CARPATHIAN J. MATH.
Volume **42** (2026), No. 3,
Pages 635-648

Online version at **https://www.carpathian.cunbm.utcluj.ro/**
ISSN 1584-2851 (Print edition); ISSN 1843-4401 (Electronic)
DOI: https://doi.org/10.37193/CJM.2026.03.10

Existence, uniqueness, and stability of solutions to a nonlinear Ψ-Hilfer fractional differential equation with anti-periodic conditions

PIYACHAT BORISUT[1,2] AND SUPAK PHIANGSUNGNOEN[1,2,*]

ABSTRACT. In this paper, we investigate the existence, uniqueness, and Ulam-Hyers stability of solutions to the nonlinear Ψ-Hilfer fractional differential equation with anti-periodic conditions by using O'Regan's fixed point theorem and the Banach contraction principle. An illustrative example is included to demonstrate the applicability of our results.

1. INTRODUCTION

Fractional differential equations have evolved into a vital tool for tackling real-world issues in biology, physics, chemistry, biophysics, electromagnetic theory, electrodynamics, electrochemistry, fluid mechanics, networking, economics, and image processing, etc.; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] for more details. There are various distinct definitions of fractional integrals and derivatives in the literature; for example, fractional derivatives in the sense of Riemann–Liouville and Caputo are two of the most prominent. The Hadamard fractional derivative, the Erdeyl–Kober fractional derivative, the Hilfer-Katugampola fractional derivative, the Caputo-Fabrizio fractional derivative, and others are some examples of similar well-known concepts.

In 2019 Salem and Alghamdi [11] studied anti-periodic and a new class of multi-point boundary conditions. They further investigated both, the existence and uniqueness of solutions for the Langevin equation that has Caputo fractional derivatives of two different orders. Existence of solutions was obtained by applying the Krasnoselskii–Zabreiko and the Leray–Schauder fixed point theorems. The Banach contraction mapping principle was used to investigate the uniqueness of the form:

$$\begin{cases} {}^C\mathcal{D}^{\beta}_{0^+}({}^C\mathcal{D}^{\alpha}_{0^+}+\lambda)u(t)=\mathcal{F}(t,u(t)),\ t\in[0,1],\\ u(0)+u(1)=0,\ ({}^C\mathcal{D}^{\alpha}_{0^+}u)(0)=0,\ ({}^C\mathcal{D}^{\alpha}_{0^+}u)(1)=\displaystyle\sum_{i=1}^{m}\gamma_i u(q_i),\end{cases}$$

where ${}^C\mathcal{D}^{\alpha}_{0^+}$ and ${}^C\mathcal{D}^{\beta}_{0^+}$ are the Caputo fractional derivative of orders $\alpha\in(0,1]$ and $\beta\in(1,2]$, $\lambda\in\mathbb{R}$, $q_i\in(0,1)$, $i=1,2,\ldots,m$ with $m\in\mathbb{N}$ and the function $\mathcal{F}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable.

Received: 21.04.2025. In revised form: 16.08.2025. Accepted: 03.01.2026
2020 *Mathematics Subject Classification.* 34A08, 34B15, 47H10.
Key words and phrases. *Ψ-Hilfer fractional differential equation, anti-periodic conditions, Banach contraction principle, O′Regan fixed point theorem, Ulam-Hyers stability.*
Corresponding author: supak.pia@mutr.ac.th

The following implicit Caputo fractional derivative and non-local fractional integral conditions of following were analyzed in the work by Borisut and Bantaojai [12] in 2021:

$$\begin{cases}({}^{C}\mathcal{D}^{q}_{0+}u)(t)=f(t,u(t),({}^{C}\mathcal{D}^{q}_{0+}u)(t)),\ t\in[0,T]\\ u(0)=\eta,\ u(T)=({}_{RL}\mathcal{I}^{p}_{0+}u)(\kappa),\end{cases}$$

where $1<q\leq 2,\ 0<p\leq 1,\ \eta\in\mathbb{R},\ {}^{C}\mathcal{D}^{q}_{0+}u(t)$ is the Caputo fractional derivative of order q, ${}_{RL}\mathcal{I}^{p}_{0+}$ is the Riemann-Liouville fractional integral of order p and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. By using the Krasnosel'skii fixed point theorem and and Boyd-Wong non-linear contraction principle, the existence and uniqueness of solutions to this problem were obtained.

In 2021, Chatthai et. al [13] investigated the existence and uniqueness of solutions for a class of Ψ-Hilfer implicit fractional integro-differential equations with mixed non-local conditions. The arguments are based on the Banach, Schaefer, and Krasnosell'skii fixed point theorems. Further, applying techniques of nonlinear functional analysis, the authors establish various kinds of the Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized stability of the following Ψ-Hilfer fractional integro-differential equation with mixed non-local conditions

$$\begin{cases}(\mathcal{D}^{\alpha,\rho;\Psi}_{0^+}u)(t)=\mathcal{F}(t,u(t),(\mathcal{D}^{\alpha,\rho;\Psi}_{0^+}u)(t),(\mathcal{I}^{\alpha;\Psi}_{0^+}u)(t)),\ t\in[0,T],\\ \displaystyle\sum_{i=1}^{m}\omega_i u(\eta_i)+\sum_{j=1}^{n}\kappa_j(\mathcal{D}^{\beta_j,\rho;\Psi}_{0^+}u)(\zeta_j)+\sum_{r=1}^{k}\sigma_r(\mathcal{I}^{\delta_r;\Psi}_{0^+}u)(\theta_r)=A,\end{cases}$$

where $\mathcal{D}^{x,\rho;\Psi}_{0^+}$ is the Ψ-Hilfer fractional derivative of order $x=\{\alpha,\beta_j\}$ with $0<\alpha,\beta_i\leq 1,\ \alpha\geq\beta_j+\rho(1-\beta_j),\ j=1,2,\ldots,n$ and $0\leq\rho\leq 1$, $\mathcal{I}^{\alpha;\Psi}$ and $\mathcal{I}^{\delta;\Psi}$ are Ψ-Riemann-Liouville fractional integrals of orders α and δ, respectively, $\omega_i,\kappa_j,\sigma_r,A\in\mathbb{R},\ \eta_i,\zeta_j,\theta_r\in\mathcal{J},\ i=1,2,\ldots,m,\ j=1,2,\ldots,n,\ r=1,2,\ldots,k,\ \mathcal{F}:\mathcal{J}\times\mathbb{R}^3\to\mathbb{R}$ is a given continuous function and $\mathcal{J}:=[0,T],\ T>0$.

Inspired by the results in the papers [11, 12, 13, 14], the aim of this study is to derive the existence and uniqueness of solutions to the Ψ-Hilfer fractional differential equation with anti-periodic conditions:

$$(1.1)\quad\begin{cases}(\mathcal{D}^{q,p;\Psi}_{\eta_1^+}u)(t)=\mathcal{F}(t,u(t)),\ q\in[1,2),\ p\in[0,1],\\ u(\eta_1)=-u(\eta_2),(\mathcal{D}^{s,p;\Psi}_{\eta_1^+}u)(\eta_1)=-(\mathcal{D}^{s,p;\Psi}_{\eta_1^+}u)(\eta_2),\ 0<s<1,\ t\in(\eta_1,\eta_2),\end{cases}$$

where $\mathcal{D}^{q,p;\Psi}_{\eta_1^+}$, $\mathcal{D}^{s,p;\Psi}_{\eta_1^+}$ are Ψ-Hilfer fractional derivatives of orders q and s, respectively, and $-\infty<\eta_1<\eta_2<\infty$, with $\mathcal{F}:(\eta_1,\eta_2)\times\mathbb{R}^+\to\mathbb{R}^+$ is increasing continuous function.

The structure of the paper is as follows. Section 2 contains some fundamental concepts of fractional derivatives and fixed point theorems. In Section 3, we first employ O'Regan's fixed point theorem to demonstrate the existence of solutions in Section 3.1, followed by the Banach contraction principle to guarantee their uniqueness in Section 3.2. Subsequently, in Section 3.3, we investigate the Ulam-Hyers stability of the solutions. Finally, this section concludes with an illustrative example that exemplifies our main findings.

2. Preliminaries

We will go through some fundamental notations, definitions, lemmas, and theorems that will be utilized to establish the major result in this section.

Definition 2.1. [15] Let $\Gamma(\cdot)$ denote the Gamma function which is defined by $\Gamma(q) = \int_0^\infty e^{-s}s^{q-1}ds$, and let a function $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ be given. Then the Riemann-Liouville fractional integral of order $q > 0$ is given by

$$({}_{RL}\mathcal{I}^q_{0^+}\mathcal{F})(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathcal{F}(s)ds.$$

Definition 2.2. [16] The Ψ-Riemann-Liouville fractional integral of order q for an integrable function $\mathcal{F} : (\eta_1, \eta_2) \to (-\infty, \infty)$ with respect to Ψ. Let $\Psi : (\eta_1, \eta_2] \to (-\infty, \infty)$ be an increasing function and continuous derivative $\Psi'(t)$ on $(\eta_1, \eta_2), \Psi'(t) \neq 0$ is defined as follows:

$$(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F})(t) = \frac{1}{\Gamma(q)}\int_{\eta_1}^t \Psi'(s)(\Psi(t)-\Psi(s))^{q-1}\mathcal{F}(s)ds \tag{2.1}$$

and the differential fractional derivative of order q is defined by

$$\begin{aligned}(\mathcal{D}^{q;\Psi}_{a^+}\mathcal{F})(t) &= \Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)^n \mathcal{I}^{n-q;\psi}_{\eta_1^+}\mathcal{F}(t)\\ &= \frac{1}{\Gamma(n-q)}\Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)^n \int_{\eta_1}^t \Psi'(s)(\Psi(t)-\Psi(s))^{n-q-1}\mathcal{F}(s)ds,\end{aligned}$$

where n is the positive integer with $n-1 < q < n$ and $-\infty \leq \eta_1 < \eta_2 \leq +\infty$.

Definition 2.3. [16] The Ψ-Caputo fractional derivative of a function $\mathcal{F} \in C^n[\eta_1, \eta_2]$ with respect to another function Ψ is defined by

$$\begin{aligned}({}^C\mathcal{D}^{q;\Psi}_{\eta_1^+}\mathcal{F})(t) &= \mathcal{I}^{n-q;\Psi}_{\eta_1^+}\Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)^n \mathcal{F}(t)\\ &= \frac{1}{\Gamma(n-q)}\int_{\eta_1}^t \Psi'(s)(\Psi(t)-\Psi(s))^{n-q-1}\mathcal{F}^{[n]}_\Psi(s)ds,\end{aligned}$$

where n is the positive integer with $n-1 < q < n$ and n is the smallest integer such that $n \geq q$, $\mathcal{F}^{[n]}_\Psi(s) = \Big(\frac{1}{\Psi'(s)}\frac{d}{ds}\Big)^n \mathcal{F}(s)$ and Ψ is as in Definition 2.2.

Definition 2.4. [3] Let $0 \leq p \leq 1$, and q a positive non-integer. Also, let $\Psi \in C^n[\eta_1, \eta_2]$ be increasing with $\Psi'(t) \neq 0$ for all $t \in (\eta_1, \eta_2]$, where $n \in \mathbb{N}$ is so that $n-1 < q < n$ and also $-\infty \leq \eta_1 < \eta_2 \leq \infty$. The Ψ-Hilfer fractional derivative of a function $\mathcal{F} \in C^n[\eta_1, \eta_2]$ of order q is determined as

$$(\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{F})(t) = \mathcal{I}^{p(n-q);\Psi}_{\eta_1^+}\Big[\frac{1}{\Psi'(t)}\frac{d}{dt}\Big]^n \mathcal{I}^{(n-q)(1-p);\Psi}_{\eta_1^+}\mathcal{F}(t),\ t > \eta_1.$$

However, since we have access to

$$(\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{F})(t) = \mathcal{I}^{(n-q)p;\Psi}_{\eta_1^+}\mathcal{D}^{r;\Psi}_{\eta_1^+}\mathcal{F}(t),\ t > \eta_1, \tag{2.2}$$

where $\mathcal{D}^{r;\Psi}_{\eta_1^+} = \Big[\frac{1}{\Psi'(t)}\frac{d}{dt}\Big]^n \mathcal{I}^{(n-q)(1-p);\Psi}_{\eta_1^+}\mathcal{F}(t)$ where $r := q + np - qp$, then in particular, the Ψ-Hilfer fractional derivative of $0 < q < 1$ and type $0 \leq p \leq 1$, can be expressed as follows:

$$(\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{F})(t) = \frac{1}{\Gamma(r-q)}\int_{\eta_1}^t (\Psi(t)-\Psi(s))^{r-q-1}\mathcal{D}^{r;\Psi}_{\eta_1^+}\mathcal{F}(s)ds,$$

where $\mathcal{I}^{r-q;\Psi}_{\eta_1^+}(\cdot)$ is as defined in Equation (2.1) and $(\mathcal{D}^{r;\Psi}_{\eta_1^+}\mathcal{F})(t) = \Big[\frac{1}{\Psi'(t)}\frac{d}{dt}\Big]\mathcal{I}^{1-r;\Psi}_{\eta_1^+}\mathcal{F}(t)$.

Lemma 2.5. [17] Let $\rho, q > 0, \mathcal{F} \in L^1(\eta_1, \eta_2), 0 \le r \le 1$.Then $\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{I}^{\rho;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{I}^{q+\rho;\Psi}_{\eta_1^+}\mathcal{F}(t),\ \forall t \in [\eta_1, \eta_2]$. In particular:

(i) If $\mathcal{F} \in C_{r;\Psi}[\eta_1, \eta_2]$, then $\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{I}^{\rho;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{I}^{q+\rho;\Psi}_{\eta_1^+}\mathcal{F}(t),\ \forall t \in [\eta_1, \eta_2]$.

(ii) If $\mathcal{F} \in C[\eta_1, \eta_2]$, then $\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{I}^{\rho;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{I}^{q+\rho;\Psi}_{\eta_1^+}\mathcal{F}(t),\ \forall t \in [\eta_1, \eta_2]$.

Lemma 2.6. [18] Let $0 \le p \le 1, q > 0, 0 \le r \le 1$.

(i) If $\mathcal{F} \in C_{r;\Psi}[\eta_1, \eta_2]$ then $\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{F}(t),\ \forall t \in (\eta_1, \eta_2]$.

(ii) If $\mathcal{F} \in C^1[\eta_1, \eta_2]$ then $\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{F}(t),\ \forall t \in (\eta_1, \eta_2]$.

Proposition 2.7. [19] The Ψ-fractional derivative and integral of a power function are considered by

$$\mathcal{D}^{q,p;\Psi}_{\eta_1^+}(\Psi(t) - \Psi(\eta_1))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho - q)}(\Psi(t) - \Psi(\eta_1))^{\rho-q-1}$$

and

$$\mathcal{I}^{q;\Psi}_{\eta_1^+}(\Psi(t) - \Psi(\eta_1))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho + q)}(\Psi(t) - \Psi(\eta_1))^{\rho+q-1},$$

where $t > \eta_1, 0 \le p \le 1, q \ge 0$ and $\rho > 0$. Additionally, if $0 < q < 1$, then $\mathcal{D}^{q,p;\Psi}_{\eta_1^+}(\Psi(t) - \Psi(\eta_1))^{q-1} = 0$.

Theorem 2.8. [20] Let $\mathcal{F} \in C^n[\eta_1, \eta_2], 0 \le p \le 1, n-1 < q < n, n \in \mathbb{N}$ and $r = q + p - qp$. Then for all $t \in (\eta_1, \eta_2]$,

$$\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{F}(t) - \sum_{i=1}^{n}\frac{[\Psi(t) - \Psi(\eta_1)]^{r-i}}{\Gamma(r - i + 1)}\mathcal{F}^{[n-i]}_{\Psi}\mathcal{I}^{(n-q)(1-p);\Psi}_{\eta_1^+}\mathcal{F}(\eta_1),$$

where $\mathcal{F}^{[n-i]}_{\Psi}(s) = \left(\frac{1}{\Psi'(s)}\frac{d}{ds}\right)^{n-i}\mathcal{F}(s),\ i = 1, 2, \ldots, n.$ If $0 < q < 1$, then in particular,

$$\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{D}^{q,p;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{F}(t) - \frac{[\Psi(t) - \Psi(\eta_1)]^{r-1}}{\Gamma(r)}\mathcal{I}^{(1-q)(1-p);\Psi}_{\eta_1^+}\mathcal{F}(\eta_1).$$

Furthermore, if $\mathcal{F} \in C_{1-r;\Psi}[\eta_1, \eta_2]$ and $\mathcal{I}^{1-r;\Psi}_{\eta_1^+}\mathcal{F} \in C^1_{1-r}[\eta_1, \eta_2]$ for $0 < r < 1$, that for all $t \in (\eta_1, \eta_2]$,

$$\mathcal{I}^{r;\Psi}_{\eta_1^+}\mathcal{D}^{r;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{F}(t) - \frac{[\Psi(t) - \Psi(\eta_1)]^{r-1}}{\Gamma(r)}\mathcal{I}^{(1-r);\Psi}_{\eta_1^+}\mathcal{F}(\eta_1).$$

Lemma 2.9. [18] Let $n - 1 \le r \le n$ and $\mathcal{F} \in C_{r;\Psi}[\eta_1, \eta_2], q > r$ the $\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(\eta_1) = \lim_{t \to \eta_1^+}\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t) = 0$.

Theorem 2.10. [15] (O′Regan Fixed Point Theorem)
Let X be a Banach space and $K \subset X$ a convex, closed set. Let $\mathcal{O}$ be an open set in K with $0 \in \mathcal{O}$. Consider an operator $\mathcal{A}$ from $\overline{\mathcal{O}}$ to K with $\mathcal{A}(\overline{\mathcal{O}})$ bounded, which is given by $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ where $\mathcal{A}_1 : \overline{\mathcal{O}} \to K$ is completely continuous and continuous, and $\mathcal{A}_2 : \overline{\mathcal{O}} \to K$ is a non-linear contraction (i.e., there exists a non-negative, non-decreasing function $\theta : [0, \infty) \to [0, \infty)$ satisfying $\theta(w) < w$ for $w > 0$, such that $\|\mathcal{A}_2u - \mathcal{A}_2v\| \le \theta(\|u - v\|)$ for all $u, v \in \mathcal{O}$). Then either
$(C1)$ $\mathcal{A}$ has a fixed point $u^* \in \overline{\mathcal{O}}$, or
$(C2)$ there exist a point $u^* \in \partial\overline{\mathcal{O}}$ and $\lambda \in (0, 1)$ with $u^* = \lambda\mathcal{A}u^*$, where $\bar{\mathcal{O}}$ and $\partial\mathcal{O}$, respectively, represent the closure and boundary of $\overline{\mathcal{O}}$.

Theorem 2.11. [15] (Contraction Mapping Principle)
Consider a Banach space $\mathcal{B}$, let $\mathcal{Q}$ be a closed set in $\mathcal{B}$ and $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ be a contraction mapping (i.e. $\|\mathcal{F}u - \mathcal{F}v\| \leq k\|u - v\|$ for some $k \in (0, 1)$ and for all $u, v \in \mathcal{Q}$). Then $\mathcal{F}$ has a unique fixed point.

3. Main results

In this section, we present the main results concerning the existence, uniqueness, and stability of solutions for the nonlinear Ψ-Hilfer fractional differential equation with anti-periodic conditions (1.1). Initially, we apply O'Regan's fixed point theorem to establish the existence of solutions in Section 3.1. Subsequently, the uniqueness of the solution is proved by utilizing the Banach contraction principle in Section 3.2. Furthermore, we investigate the Ulam-Hyers stability and its generalizations for the proposed problem in Section 3.3. Finally, we provide an illustrative example (Example 3.7) to demonstrate the applicability of our theoretical findings.

Lemma 3.1. *Let $1 < q \leq 2,\ 0 < s \leq 1,\ 0 \leq p \leq 1,\ r_1 \geq s + p(1 - s)$, where $r_1 = q + p - pq = q + p(1 - q)$ and $\mathcal{F} \in C[\eta_1, \eta_2]$. Then $\mathcal{D}^{s,p;\Psi}_{\eta_1^+}\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}(t)$.*

Proof. From $r_1 = q + p - pq = q + p(1 - q)$ and by (2.2), we have

$$\begin{aligned}
\mathcal{D}^{s,p;\Psi}_{\eta_1^+}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t)) &= \mathcal{I}^{r_1-s;\Psi}_{\eta_1^+}\mathcal{D}^{r;\Psi}_{\eta_1^+}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t)) \\
&= \mathcal{I}^{r_1-s;\Psi}_{\eta_1^+}\Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)\mathcal{I}^{1-r_1;\Psi}_{\eta_1^+}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t)) \\
&= \mathcal{I}^{r_1-s;\Psi}_{\eta_1^+}\Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)\mathcal{I}^{1-r_1+q;\Psi}_{\eta_1^+}\mathcal{F}(t)).
\end{aligned}$$

Consider

$$\begin{aligned}
\Big(\frac{1}{\Psi'(t)}\frac{d}{dt}\Big)\mathcal{I}^{1-r_1+q;\Psi}_{\eta_1^+}\mathcal{F}(t)) &= \frac{1}{\Psi'(t)\Gamma(1 - r_1 + q)}\frac{1}{dt}\int_{\eta_1}^{t}\Psi'(s)\Big(\Psi(t) - \Psi(s)\Big)^{q-r_1}\mathcal{F}(s)ds \\
&= \frac{1}{\Gamma(q - r_1)}\int_{\eta_1}^{t}\Psi'(s)\Big(\Psi(t) - \Psi(s)\Big)^{q-r_1-1}\mathcal{F}(s)ds \\
&= \mathcal{I}^{q-r_1;\Psi}_{\eta_1^+}\mathcal{F}(t).
\end{aligned}$$

Thus $\mathcal{D}^{s,p;\Psi}_{\eta_1^+}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t)) = \mathcal{I}^{r_1-s;\Psi}_{\eta_1^+}\mathcal{I}^{q-r_1;\Psi}_{\eta_1^+}\mathcal{F}(t) = \mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}(t)$. □

Lemma 3.2. *Let the function $\mathcal{F} : [\eta_1, \eta_2] \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathcal{F} \in C^2[\eta_1, \eta_2]$ for all $u \in C^2[\eta_1, \eta_2]$. If a function $u \in C^2[\eta_1, \eta_2]$ then u satisfies,*

$$\begin{cases} (\mathcal{D}^{q,p;\Psi}_{\eta_1^+}u)(t) = \mathcal{F}(t, u(t)),\ t \in (\eta_1, \eta_2), \\ u(\eta_1) = -u(\eta_2), (\mathcal{D}^{s,p;\Psi}_{\eta_1^+}u)(\eta_1) = -(\mathcal{D}^{s,p;\Psi}_{\eta_1^+}u)(\eta_2), \end{cases} \tag{3.1}$$

for some $1 \leq q < 2, 0 \leq p \leq 1$ and $0 < s < 1$, then the following integral expression holds for u,

$$\begin{cases} u(t) = (\mathcal{I}^{q,\Psi}_{\eta_1^+}\mathcal{F}_u)(t) + \frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\square\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1 + \eta_2) \\ \quad -\frac{1}{\Gamma(r-1)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1 + \eta_2)\Big] \\ \quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1 + \eta_2) \\ \quad -\frac{1}{\Gamma(r)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1 + \eta_2)\Big] \end{cases} \tag{3.2}$$

where

$$\square = \Big[-\frac{1}{\Gamma(r-s-1)\Gamma(r)}+\frac{1}{\Gamma(r-s)\Gamma(r-1)}\Big](\Psi(\eta_2)-\Psi(\eta_1))^{r-s-1},$$
$$\triangle = \Big[-\frac{1}{\Gamma(r-s)\Gamma(r-1)}+\frac{1}{\Gamma(r-s-1)\Gamma(r)}\Big](\Psi(\eta_2)-\Psi(\eta_1))^{r-s-2},$$

provided that $q \le r = q+2p-pq < 2$.

Proof. Applying $\mathcal{I}^{q;\Psi}_{\eta_1^+}$ to both sides in equation (3.1) and using Theorem 2.8, we obtain

$$\begin{aligned}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(t) &= u(t)-\frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Gamma(r)}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ &\quad -\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\Gamma(r-1)}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ u(t) &= \mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(t,u(t))+\frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Gamma(r)}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ &\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\Gamma(r-1)}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1).\end{aligned}$$

From the first condition, we have

$$(3.3)\quad \begin{cases}-(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1)=\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}(\eta_2,u(\eta_2))+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{r-1}}{\Gamma(r)}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ \qquad +\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{r-2}}{\Gamma(r-1)}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1).\end{cases}$$

By using Lemma 2.6, Proposition 2.7, and the second condition, we obtain

$$\begin{aligned}(\mathcal{D}^{s,p;\Psi}_{\eta_1^+}u)(t) &= (\mathcal{D}^{s,p;\Psi}_{\eta_1^+}\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(t)+\mathcal{D}^{s,p;\Psi}_{\eta_1^+}\frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Gamma(r)}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ &\quad +\mathcal{D}^{s,p;\Psi}_{\eta_1^+}\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\Gamma(r-1)}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ &= (\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(t)+\frac{(\Psi(t)-\Psi(\eta_1))^{r-s-1}}{\Gamma(r-s)}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ &\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-s-2}}{\Gamma(r-s-1)}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1).\end{aligned}$$

So,

$$(3.4)\quad \begin{cases}-(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1)=(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_2)+\\ \qquad \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{r-s-1}}{\Gamma(r-s)}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1)\\ \qquad +\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{r-s-2}}{\Gamma(r-s-1)}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1).\end{cases}$$

From (3.3) and (3.4), we then obtain that

$$\begin{aligned}u^{[1]}_{\Psi}(\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1) &= \frac{1}{\square}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\\ &\quad -\frac{1}{\Gamma(r-1)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big]\\ (\mathcal{I}^{(1-p)(2-q);\Psi}_{\eta_1^+}u)(\eta_1) &= \frac{1}{\triangle}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\\ &\quad -\frac{1}{\Gamma(r)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big].\end{aligned}$$

This implies that

$$
\begin{aligned}
u(t) &= (\mathcal{I}_{\eta_1^+}^{q,\Psi}\mathcal{F}_u)(t) + \frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Box\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}_{\eta_1^+}^{q;\Psi}\mathcal{F}_u)(\eta_1+\eta_2) \\
&\quad -\frac{1}{\Gamma(r-1)}(\mathcal{I}_{\eta_1^+}^{q-s;\Psi}\mathcal{F}_u)(\eta_1+\eta_2)\Big] \\
&\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}_{\eta_1^+}^{q;\Psi}\mathcal{F}_u)(\eta_1+\eta_2) \\
&\quad -\frac{1}{\Gamma(r)}(\mathcal{I}_{\eta_1^+}^{q-s;\Psi}\mathcal{F}_u)(\eta_1+\eta_2)\Big].
\end{aligned}
$$

□

Let $E = C([\eta_1,\eta_2],\mathbb{R})$, so that E is a Banach space with norm $\|u\|= \max_{t\in[\eta_1,\eta_2]}|u(t)|$ and $\mathcal{F}:(\eta_1,\eta_2)\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous function. Define an operator $\mathcal{A}:E\to E$ by

$$
(3.5)\qquad \begin{cases}
(\mathcal{A}u)(t) = (\mathcal{I}_{\eta_1^+}^{q;\Psi}\mathcal{F}_u)(t) \\
\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Box\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}_{\eta_1^+}^{q;\Psi}\mathcal{F}_u)(\eta_1+\eta_2) \\
\quad -\frac{1}{\Gamma(r-1)}(\mathcal{I}_{\eta_1^+}^{q-s;\Psi}\mathcal{F}_u)(\eta_1+\eta_2)\Big] \\
\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}_{\eta_1^+}^{q;\Psi}\mathcal{F}_u)(\eta_1+\eta_2) \\
\quad -\frac{1}{\Gamma(r)}(\mathcal{I}_{\eta_1^+}^{q-s;\Psi}\mathcal{F}_u)(\eta_1+\eta_2)\Big].
\end{cases}
$$

then the problem (1.1) has a solution if and only if the operator $\mathcal{A}$ has a fixed point. In the next step, we show the existence and uniqueness of solutions for the Ψ-Hilfer fractional differential equation with anti-periodic conditions (1.1) via the O′Regan fixed point theorem and using Banach contraction principle. An example is included as illustration.

3.1. **Existence Result via O′Regan Fixed Point Theorem.**

Theorem 3.3. *Assume that* $\mathcal{F}:(\eta_1,\eta_2)\times\mathbb{R}\to\mathbb{R}$ *is a continuous function and* $(\Psi(\eta_2)-\Psi(\eta_1))>1$, *and let* $B_R=\{u\in E:\|u\|< R\}$. *Suppose that the following hypotheses hold:*
$(H1)$ *there exists a positive constant* $\mathcal{B}$ *and non-decreasing function* $\Theta:[0,\infty)\to(0,\infty)$ *such that*

$$|\mathcal{F}(t,u)|\leq \mathcal{B}\Theta(|u|),\ \forall(t,u)\in(\eta_1,\eta_2)\times\mathbb{R}.$$

$(H2)$ *Let* $\mathcal{K}\geq R-(\eta_2+\eta_1)\mathcal{B}\Theta(R)$, *where*

$$
\begin{aligned}
\mathcal{K} &= \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r-s-1)\Gamma(q+1)}+\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r-1)\Gamma(q-s+1)}\Big] \\
&\quad +\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r-s)\Gamma(q+1)}+\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r)\Gamma(q-s+1)}\Big].
\end{aligned}
$$

Then the problem (1.1) *has at least one solution on* (η_1,η_2).

Proof. From (3.7), we can now decompose the operator $\mathcal{A}:E\to E$ by

$$\mathcal{A}u(t)=\mathcal{A}_1u(t)+\mathcal{A}_2u(t),$$

where $\mathcal{A}_1,\ \mathcal{A}_2$ are given by

$$\mathcal{A}_1u(t) = (\mathcal{I}_{\eta_1^+}^{q;\Psi}\mathcal{F}_u)(t)$$

$$\begin{aligned}
\mathcal{A}_2u(t) &= \frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Box\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\\
&\quad -\frac{1}{\Gamma(r-1)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big]\\
&\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\\
&\quad -\frac{1}{\Gamma(r)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big].
\end{aligned}$$

We divide the proof into three steps as follows:

Step 1 We will show that $\mathcal{A}_1$ is a non-linear contraction.

Let

$$|\mathcal{F}(t,u)-\mathcal{F}(t,v)|\leq \frac{y(t)|u-v|}{\mathcal{Y}^*+|u-v|}$$

for all $t\in(\eta_1,\eta_2)$,where $y(t):(\eta_1,\eta_2)\to\mathbb{R}$, $\mathcal{Y}^*$ is the constant defined by $\mathcal{Y}^*:=(\mathcal{I}^{q,\Psi}_{\eta_1^+}y)(\eta_2)$. Define a continuous non-decreasing function $\mathcal{Y}:\mathbb{R}^+\to\mathbb{R}^+$ by $\mathcal{Y}(\epsilon)=\dfrac{\mathcal{Y}^*\epsilon}{\mathcal{Y}^*+\epsilon}$, $\forall\epsilon\geq 0$. Then $\mathcal{Y}(0)=0$ and $\mathcal{Y}(\epsilon)<\epsilon,\forall\epsilon>0$. We now obtain that

$$\begin{aligned}
\Big|\mathcal{A}_1u(t)-\mathcal{A}_1v(t)\Big| &\leq \mathcal{I}^{q;\Psi}_{\eta_1^+}|\mathcal{F}(t,u(t))-\mathcal{F}(t,v(t))|\\
&\leq \frac{\|u-v\|}{\mathcal{Y}^*+\|u-v\|}\mathcal{I}^{q;\Psi}_{\eta_1^+}y(\eta_2)\\
&= \frac{\mathcal{Y}^*\|u-v\|}{\mathcal{Y}^*+\|u-v\|}\\
&= \mathcal{Y}(\|u-v\|).
\end{aligned}$$

Since $\|\mathcal{A}_1u-\mathcal{A}_1v\|\leq\mathcal{Y}(\|u-v\|)$, therefore $\mathcal{A}_1$ is a non-linear contraction.

Step 2 We will prove that $\mathcal{A}_2$ is continuous and completely continuous. Since $\Psi(t)$ is continuous, then $\mathcal{A}_2$ is continuous also. From $B_R=\{u\in E:\|u\|<R\}$ and $(H1)$ hold, $1\leq q\leq r<2$ for all $t\in(\eta_1,\eta_2)$, $\Psi(t)$ is a non-decreasing continuous function, we get

$$\begin{aligned}
\Big|\mathcal{A}_2u(t)\Big| &\leq \frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{|\Box|\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}\Big|(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_2)+(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1)\Big|\\
&\quad -\frac{1}{\Gamma(r-1)}\Big|(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_2)+(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1)\Big|\Big]\\
&\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}\Big|(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_2)+(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1)\Big|\\
&\quad -\frac{1}{\Gamma(r)}\Big|(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_2)+(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1)\Big|\Big]\\
&\leq \frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{|\Box|\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q-s}}{\Gamma(r-s-1)\Gamma(q+1)}\mathcal{B}\Theta(R)(\eta_2+\eta_1)\\
&\quad -\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q-s}}{\Gamma(r-1)\Gamma(q-s+1)}\mathcal{B}\Theta(R)(\eta_2+\eta_1)\Big]\\
&\quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q-s}}{\Gamma(r-s)\Gamma(q+1)}\mathcal{B}\Theta(R)(\eta_2+\eta_1)\\
&\quad -\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q-s}}{\Gamma(r)\Gamma(q-s+1)}\mathcal{B}\Theta(R)(\eta_2+\eta_1)\Big]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r-s-1)\Gamma(q+1)}+\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r-1)\Gamma(q-s+1)}\Big]\\
&\quad+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r-s)\Gamma(q+1)}+\frac{(\eta_2+\eta_1)\mathcal{B}\Theta(R)}{\Gamma(r)\Gamma(q-s+1)}\Big]\\
&\leq \mathcal{K}.
\end{aligned}$$

Hence $|\mathcal{A}_2 u(t)|$ is uniformly bounded, and for $\tau_1,\ \tau_2 \in (\eta_1,\eta_2),\ \tau_1<\tau_2$, we have

$$\begin{aligned}
\Big|\mathcal{A}_2u(\tau_2)-\mathcal{A}_2u(\tau_1)\Big| &= \Big((\Psi(\tau_2)-\Psi(\eta_1))^{r-1}-(\Psi(\tau_1)-\Psi(\eta_1))^{r-1}\Big)\\
&\quad\times\Bigg|\frac{1}{\Box\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\\
&\quad-\frac{1}{\Gamma(r-1)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big]\Bigg|\\
&\quad+\Big((\Psi(\tau_2)-\Psi(\eta_1))^{r-2}-(\Psi(\tau_1)-\Psi(\eta_1))^{r-2}\Big)\\
&\quad\times\Bigg|\frac{1}{\triangle\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\\
&\quad-\frac{1}{\Gamma(r)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big]\Bigg|.
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above equation tends to zero. By applying the Arzala-Ascoli theorem with $\mathcal{A}_2 : C[\eta_1,\eta_2] \to C[\eta_1,\eta_2]$ is equicontinuous and uniformly bounded, and as a consequence is completely continuous.

Step 3 We will prove that the case $(C2)$ in Theorem 2.10 does not occur. To this end let us suppose to the contrary that condition $(C2)$ holds, with corresponding $\lambda \in (0,1)$ and $B_R = \{u \in E : \|u\| < R\}$. From hypothesis $(H1)$, let $u^* \in \partial \bar{B_R}, u^* = \lambda\mathcal{A}u^*$, so we have $\|u^*\| = R$,

$$\begin{aligned}
|u^*(t)| &= \lambda|\mathcal{A}u^*(t)|\\
&= \lambda|\mathcal{A}_1u^*(t)+\mathcal{A}_2u^*(t)|\\
&\leq |\mathcal{A}_1u^*(t)|+|\mathcal{A}_2u(t)|\\
&\leq \mathcal{I}^{q;\Psi}_{\eta_1^+}|\mathcal{F}(t,u^*(t))|+\mathcal{K}\\
&\leq \mathcal{B}\Theta(R)(\eta_2+\eta_1)+\mathcal{K}.
\end{aligned}$$

Taking the supremum for all $t \in [\eta_1,\eta_2]$, then $\|u^*\| \leq (\eta_2+\eta_1)\mathcal{B}\Theta(R)+\mathcal{K}$. In consequence, we have $R-(\eta_2+\eta_1)\mathcal{B}\Theta(R) \leq \mathcal{K}$, which contradicts $(H2)$. Thus the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all the conditions O'Regan's fixed point theorem. Hence the operator $\mathcal{A}$ has at least one fixed point on (η_1,η_2), which is a solution of the problem (1.1).

□

3.2. **Uniqueness Result via The Banach contraction principle.**

Theorem 3.4. *Let $\mathcal{F} \in C[\eta_1,\eta_2]$ satisfy that for any there exists a positive constant $\mathcal{D}$ such that $|\mathcal{F}(t,u)-\mathcal{F}(t,v)| \leq \mathcal{D}|u-v|$ for any $u,v \in \mathbb{R}$ and $t \in [\eta_1,\eta_2]$. If*

$$(3.6)\quad \left\{\begin{aligned}
&\Big(\tfrac{(\Psi(\eta_2)-\Psi(\eta_1))^q}{\Gamma(q+1)}+\tfrac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\tfrac{1}{\Gamma(r-s-1)\Gamma(q+1)}+\tfrac{1}{\Gamma(r-1)\Gamma(q-s+1)}\Big]\\
&+\tfrac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\tfrac{1}{\Gamma(r-s)\Gamma(q+1)}+\tfrac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\Big)\mathcal{D}<1,
\end{aligned}\right.$$

is satisfied, then problem (1.1) *has a unique solution.*

Proof. Define an operator $\mathcal{A} : C[\eta_1, \eta_2] \to C[\eta_1, \eta_2]$ by

$$(3.7) \qquad \begin{cases} (\mathcal{A}u)(t) = (\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(t) \\ \quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{\Box\Gamma(r)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2) \\ \quad -\frac{1}{\Gamma(r-1)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big] \\ \quad +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2) \\ \quad -\frac{1}{\Gamma(r)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{F}_u)(\eta_1+\eta_2)\Big]. \end{cases}$$

Let $u, v \in C[\eta_1, \eta_2]$ and $|\mathcal{F}(t,u) - \mathcal{F}(t,v)| \leq \mathcal{D}|u-v|,\ t \in [\eta_1, \eta_2]$, then we get

$$\begin{aligned} |\mathcal{A}u(t) - \mathcal{A}v(t)| \leq\ & \mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{D}|u-v|(t) \\ & +\frac{(\Psi(t)-\Psi(\eta_1))^{r-1}}{|\Box\Gamma(r)|}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s-1)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{D}|u-v|)(\eta_2) \\ & +\frac{1}{\Gamma(r-1)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{D}|u-v|)(\eta_2)\Big] \\ & +\frac{(\Psi(t)-\Psi(\eta_1))^{r-2}}{|\triangle\Gamma(r-1)|}\Big[\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{-s}}{\Gamma(r-s)}(\mathcal{I}^{q;\Psi}_{\eta_1^+}\mathcal{D}|u-v|)(\eta_2) \\ & +\frac{1}{\Gamma(r)}(\mathcal{I}^{q-s;\Psi}_{\eta_1^+}\mathcal{D}|u-v|)(\eta_2)\Big] \\ \leq\ & \Bigg\{\frac{1}{\Gamma(q+1)} + \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{1}{\Gamma(r-s-1)\Gamma(q+1)} \\ & +\frac{1}{\Gamma(r-1)\Gamma(q-s-1)}\Big] + \frac{(\Psi(t)-\Psi(\eta_1))^{r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{1}{\Gamma(r-s)\Gamma(q+1)} \\ & +\frac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\Bigg\} \times (\Psi(\eta_2)-\Psi(\eta_1))^q\mathcal{D}|u-v|. \end{aligned}$$

This implies that

$$\begin{aligned} \|\mathcal{A}u - \mathcal{A}v\| \leq\ & \Bigg\{\frac{(\Psi(\eta_2)-\Psi(\eta_1))^q}{\Gamma(q+1)} + \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{1}{\Gamma(r-s-1)\Gamma(q+1)} \\ & +\frac{1}{\Gamma(r-1)\Gamma(q-s+1)}\Big] + \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{1}{\Gamma(r-s)\Gamma(q+1)} \\ & +\frac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\Bigg\}\mathcal{D}\|u-v\|. \end{aligned}$$

From the Banach contraction principle, $\mathcal{A}$ has a unique fixed point, which is the unique solution of the problem (1.1). □

3.3. **Ulam-Hyers Stability.** We discuss the Ulam-Hyers stability of solution to the problem 1.1.

Definition 3.5. The problem 1.1 has the Ulam-Hyers stability if there exists a real number $\gamma > 0$ with the following property. For every $\epsilon > 0,\ u \in E$, if

$$\left|D^{q,p;\psi}_{\eta_1^+}u(t) - f(t, u(t))\right| \leq \epsilon$$

then there exists some $u^* \in E$ satisfying

$$(3.8)\qquad \begin{cases} (\mathcal{D}^{q,p;\Psi}_{\eta_1^+} u^*)(t) = \mathcal{F}(t, u^*(t)),\ t \in (\eta_1, \eta_2), \\ u^*(\eta_1) = -u^*(\eta_2), (\mathcal{D}^{s,p;\Psi}_{\eta_1^+} u^*)(\eta_1) = -(\mathcal{D}^{s,p;\Psi}_{\eta_1^+} u^*)(\eta_2), \end{cases}$$

such that $|u(t) - u^*(t)| \leq \gamma\epsilon,\ t \in (\eta_1, \eta_2)$.

Theorem 3.6. *Suppose that Theorem* 3.4 hold, then the unique solution $u \in E$ is given by equation (3.2) is Ulam-Hyers stable if there exist $\gamma > 0$, for each $\epsilon > 0$ and the following inequalities

$$\begin{aligned} \Big| u(t) \quad &- \quad (\mathcal{I}^{q,\Psi}_{\eta_1^+} \mathcal{F}_u)(t) - \frac{(\Psi(t) - \Psi(\eta_1))^{r-1}}{\Box\Gamma(r)} \Big[\frac{(\Psi(\eta_2) - \Psi(\eta_1))^{-s}}{\Gamma(r-s-1)} (\mathcal{I}^{q;\Psi}_{\eta_1^+} \mathcal{F}_u)(\eta_1 + \eta_2) \\ &- \quad \frac{1}{\Gamma(r-1)} (\mathcal{I}^{q-s;\Psi}_{\eta_1^+} \mathcal{F}_u)(\eta_1 + \eta_2) \Big] \frac{(\Psi(t) - \Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)} \Big[\frac{(\Psi(\eta_2) - \Psi(\eta_1))^{-s}}{\Gamma(r-s)} (\mathcal{I}^{q;\Psi}_{\eta_1^+} \mathcal{F}_u)(\eta_1 + \eta_2) \\ &- \quad \frac{1}{\Gamma(r)} (\mathcal{I}^{q-s;\Psi}_{\eta_1^+} \mathcal{F}_u)(\eta_1 + \eta_2) \Big] \Big| \leq \epsilon^* \end{aligned}$$

and inequality (3.6) hold, where $\epsilon^* = \epsilon$. There exists a solution $u^* E$ of problem (1.1) such that $|u(t) - u^*(t)| \leq \gamma\epsilon$, $t \in (\eta_1, \eta_2)$.

Proof. In view of Theorem 3.4, let $u \in E$ be a unique solution of (1.1), which is given by equation (3.2) and let $u^* \in E$ satisfy equation (3.8) that is

$$\begin{aligned} u^*(t) \quad = \quad & (\mathcal{I}^{q,\Psi}_{\eta_1^+} \mathcal{F}_{u^*})(t) + \frac{(\Psi(t) - \Psi(\eta_1))^{r-1}}{\Box\Gamma(r)} \Big[\frac{(\Psi(\eta_2) - \Psi(\eta_1))^{-s}}{\Gamma(r-s-1)} (\mathcal{I}^{q;\Psi}_{\eta_1^+} \mathcal{F}_{u^*})(\eta_1 + \eta_2) \\ & - \frac{1}{\Gamma(r-1)} (\mathcal{I}^{q-s;\Psi}_{\eta_1^+} \mathcal{F}_{u^*})(\eta_1 + \eta_2) \Big] \\ & + \frac{(\Psi(t) - \Psi(\eta_1))^{r-2}}{\triangle\Gamma(r-1)} \Big[\frac{(\Psi(\eta_2) - \Psi(\eta_1))^{-s}}{\Gamma(r-s)} (\mathcal{I}^{q;\Psi}_{\eta_1^+} \mathcal{F}_{u^*})(\eta_1 + \eta_2) \\ & - \frac{1}{\Gamma(r)} (\mathcal{I}^{q-s;\Psi}_{\eta_1^+} \mathcal{F}_{u^*})(\eta_1 + \eta_2) \Big]. \end{aligned}$$

By the above equation, we have

$$\begin{aligned} \Big| u(t) - u^*(t) \Big| \quad = \quad & \epsilon^* + \mathcal{I}^{q;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(t) \\ & + \frac{(\Psi(t) - \Psi(\eta_1))^{r-1}}{|\Box\Gamma(r)|} \Big[\frac{(\Psi(\eta_2) - \Psi(\eta_1))^{-s}}{\Gamma(r-s-1)} \Big(\mathcal{I}^{q;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_1) \\ & + \mathcal{I}^{q;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_2) \Big) + \frac{1}{\Gamma(r-1)} \Big(\mathcal{I}^{q-s;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_1) \\ & + \mathcal{I}^{q-s;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_2) \Big) \Big] \\ & + \frac{(\Psi(t) - \Psi(\eta_1))^{r-2}}{|\triangle\Gamma(r-1)|} \Big[\frac{(\Psi(\eta_2) - \Psi(\eta_1))^{-s}}{\Gamma(r-s)} \Big(\mathcal{I}^{q;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_1) \\ & + \mathcal{I}^{q;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_2) \Big) + \frac{1}{\Gamma(r)} \Big(\mathcal{I}^{q-s;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_1) \\ & + \mathcal{I}^{q-s;\Psi}_{\eta_1^+} |\mathcal{F}_u - \mathcal{F}_{u^*}|(\eta_2) \Big) \Big] \end{aligned}$$

$$\Big(1- \qquad \left\{\frac{(\Psi(\eta_2)-\Psi(\eta_1))^q}{\Gamma(q+1)}+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{1}{\Gamma(r-s-1)\Gamma(q+1)} +\frac{1}{\Gamma(r-1)\Gamma(q-s+1)}\Big]+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{1}{\Gamma(r-s)\Gamma(q+1)} +\frac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\right\}\mathcal{D}\Big)\Big|u(t)-u^*(t)\Big|\leq\epsilon^*,$$

so;

$$\|u(t)-u^*(t)\|\leq\epsilon^*/\Big(1- \qquad \left\{\frac{(\Psi(\eta_2)-\Psi(\eta_1))^q}{\Gamma(q+1)}+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{1}{\Gamma(r-s-1)\Gamma(q+1)} +\frac{1}{\Gamma(r-1)\Gamma(q-s+1)}\Big]+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{1}{\Gamma(r-s)\Gamma(q+1)} +\frac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\right\}\mathcal{D}\Big)\Big|,$$

hold, where $\epsilon^*=\epsilon$. Take

$$\gamma\leq 1/\Big(1- \qquad \left\{\frac{(\Psi(\eta_2)-\Psi(\eta_1))^q}{\Gamma(q+1)}+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{1}{\Gamma(r-s-1)\Gamma(q+1)} +\frac{1}{\Gamma(r-1)\Gamma(q-s+1)}\Big]+\frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{1}{\Gamma(r-s)\Gamma(q+1)} +\frac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\right\}\mathcal{D}\Big)\Big|,$$

it follows that $\|u-u^*\|\leq\gamma\epsilon$. Therefor problem (1.1) solution is Ulam-Hyers stable.

□

Example 3.7. Consider the following Ψ-Hilfer fractional differential equation and anti-periodic conditions

$$\begin{cases}\mathcal{D}_{2^+}^{\frac{7}{4},\frac{1}{2};t^{\frac{1}{3}}}u(t)=\frac{\sin(t)}{5(t+2)}\left(\frac{|u(t)|}{1+|u(t)|}\right)-\frac{1}{100},\\ u(2)=-u(12),\ \mathcal{D}_{2^+}^{\frac{3}{4},\frac{1}{2};t^{\frac{1}{3}}}u(2)=-\mathcal{D}_{2^+}^{\frac{3}{4},\frac{1}{2};t^{\frac{1}{3}}}u(12),\ t\in(2,12).\end{cases}\tag{3.9}$$

By comparing problems (1.1) and (3.9), we obtain $q=\frac{7}{4},\ s=\frac{3}{4},\ p=\frac{1}{2},\ \Psi(t)=t^{\frac{1}{3}},\ \mathcal{F}(t,u(t))=\frac{\sin(t)}{5(t+2)}\left(\frac{|u(t)|}{1+|u(t)|}\right)-\frac{1}{100},\ a=2,\ b=12,\ r=q+2p-pq=\frac{15}{8}$. As $(\Psi(12)-\Psi(2))=12^{\frac{1}{3}}-2^{\frac{1}{3}}\approx 1.0295$. By setting $\Theta(\epsilon)=\epsilon,\ \epsilon>0,\ |u(t)|<R,\ |f(t,u(t))|\leq\frac{1}{20}|u(t)|<\frac{1}{20}R$. We then obtain the following result:

$$R-(\eta_2+\eta_1)\mathcal{B}\Theta(R)\approx 0.3R.$$

From (3.9), we can calculate that $\mathcal{K}\approx 0.4637R$. This implies that $\mathcal{K}\geq R-(\eta_2+\eta_1)\mathcal{B}\Theta(R)$, and

$$\begin{aligned}|\mathcal{F}(t,u(t))-\mathcal{F}(t,v(t))| &\leq \frac{1}{20}\Big|\frac{|u(t)|}{1+|u(t)|}-\frac{|v(t)|}{1+|v(t)|}\Big|\\ &\leq \frac{1}{20}|u-v|,\end{aligned}$$

with $\mathcal{D} = \frac{1}{20}$ and following

$$\Big(\frac{(\Psi(\eta_2)-\Psi(\eta_1))^q}{\Gamma(q+1)} + \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-1}}{|\Box|\Gamma(r)}\Big[\frac{1}{\Gamma(r-s-1)\Gamma(q+1)} + \frac{1}{\Gamma(r-1)\Gamma(q-s+1)}\Big] + \frac{(\Psi(\eta_2)-\Psi(\eta_1))^{q+r-s-2}}{|\triangle|\Gamma(r-1)}\Big[\frac{1}{\Gamma(r-s)\Gamma(q+1)} + \frac{1}{\Gamma(r)\Gamma(q-s+1)}\Big]\Big)\mathcal{D} \approx 0.1984 < 1,$$

we can conclude that problem (3.9) has a unique solution. Moreover, the solution is Ulam-Hyer stable from the theorem 3.6, this is there exists a unique solution $u^* \in E$, such that $\|u - u^*\| \leq \gamma\epsilon$, $t \in (2, 12)$ where $\gamma \approx 1.2474 > 0$.

4. Conclusions

We have proved the existence and uniqueness of a solution for a class of Ψ-Hilfer fractional differential equation with anti-periodic conditions. We used the fixed point theorem of O′Regan and the Banach contraction principle to investigate the existence, uniqueness and stability of the solution. Our results are not only new in the given setting but also provide some new special cases: by fixing $p = 0$ and $\Psi = t$ then our equation and conditions will be reduced to a Caputo fractional differential equation of order q, s and by setting $p = 1$ and $\Psi = t$ then our equation and conditions will be reduced to a Riemann-Liouville fractional differential equation of order q, s. The obtained results are well illustrated by example.

The work accomplished in this paper is new and enriches the literature on Ψ-Hilfer fractional differential equation and anti-periodic conditions.

Acknowledgements

The authors gratefully acknowledge the financial support provided by the Faculty of Liberal Arts and the Institute of Research and Development, Rajamangala University of Technology Rattanakosin. This research was supported by the National Science and Research Fund (NSRF) through the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [Grant No. B41G680025]. In addition, the first author received funding support from the Thailand Science Research and Innovation (TSRI) and the Fundamental Fund of Rajamangala University of Technology Rattanakosin under Contract No. FRB 68009/2568 No.202969.

References

[1] Wongcharoen, A.; Ntouyas, S.K.; Tariboon, J. Boundary Value Problems for Hilfer Fractional Differential Inclusions with Nonlocal Integral Boundary Conditions. *Mathematics.* **8** (2020), no. 11, 13–22.

[2] Miller, K. S.; Ross, B. Fractional Calculus and Fractional Differential Equations. *Journal of Fractional Calculus and Applications.* **3** (1993), 49 – 57.

[3] Sousa, J.V.C.; De Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Communications in Nonlinear Science and Numerical Simulation.* **2018** (2018), no. 1, 72 – 91.

[4] Bakakham, A.; Geji, V. D. Existence of positive solutions of nonlinear differential equations. *Journal of Mathematical Analysis and Applications.* **278** (2003), 434 – 442.

[5] Dellbosco, D. Fractional calculus and function spaces. *Journal of Fractional Calculus and Applications.* **6** (1996), no. 1, 45 – 53.

[6] Podlubny, I. *Mathematics in Science and Engineering*. Academic Press, New York, 1999.

[7] Qiao, Y.; Zhou, Z. Existence of positive solutions of singular fractional differential equations with infinite-point boundary conditions. *Advances in Difference Equations.* **8** (2017), no.1, 1 – 9.

[8] Rezapour, Sh.; V. Hedayati, V. On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex compact valued multifunctions. *Kragujevac Journal of Mathematics.* **41** (2017), no.1, 143 –158.

[9] Deng, J.; Ma, L. Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations. *Applied Mathematics Letters* **2010** (2010), 676 – 680.

[10] Borisut, P.; Kumam, P.; Ahmed I.; Jirakitpuwapat W. Existence and uniqueness for ψ-Hilfer fractional differential equation with nonlocal multi-point condition. *Mathematical Methods in the Applied Sciences.* **44** (2021), no.1, 2506 – 2520.

[11] Salem, A.; Alghamdi, B. Multi-point conditions for generalized Langevin equation with two fractional order. *Fractal and Fractional.* **5** (2019), no. 51, 1 –14.

[12] Borisut, P.; Bantaojai, T. Implicit fractional differential equation with nonlocal fractional integral conditions. *Thai Journal of Mathematics.* **19** (2021), no. 3, 993 –1003.

[13] Chatthai, T.; Weerawat, S.; Sotiris K. N. Mixed nonlocal boundary value problem for implicit fractional integro-differential equations via Ψ-Hilfer fraction derivative. *Advances in Difference Equations.* **50** (2014), no.1, 1–24.

[14] Borisut, P.; Bantaojai, T. Leray–Schauder Theorem for implicit fractional differential equation and non-local multi-point conditions. *Fixed Point Theory and Fractional Calculus Forum for Interdisciplinary Mathematics* **2022** (2022), 297 – 309.

[15] Ahmad, B.; Alsacdi, A.; Ntouyas, S.K.; Tariboon, J. Hadamard-Type Fractional Differential Equations Inclusions and Inequalities. *Springer International Publishing*, 2017.

[16] Kilbas, A.A.; Srivastava H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. *North-Holland Math Study*, 1 – 20, Amsterdam, 2006.

[17] Furati, K.M.; Kassim, M.D. Existence and uniqueness for a problem involving Hilfer fractional derivative. *Computers and Mathematics with Applications.* **64** (2012), no. 6, 1616 –1626.

[18] Abdo, M.S.; Satish K.P. Fractional integro-differential equations involving ψ-Hilfer fractional derivative. *Advances in Applied Mathematics and Mechanics.* **11** (2019), no. 1, 1 – 22.

[19] Sousa, J., Vanterler da C.; De Oliveira, E Capelas. Leibniz type rule: ψ-Hilfer fractional operator. *Communications in Nonlinear Science and Numerical Simulation.* **77** (2019), 305 – 311.

[20] Almeida, R. A. Caputo fractional derivative of a function with respect to another function. *Communications in Nonlinear Science and Numerical Simulation* **44** (2017), 460 – 481.

[1] DEPARTMENT OF GENERAL EDUCATION (MATHEMATICS), FACULTY OF LIBERAL ARTS, RAJAMANGALA UNIVERSITY OF TECHNOLOGY RATTANAKOSIN, SAMPHANTHAWONG, BANGKOK 10100, THAILAND
Email address: piyachat.b@rmutr.ac.th
Email address: supak.pia@rmutr.ac.th

[2] INSTITUTE OF RESEARCH AND DEVELOPMENT RAJAMANGALA UNIVERSITY OF TECHNOLOGY RATTANAKOSIN, RAJAMANGALA UNIVERSITY OF TECHNOLOGY RATTANAKOSIN, 96 MU 3 PHUTTHAMONTHON SAI 5 RAOD, SALAYA, PHUTTHAMONTHON, NAKHON PATHOM 73170, THAILAND
Email address: piyachat.b@rmutr.ac.th
Email address: supak.pia@rmutr.ac.th