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The fruit of the meditations of a young Lieutenant of the Engineers: A Theorem of Major Impact and Its Applications. Two Centuries from the Discovery of the Nine-Point Circle

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ABSTRACT. The Nine-Point Circle Theorem was first obtained in a joint work written by C.-J. Brianchon and J.-V. Poncelet. Since the authors of this Theorem presented it in their original work as a principle in geometry, we are invited to reflect whether there are some *direct* applications of this Theorem in advanced Euclidean geometry. Our exploration only confirms the original authors' foretelling intuition, since *un nouveau principe à la géométrie élémentaire* is expected to admit applications. We regard the Nine-Point Circle Theorem in the overall context of the history of ideas and, pursuing Philip Davis' thoughts, we regard the geometry of triangle as a chapter in the history of mathematical cultures.

1. FROM THE BATTLEFIELD OF KRASNOY TO METZ

For the readers of *War and Peace*, it is impossible to forget the permeating atmosphere and the revealing details of the battlefield scenes of Krasnoy, so masterfully described by Lev Tolstoy's limpid prose. However impressive, *War and Peace* is a work of fiction. On the other hand, it is factually true that Jean-Victor Poncelet (July 1, 1788 – December 22, 1867) was left for dead on the battlefield after the battle of Krasnoy. Fortunately for mathematics, he survived the ordeal of injury and captivity, to return to France and pursue a career as an engineer and mathematician, whose culmination towards the end of his career was his term as the Commanding General of the École Polytechnique, from 1848 until his retirement in 1850 (at 62). One of his major works is *Applications d'analyse et de géométrie,* whose first version was written while he was a war prisoner in Russia in 1813 and 1814. In the *Preface* of this impressive monument of mathematics [10], we read several unusual paragraphs, quite emotional for a foreword of a mathematical text:

> I could have entitled this work, which is purely mathematical, *Memoirs from beyond the tomb*. It is, in fact, the fruit of the meditations of a young lieutenant of the engineers, left for dead on the fatal battlefield of Krasnoy, not far from Smolensk, and for a long time strewn with the bodies of the French army. There, in that terrible retreat from Moskow, seven thousand Frenchmen, exhausted by hunger, cold and fatigue, under the orders of the unfortunate Marshal Ney, came, deprived of all artillery, on the 18th of November 1812, the anniversary of the Russian Saint Michael, to fight a furious, bloody and final combat with twenty-five thousand soldiers, fresh and equipped with forty cannons of Field-Marshall Prince Miliradowitch, who himself would soon became the victim of a military conspiracy hatched in the bosom of the modern capital of the Muscovite Tsars.

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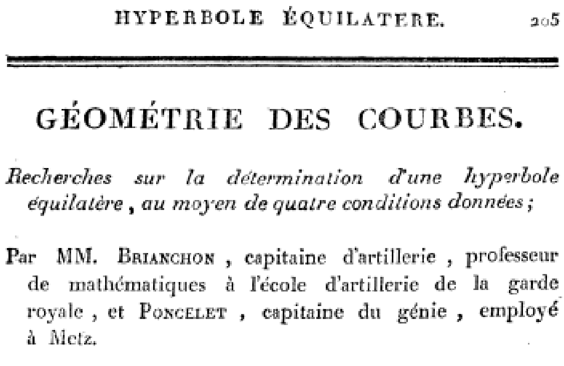
Was Poncelet exaggerating his Russian adventure? Not at all, seem to conclude the historians. Jeremy Gray, writing about Poncelet, confirmed that [7], p.13,

> he fought in the terrible battle at Krasnoy but was taken prisoner, and survived the years that followed only by luck. In the winter of 1812 it became so cold that even the mercury in the thermometers froze (which occurs at a temperature of -39^oC). He managed to get to the hospital at Saratov, where he remained a prisoner until the defeat of Napoleon and the Treaty of Paris was signed on 30 May 1814; his journey home took two and a half months, and he arrived in Metz on 7 September. In prison there there was nothing to restore him to health but the April sun, and there, to distract his spirits, he resumed this study of mathematics.

Poncelet is considered today one of the founders of modern projective geometry, simultaneously discovered by Joseph Gergonne.

From among Poncelet's many contributions, we focus our attention today on one particular paper, published about two centuries ago. It was Poncelet's collaboration with Charles-Julien Brianchon (December 19, 1783 – April 29, 1864). Their joint work is published in January 1821, and is titled *Recherches sur la détermination d'une hyperbole équilatere, au moyen de quatre conditions données.* [1].



HYPERBOLE ÉQUILATERE. 205

GÉOMÉTRIE DES COURBES.

Recherches sur la détermination d'une hyperbole équilatère, au moyen de quatre conditions données;

Par MM. BRIANCHON, capitaine d'artillerie, professeur de mathématiques à l'école d'artillerie de la garde royale, et PONCELET, capitaine du génie, employé à Metz.

FIGURE 1. The author's presentation in Brianchon and Poncelet's original paper where the Nine-Point Circle appears for the first time.

In this paper, Theorem IX deals with a very interesting property of the classical geometry of the triangle, namely that the feet of the altitudes, midpoints of sides, and the so-called Euler's points (the points on the altitudes equidistant from the orthocenter and the vertices of the triangle) of the triangle lie on the same circle. The authors were aware immediately they have reached an important idea, since they describe it as "un nouveau principe à la géométrie élémentaire." The authors must have felt from the very moment when they wrote their work that they discovered a fact of unusual importance.

Much can be said about the perennial historical importance of the Nine-Point Circle Theorem. It plays a central role in several monographs on advanced Euclidean geometry, see e.g. [3], pp.20-22; or the classical [9]. Historically, together with Euler's work [5], the paper [1] motivated further advances in Euclidean geometry, and helped shape the interest as well as the taste for geometry in the two centuries that passed since its publication.

The question we propose in this paper is the following.

Does the Nine-Point Circle Theorem have any **direct** *elementary applications?*

The question is quite important for the following reason. In a visionary paper written a quarter of century ago, Philip J. Davis [4] discussed the evolutions of the geometry of the triangle as a casebook in the history of ideas. He is pointing out that, as a research area, in a way, the geometry triangle became more of a history piece, although one should point out that notable efforts are undergoing even in our times to pursue this noble academic tradition. On the other hand, never downplaying the importance of this theory, Davis points out that in a "sophisticated definition" the triangle geometry is "the invariant theory of five points under the projective group". There are profound nuances related to this approach, all connected to Felix Klein's Erlangen Program (see e.g. [6]).

2. The theorem

First and foremost, we should present the facts. We start with the following elementary

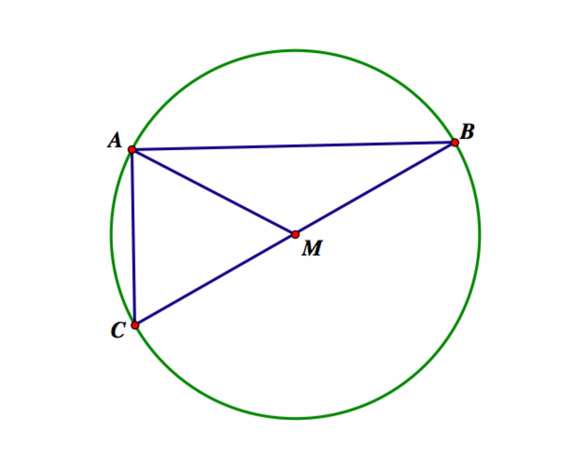


FIGURE 2. For Lemma 2.1.

Lemma 2.1. *Let ΔABC be a triangle in the Euclidean plane with the property that the median $[AM]$ is half the side $[BC]$. Then $m(\angle BAC) = \frac{\pi}{2}$.*

Proof. Since $MA = MB = MC$, consider the circle $\mathcal{C}$, of radius MA, centered in M. In this circle, the segment $[BC]$ has the length equal to two radii. Then $\angle BAC$ cuts the circle in the same points as the diameter $[BC]$. Therefore $[BA] \perp [AC]$. $\square$

THÉORÈME IX. Le cercle qui passe par les pieds des perpendiculaires abaissées des sommets d'un triangle quelconque sur les côtés qui leur sont opposés, passe aussi par les milieux de ces trois côtés, ainsi que par les milieux des distances qui séparent les sommets du point de croisement des perpendiculaires.

FIGURE 3. The statement of the Nine Point Circle Theorem, as it was published originally in Brianchon and Poncelet's paper.

Theorem 2.1. *In ΔABC in the Euclidean plane, consider M, N, P the midpoints of sides $[BC]$, $[AC]$, $[AB]$, the feet of altitudes A', B', C', and the orthocenter H. Denote by A'', B'', C'' the midpoints of $[AH]$, $[BH]$, and $[CH]$, respectively. Then M, N, P, A', B', C', A'', B'', C'' lie on the same circle.*

Proof. Consider the circle passing through the midpoints M, N, P. We show that all the other six points lie on this circle.

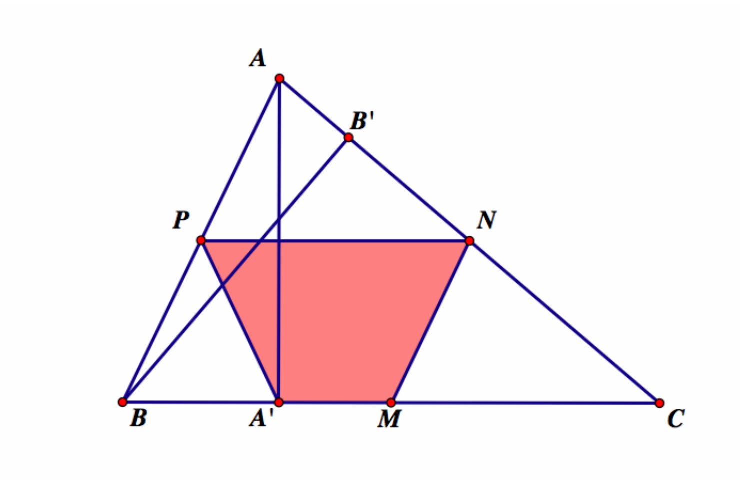


FIGURE 4. The first step in proving the theorem is discovering an isosceles trapezoid which is cyclic.

The segment $[MN]$ is midline in ΔABC, thus $MN \| AB$, and $MN = \frac{AB}{2}$. Since $\Delta AA'B$ is a right triangle, $A'P = \frac{AB}{2}$. Therefore $A'P = MN$. In fact, we proved that $A'MNP$ is an isosceles trapezoid, therefore a cyclic quadrilateral. Similarly we prove that B' and C' lie on the circle (MNP).

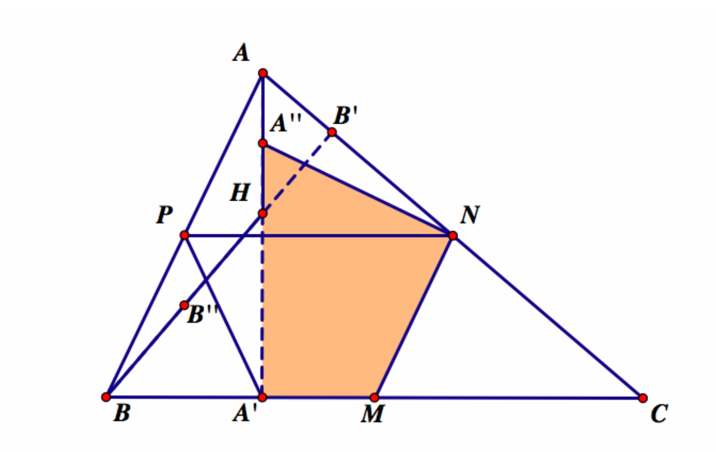


FIGURE 5. The second step in the proof is revealing a cyclic quadrilateral that has two right opposite angles.

It remains to prove that A'', B'' and C'' lie on the circle (MNP). In the quadrilateral $A'MNA''$, the angle $\angle A''A'M$ is right. In ΔAHC, $A''N$ is the midline parallel to the side CH. Since $MN \| AB$, $A''N \| CH$ and $CH \perp AB$, we obtain that the $\angle A''NM = \dfrac{\pi}{2}$. Therefore the quadrilateral $A'MNA''$ is cyclic. Thus, the point A'' is on the circle (MNP). Similarly, B'' and C'' are on the circle (MNP). $\square$

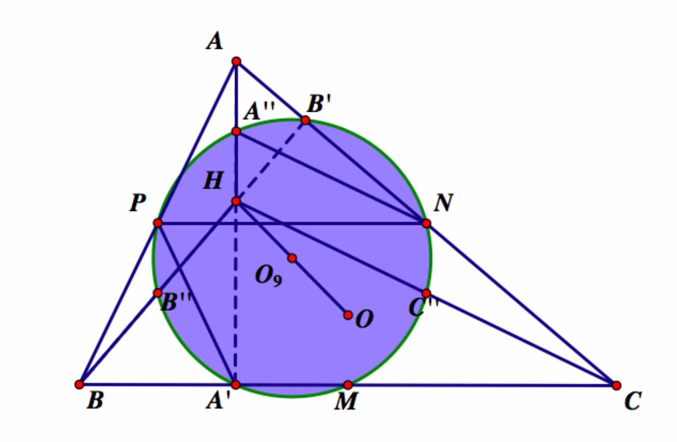


FIGURE 6. As Brianchon and Poncelet felt, it is a "principle": the Nine Point Circle revealed. The circle, denoted in some references O_9, lies at the midpoint of the segment determined by the orthocenter H and the circumcenter O. The segment $[OH]$ and the relative distances related to it were investigated successfully first by Leonhard Euler in [5].

3. EXAMPLES OF DIRECT APPLICATIONS

The examples we present here illustrate the following categories.

- Direct applications of the Nine-Point Circle configurations on different triangles than the reference triangle;
- Configurations that are "hiding" the original Brianchon-Poncelet configuration;
- Results that are helping locating points of interest within the geometry of triangle in the Euclidean plane.

Of course, these are not the only possible classes of applications, but they are undoubtedly *direct* applications, in every sense of the meaning. The very fact they exist and are of recent nature illustrate that Brianchon and Poncelet's original assessment (that the theorem represents "a principle") remains of interest within the field of triangle geometry.

Application 1. [13] *Let ABC be an acute triangle with incenter I, and let D, E, and F be the points where the circle inscribed in ABC touches $[BC]$, $[CA]$, and $[AB]$, respectively. Let M be the intersection of the line through A parallel to BC and DE, and let N be the intersection of the line through A parallel to BC and DF. Let P and Q be the midpoints of $[DM]$ and $[DN]$, respectively. Prove that A, E, F, I, P, and Q are on the same circle.*

Note: The geometric configuration investigated in this problem gave rise to a projective criterion of cyclicity [2]. See also various exploration of this geometric configuration in [12, 15] and especially the generalization in [16].

Solution: First, by using the congruence of alternate internal angles, note that $\angle ANQ \equiv \angle BDF \equiv \angle DFB \equiv \angle AFN$, therefore $[AN] \equiv [AE]$. Similarly, $[AF] \equiv [AM]$, therefore $[AM] \equiv [AN]$. Since, A, P and Q are midpoints of sides in the triangle DMN, it is natural to regard DNM as the reference triangle. The circle determined by the medial triangle of $\Delta\, DMN$ is the Nine Point circle in $\Delta\, DMN$.

Henceforth, E and F lie also on this circle, since they are the feet of the altitudes from N and M, respectively, in ΔDNM. To see this, remark that in the triangle NEM the median

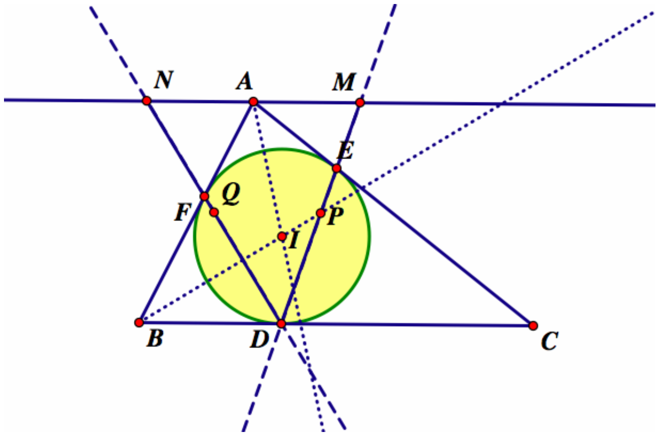


FIGURE 7. For Application 1.

$[AE]$ is half of the side $[NM]$, therefore ΔNEM is a right triangle with a right angle in E. A similar argument shows that F is the foot of the altitude from M.

For the last step of the proof, we investigate the position of I. Denote by $\{H\} = FM \cap NE$. Since $\angle(HED) \equiv \angle(HFD)$, as right angles, we see that H lies on the circle centered in I and of radius $IE = IF = ID$. However, as intersection of two altitudes in $\Delta\, DMN$ the point H is the orthocenter of this triangle. Thus, HD is a diameter in the incircle of $\Delta\, ABC$. It follows that I is the midpoint of $[HD]$, and it belongs to the Nine Point circle in $\Delta\, DMN$.

Application 2. (Nicuşor Minculete) *In ΔABC consider the altitudes $[AA']$, $[BB']$, $[CC']$, and denote by H the orthocenter. Let A'', B'', and C'' be the midpoints of the segments $[AH]$, $[BH]$, $[CH]$, respectively. Let A_2, B_2, C_2 be the midpoints of the sides of the orthic triangle $[B'C']$, $[C'A']$, $[B'A']$, respectively. Then the lines $A''A_2$, $B''B_2$, $C''C_2$ are concurrent.*

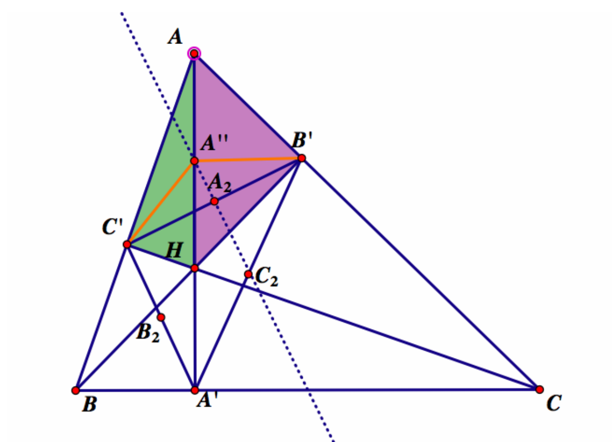


FIGURE 8. For Application 2.

Solution: Since $\Delta AC'H$ is right, the length of the median $C'A''$ is half the hypotenuse. Similarly, $B'A'' = \frac{AH}{2}$ in the right triangle $AB'H$. Therefore, $C'A'' = A''B'$. We proved that $\Delta A''B'C'$ is isosceles. In consequence, its median $[A''A_2]$ is also altitude. Thus

$A''A_2 \perp B'C'$, or furthermore $A''A_2$ is the perpendicular bisector of $B'C'$. Similarly, $B''B_2$ is perpendicular bisector for $[A'C']$ and $C''C_2$ for $[A'B']$. The perpendicular bisectors of $\Delta A'B'C'$ must intersect in the circumcenter of this triangle, which must be the center of the Nine-Point circle, as A', B', and C' lie on it.

We present now our last example.

Application 3. ([14]) *Consider the acute triangle ABC. Denote by T the intersection of the angle bisector of $\angle BAC$ with the circumcircle of ΔABC. The projections of A and T on BC are respectively G and K, the projections of B and C on AT are respectively H and L. Then:*

(i) the points G, H, K, L lie on a circle,

(ii) $KH \parallel AC, GL \parallel BT, GH \parallel TC$, and $LK \parallel AB$,

(iii) then the center of the circle determined by G, H, K, L lies on the Nine-Point circle in ΔABC.

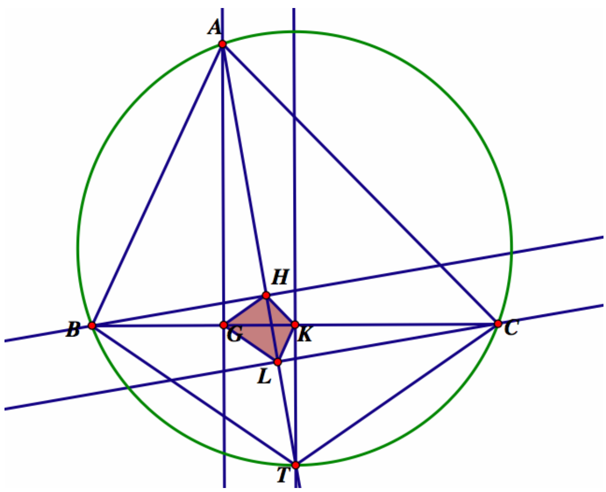


FIGURE 9. For Application 3.

Solution: (i) Let P be the intersection of AT with BC. The quadrilateral $ABGH$ is cyclic (since $m(\angle BHA) = m(\angle AGB) = \frac{\pi}{2}$.) Therefore $m(\angle HGP) = m(\angle BAH)$. Similarly, from the cyclic quadrilateral $CKLT$ one gets $m(\angle KCT) = m(\angle PLK)$ and since $m(\angle KCT) = m(\angle BAT)$ the conclusion is $m(\angle HGK) = m(\angle HLK)$, therefore $GLKH$ is cyclic.

(ii) The quadrilateral $AGLC$ is cyclic since $m(\angle AGC) = m(\angle ALC) = \frac{\pi}{2}$, therefore $m(\angle GLA) = m(\angle GKH)$ and thus $m(\angle GKH) = m(\angle BCA)$ or $HK \parallel AC$.

To see that $GL \parallel BT$, let's look first to the cyclic quadrilateral $BTKH$. We have $m(\angle BTH) = m(\angle BKH)$. Using (i), in the quadrilateral $KHGL$ we have $m(\angle BKH) = m(\angle HLG)$. Therefore $m(\angle HLG) = m(\angle ATB)$, so $GL \parallel BT$.

We now prove that $HG \parallel TC$. By (i), $m(\angle GHL) = m(\angle GKL)$. From the cyclic quadrilateral $KLTC$ we have $m(\angle LTC) + m(\angle LKC) = \pi$. But $m(\angle LKB) + m(\angle LKC) = \pi$, so $m(\angle GHL) = m(\angle GKL) = m(\angle LTC)$. Therefore $HG \parallel TC$.

Since $m(\angle ABC) = m(\angle ATC)$, the above arguments yield also $m(\angle ABC) = m(\angle BKL)$, therefore $AB \parallel KL$.

(iii) Let us denote by O' the center of the circle $(GHKL)$. We remark first that

$$m(\angle GLK) = m(\angle BCA) + \frac{1}{2}m(\angle BAC).$$

In the circle centered in O' we have $m(\angle KO'G) = 2m(\angle KLG)$, therefore

$$m(\angle KO'G) = 2\left[m(\angle BCA) + \frac{m(\angle BAC)}{2}\right] = 2m(\angle BCA) + m(\angle BAC).$$

On the other hand $m(\angle KEG) = m(\angle ABC) - m(\angle ACB)$ therefore, adding term by term these last two relations

$$m(\angle KO'G) + m(\angle KEG) = m(\angle ABC) + m(\angle BCA) + m(\angle BAC) = \pi.$$

Therefore the points K, O', G, E lie on the same circle, the Nine Point circle of the triangle ABC.

4. Conclusions

More than two centuries have passed since the war scenes described in the first section. The geometry of triangle, whose evolution is described by Philip Davis in [4] as a casebook in the history of mathematics, represents a stellar case in the history of culture. It had time to rise with all the enthusiasm of the beginnings in the 18th century, to flourish and develop in the 19th century, and to enter into a decline as research area when its methods apparently exhausted, to remain today the focus of interests of the many enthusiasts, as the classical music attracts in every generation the interested connoisseurs. Nevertheless, the geometry of triangle represents part of mankind's most precious heritage [8]. The very direct applications of the quintessential theorems that inspired and guided the interests and taste of many authors represent a casebook of ideas in itself, and it illustrates the logical ramifications of a theory that preserves today both historical and mathematical interest.

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