) \leq (\phi^{-r}(0) + k_0 r[t-1]_+)^{-\frac{1}{r}}$*,*
(2) if $r = 0$*, then* $\phi(t) \leq \phi(0)e^{-k_1[t-1]_+}$*, where*

$$[t-1]_+ = \max\{t-1, 0\}, \quad k_1 = \log\left(\frac{k_0}{k_0 - 1}\right).$$

Lemma 5.8. *The energy functional satisfies the following estimate*

$$\int_{t_1}^{t_2} E(t)\, dt \leq c\psi(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} + c\psi^{q+1}(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}} + 2\psi^2(t), \tag{5.61}$$

where $\psi(t) = \left(\int_t^{t+1} \|u_t\|_2^{q+2} dt\right)^{\frac{1}{q+2}}$*, and* t_1, t_2 *are defined in the proof of the lemma.*

Proof. Using Hölder's inequality, we obtain

$$\int_t^{t+1} \|u_t\|_2^2 dt \le \left(\int_t^{t+1} 1^{\frac{q+2}{q}} dt\right)^{\frac{q}{q+2}} \left(\int_t^{t+1} \|u_t\|_2^{q+2} dt\right)^{\frac{2}{q+2}} = \psi^2(t). \tag{5.62}$$

By the mean value theorem, we deduce from (5.62) that there exist $t_1 \in [t, t+\frac{1}{4}]$, $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|_2^2 \le 4\psi^2(t), \quad i = 1, 2. \tag{5.63}$$

Now, multiplying the equation in (1.5) by u, integrating it over $\Omega \times (t_1, t_2)$, and using integration by parts, we have

(5.64)

$$\begin{aligned}\int_{t_1}^{t_2} \left(\|\Delta u\|_2^2 + \|u\|_2^2 - \int_\Omega u^2 \ln |u|^k dx\right) dt &= -\int_{t_1}^{t_2}\int_\Omega u u_{tt}\, dx dt - \int_{t_1}^{t_2}\int_\Omega \|u_t\|_2^q\, u_t u\, dx dt \\ &\le \sum_{i=1}^{2} \|u_t(t_i)\|_2 \|u(t_i)\|_2 + \int_{t_1}^{t_2} \|u_t\|_2^2 dt \\ &\quad - \int_{t_1}^{t_2}\int_\Omega \|u_t\|_2^q\, u_t u\, dx dt.\end{aligned}$$

Using (5.63) and (4.49), we get, for $i = 1, 2$ and some positive constant c,

$$\|u_t(t_i)\|_2 \|u(t_i)\|_2 \le 2c\psi(t) \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}. \tag{5.65}$$

Similarly, using Hölder's inequality, we obtain

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_\Omega \|u_t\|_2^q u_t u\, dx\, dt\right| &\le \int_{t_1}^{t_2} \|u_t\|_2^{q+1} \|u\|_2\, dt \\ &\le c \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}} \int_{t_1}^{t_2} \|u_t\|_2^{q+1}\, dt = c\psi^{q+1}(t) \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}.\end{aligned}$$

Hence, (5.64) can be written as

$$\int_{t_1}^{t_2} I(u(t))\, dt \le c\psi(t) \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}} + c\psi^{q+1}(t) \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}} + \psi^2(t). \tag{5.66}$$

Using (4.49) and (4.58), we deduce that

$$E(t) \le \frac{1}{2}\|u_t\|_2^2 + cI(u(t)), \tag{5.67}$$

for some positive constant c. Integrating (5.67) over (t_1, t_2), we get

$$\int_{t_1}^{t_2} E(t)\, dt \le \frac{1}{2}\int_{t_1}^{t_2} \|u_t\|_2^2\, dt + c\int_{t_1}^{t_2} I(u(t))\, dt. \tag{5.68}$$

Thus, combining (5.62),(5.66) and (5.68), we obtain (5.61). □

Theorem 5.2. *Let $u_0 \in W$, $u_1 \in L^2(\Omega)$ and $E(0) < d$. Assume that (A) holds. Then there exist positive constants c_1 and c_2 such that*

$$E(t) \le \frac{1}{(c_1 + c_2[t-1]_+)^{\frac{2}{q}}}, \tag{5.69}$$

where $[t-1]_+ = \max\{t-1, 0\}$.

Proof. Integrating (2.8) over (t, t_2), we obtain

$$E(t) = E(t_2) + \int_t^{t_2} \|u_t\|_2^{q+2}\, d\tau.$$

Noticing $t_2 - t_1 \geq \frac{1}{2}$, (where t_1, t_2 are defined in the proof of the previous lemma), we see that

$$E(t_2) \leq 2 \int_{t_1}^{t_2} E(t)\, dt.$$

Again, integrating (2.8) over $(t, t+1)$, $t \geq 0$, we see that

$$\int_t^{t+1} \|u_t\|_2^{q+2} dt = E(t) - E(1+t) = \psi^{q+2}(t). \tag{5.70}$$

Then, combining (2.8), (5.70) and (5.61), we have, for some positive constant c,

$$\begin{aligned} E(t) &\leq 2 \int_{t_1}^{t_2} E(t)\, dt + \int_{t_1}^{t_2} \|u_t\|_2^{q+2}\, dx \leq 2 \int_{t_1}^{t_2} E(t)\, dt + \psi^{q+2}(t) \\ &\leq c\psi^2(t) + c(\psi(t) + \psi^{q+1}(t)) \sup_{t_1 \leq s \leq t_2 E(s)^{\frac{1}{2}}} + \psi^{q+2}(t) \\ &\leq c\psi^2(t) + c(\psi(t) + \psi^{q+1}(t)) E(t)^{\frac{1}{2}} + \psi^{q+2}(t), \end{aligned} \tag{5.71}$$

Hence, by using Young's inequality, we get for $\varepsilon > 0$,

$$E(t) \leq c\psi^2(t) + 2\varepsilon E(t) + c_\varepsilon \psi^{2(q+1)}(t) + c_\varepsilon \psi^2(t) + \psi^{q+2}(t).$$

Choosing ε small enough, we arrive at

$$E(t) \leq c\left(\psi^2(t) + \psi^{2(q+1)}(t) + \psi^{q+2}(t)\right). \tag{5.72}$$

In view of (5.70) and (5.72), we obtain

$$E(t) \leq c\left(1 + \psi^{2q}(t) + \psi^q(t)\right)\psi^2(t) \leq c\left(1 + E(0)^{\frac{2q}{q+2}} + E(0)^{\frac{q}{q+2}}\right)\psi^2(t),$$

which implies that

$$E^{\frac{q+2}{2}}(t) \leq c(E(0))^{\frac{q+2}{2}} \psi(t)^{q+2} = c\left((E(0))\right)^{\frac{q+2}{2}} \left(E(t) - E(t+1)\right).$$

Then, using Nakao's inequality (Lemma 5.7), we get the energy estimate (5.69). □

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