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A fractional reduced differential transform method for solving time fractional Black Scholes American option pricing equation

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ABSTRACT. In this paper, fractional reduced differential transform method (FRDTM) is operated to solve time fractional Black-Scholes American option pricing equation paying no dividends.The Black-Scholes model plays a significant role in the evaluation of European or American call and put options. The advantage of the proposed method to other existing methods is that it finds the solution without discretization or transformation. While using this method, no recommended assumptions are needed and hence the computational work reduces to a greater extent. Numerical experiments prove that the proposed method is efficient and valid for obtaining the solution of time fractional Black-Scholes equation governing American options. This method proves to be powerful for solving general fractional order partial differential equations (PDEs) existing in the field of Science, Engineering and other related fields.

1. INTRODUCTION

An option over an underlying asset is defined as a contract that provides the holder the right to sell or buy the underlying asset at a specific price on or before a specified date known as the expiry date. Financial derivatives notably options have been studied extensively during the past two or three decades. Thus, the pricing of options have attained enough attention and evolved as an essential and key subject in applied Mathematics. In this response, Black and Scholes in the year 1973, derived a very important mathematical formula for the pricing of options and their assessment [5]. This formula is popularly known as the Black-Scholes model and is employed to valuate options of both European-type and American-type. European options may be exercised only at the time of expiry. On the other hand American options can be exercised at any date up to expiry and hence are more valuable and have gained a lot of attention. Various researchers have premeditated different analytical and numerical methods to arrive at the solution of Black-Scholes equation [2, 3, 4, 6, 10, 12, 13, 16, 20].

With the growth of fractal structure for financial markets, fractional calculus and fractional differential equations have witnessed remarkable development over the years. Many important phenomenon containing fractional order derivatives can be well used to model a variety of systems like electro-magnetic waves, diffusion equations, visco-elasticity, electro chemistry, heat conduction and material sciences.

In recent times, fractional partial differential equations have been introduced to financial theory. Fractional Black-Scholes models are derived in this field. Kumar *et al.* managed to obtain the analytic solution of fractional Black-Scholes equation for European options by extending the application of homotopy analysis method (HAM) and homotopy perturbation method (HPM) [25]. Akrami *et al.* implemented the modified version of

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variational iteration method known as the reconstruction of variational iteration method to find the solution of time fractional Black-Scholes equation [1]. Jumarie applied the fractional Taylor formula to solve time and space fractional Black-Scholes equation and derived the optimal fractional Merton's portfolio [23]. The numerical solution of various heat and wave partial differential equations have been obtained through fractional reduced diferential transform method (FRDTM) [26].

The regular Black-Scholes equation used for the pricing of options is stated as:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0, \tag{1.1}$$

where $C(S,t)$ denotes the price of option or option premium at asset price S and time t, $S(t)$ is the asset price at time t, r denotes the short term interest rate, and σ represents the volatility.

Options are categorized into two main brands as call options and put options. The right to buy the underlying asset is known as a call option. On the other hand, put option is the right or option to sell the underlying asset. Thus, the payoff functions for call options and put options are given as:

$$C_c(S,t)=\max\,(S-K,0) \text{ and } \mathrm{C_p(S,t)=\max\,(K-S,0)}, \tag{1.2}$$

where $C_c(S,t)$ and $C_p(S,t)$ are the values of call option and put option respectively.

In this paper we consider the following time fractional Black-Scholes equation given as:

$$\frac{\partial^\alpha C}{\partial t^\alpha}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0,\ 0<\alpha\leq 1 \tag{1.3}$$

The option pricing problem is usually solved in two ways; numerically and analytically. Researchers have applied different mathematical methods to obtain the numerical and analytical solution of European options. The focus is to find the numerical and analytical solution of American options. Macmilan and Johnson obtained the analytical approximation of American puts on a non dividend paying stock [22], [27]. While Geske and Johnson [19] gave the analytical solution of American options on a dividend paying stock. The obtained solution is given in a series form. Cox, Ross and Rubinstein [14] proposed a simple and easier numerical method known as the Binomial method to solve American options numerically. Brennan and Schwartz [7, 8] and Schwartz [33] introduced the finite difference methods for obtaining the numerical solution of American options.

Researchers have proposed various analytical and semi-analytical methods to arrive at the solution of fractional Black-Scholes equation. The important ones are listed as, fractional Green function method [18], Mellin transform method [30], finite difference method [34], He's variational iteration method [35], Laplace transform method [24], homotopy analysis method and homotopy perturbation method [25] and differential transform method [11, 38].

In this paper, we have applied the fractional reduced differential transform method (FRDTM) to obtain the approximate analytical solution of time fractional Black Scholes equation governing American options paying no dividends on underlying assets. The rest part of the paper is organized as follows: The basic definitions, mathematical preliminaries, main notations of fractional calculus and analysis of fractional reduced differential transform method (FRDTM) are given in section 2. In section 3 the solution of time fractional Black-Scholes equation for American options is carried out through FRDTM. The results and discussion are presented in section 4. Moreover, the graphical presentation for interpretation of results is given in this section. Finally, the concluding remarks are given in section 5.

2. Fractional calculus and fractional reduced differential transform method

The basic definitions of Riemann-Liouville and Caputo fractional order integrals and derivatives used in this work are given as follows [15, 32]:

Definition 2.1. The Riemann-Liouville definition of fractional order integral operator $J_a^\alpha f(z)$ is given as [32]:

$$J_a^\alpha f(z) = \frac{1}{\Gamma\alpha}\int_a^z (z-t)^{\alpha-1} f(t)dt,\ \alpha>0,\ z>0 \tag{2.4}$$

Definition 2.2. The Riemann-Liouville fractional differential operator $D_a^\alpha f(z)$ of order $\alpha > 0$ is stated as [32]:

$$D_a^\alpha f(z) = \begin{Bmatrix} \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^z (z-t)^{m-\alpha-1} f(t)dt,\ m-1<\alpha<m\in\mathbb{N} \\ (\frac{d}{dz})^m f(z),\ \alpha = m\in N \end{Bmatrix} \tag{2.5}$$

Definition 2.3. The Caputo definition of fractional order derivative $D_a^\alpha(\alpha>0)$ of $f(z)$ is given as [9]:

$$D_a^\alpha f(z) = \begin{Bmatrix} \frac{1}{\Gamma(m-\alpha)}\int_a^z (z-t)^{m-\alpha-1} f^{(m)}(t)dt,\ m-1<\alpha<m\in\mathbb{N} \\ (\frac{d}{dz})^m f(z),\ \alpha = m\in\mathbb{N} \end{Bmatrix}, \tag{2.6}$$

where a is the initial value of function f and α defines the order of the derivative.

Definition 2.4. The Caputo definition of time fractional derivative of order $\alpha > 0$ is given as:

$$D_t^\alpha v(x,y,z,t) = \frac{\partial^\alpha v(x,y,z,t)}{\partial t^\alpha} = \begin{Bmatrix} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{(m-\alpha-1)}\frac{\partial^m v(x,y,z,\xi)}{\partial\xi^m}d\xi,\ m-1<\alpha<m \\ \frac{\partial^m v(x,y,z,t)}{\partial t^m},\ \alpha = m\in N \end{Bmatrix} \tag{2.7}$$

Some important properties of Caputo's fractional derivative are listed below:

$$D_a^\alpha C = 0,$$

where C is constant.

$$D_a^\alpha z^\gamma = \begin{Bmatrix} 0,\ \gamma\le\alpha-1 \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}z^{\gamma-\alpha},\ \gamma>\alpha-1 \end{Bmatrix} \tag{2.8}$$

2.1. Fractional reduced differential transform method.

Definition 2.5. Let $v(x,t)$ be an analytic and continuously differentiable function with respect to time variable t, then the Taylor series expansion of function $v(x,t)$ with respect to $t=t_0$ is given by,

$$v(x,t) = \sum_{h=0}^{\infty}\frac{1}{\Gamma(\alpha h+1)}[(D_t^\alpha)^h v(x,t)]_{t=t_0}(t-t_0)^{\alpha h} \tag{2.9}$$

The fractional reduced differential transform $V_h^\alpha(x)$ of $v(x,t)$ at $t = t_0$ is given as

$$V_h^\alpha(x) = \frac{1}{\Gamma(\alpha h+1)}[(D_t^\alpha)^h v(x,t)]_{t=t_0}, \tag{2.10}$$

where $0<\alpha\le 1$ and D_t^α represents the fractional differential operator with respect to time of order α. The fractional reduced differential inverse transform of $V_h^\alpha(x)$ is given as

$$v(x,t) = \sum_{h=0}^{\infty} V_h^\alpha(x)(t-t_0)^{\alpha h} \tag{2.11}$$

Substituting equation (2.10) in equation (2.11)

$$v(x,t)=\sum_{h=0}^{\infty}\frac{(t-t_0)^{\alpha h}}{\Gamma(\alpha h+1)}[(D_t^{\alpha})^h v(x,t)]_{t=t_0} \tag{2.12}$$

In real application the function $v(x,t)$ can be approximated by a finite series

$$v_m^*(x,t)=\sum_{h=0}^{m}V_h^{\alpha}(x)(t-t_0)^{\alpha h}, \tag{2.13}$$

where m denotes the order of the approximation. Therefore the exact solution is obtained by the following relation

$$v(x,t)=\lim_{m\to\infty}v_m^*(x,t)=\sum_{h=0}^{\infty}V_h^{\alpha}(x)(t-t_0)^{\alpha h} \tag{2.14}$$

In above equation (2.14) if we put $\alpha=1$, the fractional reduced differential transform method (FRDTM) converts to regular differential transform method (DTM) [11, 17, 21, 38].

Theorem 2.1. *From above definitions , some basic properties and fundamental operations of FRDTM are listed below* [29, 31].
(*a*) *If* $v(X,t)=u(X,t)\pm w(X,t)$, *then* $V_h^{\alpha}(X)=U_h^{\alpha}(X)\pm W_h^{\alpha}(X)$.
(*b*) *If* $v(X,t)=\lambda u(X,t)$, *then* $V_h^{\alpha}(X)=\lambda U_h^{\alpha}(X)$.
(*c*) *If* $v(X,t)=\frac{\partial u(X,t)}{\partial x}$, *then* $V_h^{\alpha}(X)=\frac{\partial U_h^{\alpha}(X)}{\partial x}$.
(*d*) *If* $v(X,t)=D_{t_0}^{\alpha}u(X,t)$, *then* $V_h^{\alpha}(x)=\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{h+1}^{\alpha}(X)$.
(*e*) *If* $v(X,t)=x_1^{l_1}x_2^{l_2}x_3^{l_3}t^m$, *then* $V_h^{\alpha}(X)=x_1^{l_1}x_2^{l_2}x_3^{l_3}\delta(\alpha h-m)$, *where*

$$\delta(\alpha h-m)=\begin{Bmatrix}1\ if\alpha h=m\\ 0\ if\alpha h\neq m\end{Bmatrix}.$$

(*f*) *If* $v(X,t)=D_{t_0}^{N\alpha}u(X,t)$, *then* $V_h^{\alpha}(X)=\frac{\Gamma(\alpha h+N\alpha+1)}{\Gamma(\alpha h+1)}U_{h+N}^{\alpha}(X)$.
(*g*) *If* $v(X,t)=(\frac{\partial^{\beta}}{\partial x_i{}^{\beta}})u(X,t)$, *then* $V_h^{\alpha}(X)=(\frac{\partial^{\beta}}{\partial x_i{}^{\beta}})U_h^{\alpha}(X)$.

3. Time fractional Black-Scholes equation governing American Options

In this section, the proposed method is employed on time fractional Black-Scholes equation governing American options with non dividend paying assets. As it is evident that the Mittag-Leffler function exists in the solution of the problem. The fair price of American options on a single underlying asset is given by the following time fractional Black-Scholes partial differential equation.

$$\frac{\partial^{\alpha}C}{\partial t^{\alpha}}+\frac{1}{2}\sigma^2S^2\frac{\partial^2C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0,\ 0<\alpha\leq 1, \tag{3.15}$$

where $C(S,t)$ denotes the option price or option premium, $S(t)$ denotes the asset price at time t, σ represents the volatility, r denotes the risk free rate of interest.
The pay off functions for American call and put options are given by:

$$C_c(S,t)=\max\,(S-K,0)\text{ and }\mathrm{C_p(S,t)=\max\,(K-S,0)}, \tag{3.16}$$

where K denotes the strike price.
The early exercise constraint for call and put options is given as:

$$C_c(S,t)\geq\ \max\,(S-K,0)\text{ and }\mathrm{C_p(S,t)\geq\ \max\,(K-S,0)} \tag{3.17}$$

Equation (3.15) can be turned into a forward equation by making certain transformations. We set:

$$x = \ln(\frac{S}{K}),\ \tau = \frac{1}{2}\sigma^2(T-t) \text{ and } \mathrm{C(S,t) = k}\upsilon(\mathrm{x},\tau)$$

Applying the above transformations in equations (3.15) to (3.17) the Black-Scholes model, Eq. (3.15) reduces to:

$$\frac{\partial^\alpha \upsilon}{\partial \tau^\alpha} = \frac{\partial^2 \upsilon}{\partial x^2} + (k-1)\frac{\partial \upsilon}{\partial x} - k\upsilon,\ 0 < \alpha \leq 1,\ \text{where k} = \frac{\mathrm{2r}}{\sigma^2} \tag{3.18}$$

with initial condition

$$\upsilon(x,0) = \begin{Bmatrix} \max(e^x - 1, 0), \text{ for call options} \\ \max(1 - e^x, 0), \text{ for put options} \end{Bmatrix} \tag{3.19}$$

and early exercise constraints

$$\upsilon(x,0) \geq \begin{Bmatrix} \max(e^x - 1, 0), \text{ for call options} \\ \max(1 - e^x, 0), \text{ for put options} \end{Bmatrix} \tag{3.20}$$

By putting

$$\upsilon(x,t) = \exp\left[-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau\right]u(x,t)$$

equation (3.18) reduces to

$$\frac{\partial^\alpha u}{\partial \tau^\alpha} = \frac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ \alpha > 0 \tag{3.21}$$

with initial condition

$$u(x,0) = \begin{Bmatrix} \max(\exp[\frac{1}{2}(k+1)x] - \exp[\frac{1}{2}(k-1)x], 0), \text{ for call options} \\ \max(\exp[\frac{1}{2}(k-1)x] - \exp[\frac{1}{2}(k+1)x], 0), \text{ for put options} \end{Bmatrix} \tag{3.22}$$

and early exercise constraints

$$u(x,\tau) \geq \begin{Bmatrix} \exp\{\frac{1}{4}(k+1)^2\tau\} \max(\exp[\frac{1}{2}(k+1)x] - \exp[\frac{1}{2}(k-1)x], 0), \text{ for call options} \\ \exp\{\frac{1}{4}(k+1)^2\tau\} \max(\exp[\frac{1}{2}(k-1)x] - \exp[\frac{1}{2}(k+1)x], 0), \text{ for put options} \end{Bmatrix} \tag{3.23}$$

The option price in terms of financial variables can be obtained by

$$V(S,t) = K\upsilon(x,\tau) = K\exp[-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau]u(x,\tau) \tag{3.24}$$

Applying the fractional reduced differential transform method on equation(3.21) we have

$$U^\alpha_{h+1}(x) = \frac{\Gamma(\alpha h + 1)}{\Gamma(\alpha(h+1)+1)}[\frac{\partial^2}{\partial x^2}U^\alpha_h(x)] \tag{3.25}$$

Here we consider the initial condition as

$$U^\alpha_0(x) = \exp[\frac{1}{2}(k+1)x] - \exp[\frac{1}{2}(k-1)x],\ x > 0 \tag{3.26}$$

For $h = 0, 1, 2, 3, ...$ using recurrence relation, equation (3.25) and transformed initial condition, equation (3.26) we get,
for $h = 0$;

$$U^\alpha_1(x) = \frac{1}{\Gamma(\alpha+1)}[\frac{\partial^2}{\partial x^2}U^\alpha_0(x)] = \frac{1}{\Gamma(\alpha+1)}[\frac{\partial^2}{\partial x^2}\{\exp[\frac{1}{2}(k+1)x] - \exp[\frac{1}{2}(k-1)x]\}]$$

or

$$U^\alpha_1(x) = \frac{1}{4}\frac{1}{\Gamma(\alpha+1)}[(k+1)^2 e^{\frac{1}{2}(k+1)x} - (k-1)^2 e^{\frac{1}{2}(k-1)x}], \tag{3.27}$$

for $h = 1$;

$$U_2^\alpha(x) = \frac{1}{\Gamma(\alpha+1)}[\frac{\partial^2}{\partial x^2}U_1^\alpha(x)] = \frac{1}{16}\frac{1}{\Gamma(2\alpha+1)}[(k+1)^4e^{\frac{1}{2}(k+1)x} - (k-1)^4e^{\frac{1}{2}(k-1)x}] \quad (3.28)$$

for $h = 2$;

$$U_3^\alpha(x) = \frac{1}{\Gamma(\alpha+1)}[\frac{\partial^2}{\partial x^2}U_2^\alpha(x)] = \frac{1}{64}\frac{1}{\Gamma(3\alpha+1)}[(k+1)^6e^{\frac{1}{2}(k+1)x} - (k-1)^6e^{\frac{1}{2}(k-1)x}] \quad (3.29)$$

for $h = 3$;

$$U_4^\alpha(x) = \frac{1}{\Gamma(\alpha+1)}[\frac{\partial^2}{\partial x^2}U_3^\alpha(x)]$$

$$U_4^\alpha(x) = \frac{1}{256}\frac{1}{\Gamma(4\alpha+1)}[(k+1)^8e^{\frac{1}{2}(k+1)x} - (k-1)^8e^{\frac{1}{2}(k-1)x}] \quad (3.30)$$

In this way, we get,

$$U_n^\alpha(x) = \frac{1}{2^{2n}}\frac{1}{\Gamma(n\alpha+1)}[(k+1)^{2n}e^{\frac{1}{2}(k+1)x} - (k-1)^{2n}e^{\frac{1}{2}(k-1)x}] \quad (3.31)$$

Using fractional reduced differential inverse transform as

$$u(x,\tau) = \sum_{n=0}^{\infty} U_n^\alpha(x)\tau^{\alpha n} \quad (3.32)$$

The solution of the problem (3.15) through fractional reduced differential transform method is given as;

$$u(x,\tau) = \sum_{n=0}^{\infty}\frac{\tau^{n\alpha}}{2^{2n}\Gamma n\alpha+1}[(k+1)^{2n}e^{\frac{1}{2}(k+1)x} - (k-1)^{2n}e^{\frac{1}{2}(k-1)x}] \quad (3.33)$$

$$= e^{\frac{1}{2}(k+1)x}\sum_{n=0}^{\infty}\frac{\tau^{n\alpha}}{\Gamma n\alpha+1}(\frac{(k+1)}{2})^{2n} - e^{\frac{1}{2}(k-1)x}\sum_{n=0}^{\infty}\frac{\tau^{n\alpha}}{\Gamma n\alpha+1}(\frac{(k-1)}{2})^{2n}$$

$$= e^{px}E_\alpha(p^2t^\alpha) - e^{qx}E_\alpha(q^2t^\alpha), \quad (3.34)$$

where $p = \frac{(k+1)}{2}$ and $q = \frac{(k-1)}{2}$. Moreover $E_\alpha(x)$ is the one parameter Mittag-Leffler function defined in [28].

Equation (3.34) gives the closed form solution of equation (3.15). It is well known that the above solution is obtained from a recursive relation and the exact solution can be obtained by adding more terms to the series. Once the solution is obtained and the option value is calculated we implement Bellman's principle to ensure that

$$u(x,\tau) \geq \begin{Bmatrix} \exp\{\frac{1}{4}(k+1)^2\tau\}\max(\exp[\frac{1}{2}(k+1)x] - exp[\frac{1}{2}(k-1)x], 0), \text{ for call options} \\ \exp\{\frac{1}{4}(k+1)^2\tau\}\max(\exp[\frac{1}{2}(k-1)x] - \exp[\frac{1}{2}(k+1)x], 0), \text{ for put options} \end{Bmatrix} \quad (3.35)$$

4. Results and discussion

In this section, the graphical view of the series solution given in equation (3.34) for time fractional Black-Scholes equation governing American options paying no dividends are presented. The graphical solutions are computed using MATLAB programming. In figures 1 and 2, the interval used for parameters for x and τ is same while different values of α and k are taken. Figures 3 and 4, show the graphical representation of the solution given in equation (3.34). For each case, different intervals are used for the time parameter τ, α and k while the same interval is used for x.

Figure 1, represents the surface plot of call option $u(x,\tau)$ corresponding to asset price x and time τ. Effects of fractional order α on option price are shown in Figure 1, left and

TABLE 1. Numerical solution of American Call options through FRDTM

τ	x	$\alpha = 1, k = 1$	$\alpha = 0.9, k = 1$	$\alpha = 1, k = 2$	$\alpha = 0.9, k = 2$
0.25	0.3	0.3375	0.4030	0.8095	0.9669
	0.6	0.4555	0.5440	1.2990	1.5515
	0.9	0.6148	0.7343	2.0715	2.4741
0.50	0.3	0.6749	0.7521	1.6190	1.8042
	0.6	0.9110	1.0152	2.5981	2.8952
	0.9	1.2297	1.3703	4.1430	4.6168
0.75	0.3	1.0124	1.0883	2.4286	2.5988
	0.6	1.3665	1.4623	3.8971	4.1703
	0.9	1.8445	1.9738	6.2144	6.6501
1.0	0.3	1.3498	1.4035	3.2381	3.3668
	0.6	1.8220	1.8944	5.1961	5.4027
	0.9	2.4594	2.5571	8.2859	8.6153

right. The comparison analysis show that for greater order of α, call option has a lower value, while for lower orders of α call option has a higher value.

In Figure 2, the surface plot of option value is identical to that of Figure 1. It is noted that for higher value of k, the option value increases exponentially. Moreover for a higher value of α, the value of option price decreases and increases for lower value of α.

In Figures 3 and 4, prices are investigated for various orders of $\alpha = 0.1, 0.2, ..., 1$. Effects of fractional order α are shown. It can be deduced from the comparison test that the a lower call option value is obtained for higher order of α and a higher call option value is obtained for lower value of α. Moreover, in Figures 3 and 4 (left and right), the time parameter τ and k also varies, the option price increases with increasing k and τ.

The numerical solution at different values of α, k, x and τ are shown in Table 1 given below. Table 2 lists the computational values of American call options through fractional reduced differential transform method (FRDTM) and is compared with the results obtained by other numerical methods such as the Laplace transform method (LTM) developed by Wong and Zhao [36] and finite difference method (FDM) [7, 33, 37]. At maturity $\tau = 3$, it can be well observed that option price decreases with decreasing α. By comparison with other methods, the advantages of this method can be observed.

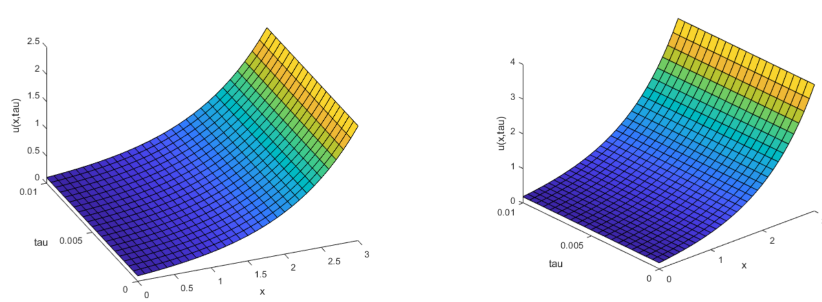

FIGURE 1. demonstrates the solution of equation (3.15): (Left) for $\alpha = 1$, $k = 1$, (Right) for $\alpha = 0.9$, $k = 1$.

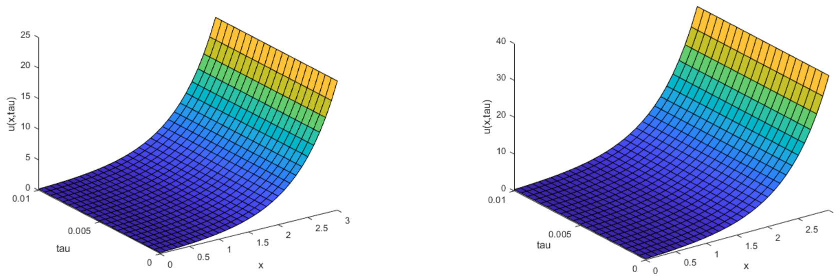

FIGURE 2. demonstrates the solution of equation (3.15): (Left) for $\alpha = 1$, $k = 2$, (Right) for $\alpha = 0.9$, $k = 2$.

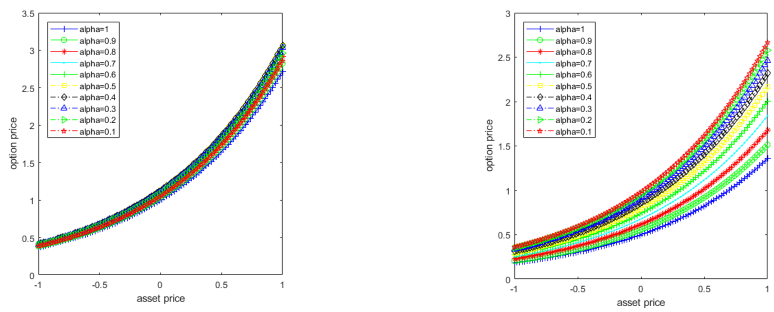

FIGURE 3. demonstrates the solution of equation (3.15): (Left) for $\alpha = 0.1, 0.2, ..., 1$, $k = 1$, $\tau = 1$, (Right) for $\alpha = 0.1, 0.2, ..., 1$, $k = 1$, $\tau = 0.5$.

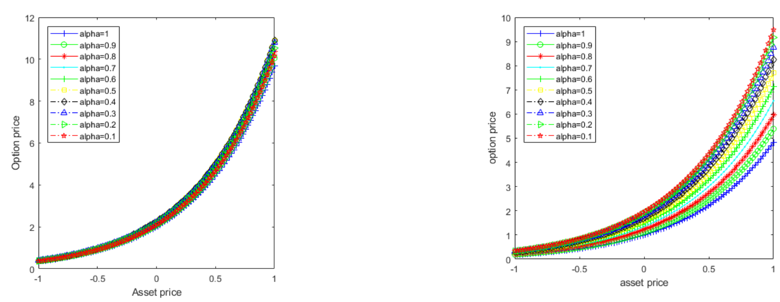

FIGURE 4. demonstrates the solution of equation (3.15): (Left) for $\alpha = 0.1, 0.2, ..., 1$, $k = 2$, $\tau = 1$ (Right) for $\alpha = 0.1, 0.2, ..., 1$, $k = 2$, $\tau = 0.5$.

TABLE 2. Numerical solution of American Call options through FRDTM, LTM and FDM

	$\tau = 3$	$x = 0.5$	k=1
α	$FRDTM$	LTM	FDM
0.2	2.2368	2.2376	2.2398
0.4	2.8835	2.8712	2.8762
0.5	3.2221	3.2168	3.2213
0.7	3.9149	3.8926	3.9014
0.9	4.6075	4.5834	4.5891
1.0	4.9459	4.7987	4.8165

5. Conclusion

With the growth of fractional order differential equations in different fields of applied Mathematics, it is necessary to modify solution techniques to approach such problems. In this work we have introduced the fractional reduced differential transform method coupled with Bellman's principle of optimality to obtain the solution of American option pricing problem based on time fractional Black-Scholes equation. By using FRDTM, the solution can be obtained in a series form and the Mittag-Leffler function appears in the solution. The successful application of the FRDTM proves that this technique is effective and requires less computational work to solve fractional partial differential equation. The advantage of using FRDTM is that it enhances the effectiveness of the computational work without utilizing linearization, discretization and restrictive assumptions. Numerical results show that FRDTM is an efficient and useful technique in order to find the exact and approximate solutions of fractional differential equations. Numerical results are presented to illustrate the efficacy and performance of the proposed method.

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